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An adaptation of adomian decomposition for numeric–analytic integration of strongly nonlinear and chaotic oscillators

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Abstract

A novel form of an explicit numeric-analytic technique is developed for solving strongly nonlinear oscillators of engineering interest. The analytic part of this technique makes use of Adomian Decomposition Method (ADM), but unlike other analytical solutions it does not rely on the functional form of the solution over the whole domain of the independent variable. Instead it discretizes the domain and solves multiple IVPs recursively. ADM uses a rearranged Taylor series expansion about a function and finds a series of functions which add up to generate the required solution. The present method discretizes the axis of the independent variable and only collects lower powers of the chosen step size in series solution. Each function constituting the series solution is found analytically. It is next shown that the modified ADM can be used to obtain the analytical solution,in a piecewise form. For nonlinear oscillators such a piecewise solution is valid only within a chosen time step. An attempt has been made to address few issues like the order of local error and convergence of the method. Emphasis has been on the application of the present method to a number of well known oscillators. The method has the advantage of giving a functional form of the solution within each time interval thus one has access to finer details of the solution over the interval. This is not possible in purely numerical techniques like the Runge–Kutta method, which provides solution only at the two ends of a given time interval, provided that the interval is chosen small enough for convergence. It is shown that the present technique successfully overcomes many limitations of the conventional form of ADM. The present method has the versatility and advantages of numerical methods for being applied directly to highly nonlinear problems and also have the elegance and other benefits of analytical techniques.

Keywords: Adomian decomposition method; Nonlinear oscillators; Chaos

1. Introduction

In many practical situations a system of coupled, possibly damped, nonlinear ordinary differential equations model the dynamical behavior of mechanical systems. For example, these equations arise (following some discretization procedure) while studying the mechanical response of systems such as strings, beams, absorbers, plates, and so on. In general, exact solutions of such equations are unknown and thus numerical integration, perturbation techniques or geometrical methods (see [1–4] and references there in) have been applied to obtain their approximate solutions. However, in many of the analytical techniques, it becomes necessary to resort to linearization techniques or assumption of weak nonlinearity, except for a small class of low-dimensional problems which can be transformed to linear equations. This so-called weak nonlinearity or small parameter assumption greatly restricts applications of perturbation techniques known that an overwhelming majority of nonlinear problems have no small parameters at all. Therefore such analytic routes may not be able to treat strongly nonlinear problems. Recently there are few attempts to overcome this restriction of weak nonlinearity (see, for instance [5–7]), but they

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either use the small parameter assumption indirectly or lack the versatility and applicability to all kinds of problems. Numerical integration methods, even though more versatile than their analytical counterparts, sometimes respond too sensitively to the choice of time-step size to be reliable (see, for instance, Yamaguti and Ushiki [8]; Lorenz [9]). Moreover, numerical integration schemes always offer approximate solutions in a discretized form thereby making it difficult to obtain a continuous representation. A third route, based on numeric-analytical algorithms, has also been tried out for nonlinear deterministic and stochastic initial value problems (IVPs). The phase-space linearization (PSL) method, proposed by Roy [10,11] and Iyengar and Roy [12,13] and implicit local linearization method termed as the Locally Transversal Linearization (LTL) [4,14] are two such ways. The present objective is to follow them up with yet another numerical-analytic technique based on Adomian Decomposition technique. George Adomian introduced a method in [15] for solving nonlinear functional equations of various kinds (algebraic, differential, delay differential, etc.), and the method is known as Adomian Decomposition Method (ADM). This method and its application are well documented in [16–22]. The technique uses a decomposition of the nonlinear term into a series of functions. Each term of this series is a generalized polynomial called Adomian's polynomial. These polynomials generate an infinite set of functions whose sum determines the actual solution. This method has significant advantages: it is not restricted with weak nonlinearity assumption, and provides a rapidly convergent series [19]. A more detailed description about the mathematical background of this method is available in [22]. Another important work by Rach [20] is worthy of mention here. It makes a comparison between Picard's method and ADM, and concludes that these two methods are not the same, with Picard's method being applicable only if the vector field satisfies the Lipschitz condition. The concept of noise terms is introduced in [23] for a faster convergence of decomposed series solution. Noise terms have been defined as the identical terms with opposite signs that appear in the first two terms of the series solution. It is also shown that noise terms appear always for inhomogeneous equations. Though easy computations of series terms are closely related to global accuracy and applicability of the method, this aspect has never been explored in previous works. In Section 2 of this study, it is demonstrated that the finite series approximation is inaccurate. Unless the decomposed solution is valid over long time intervals, easier computations of series terms are not of importance. Indeed, in the context of integration of equations of motion of nonlinear oscillators, the global accuracy of the method needs to be addressed rigorously. To the authors' knowledge such issues have not been addressed, especially for simulations of strongly nonlinear and possibly chaotic oscillators using ADM.

It has been stated that the proof of convergence of Adomian series may be based on

(1) the fixed point theorem [19,22];
(2) the assumption that the nonlinear function is replaceable by an infinite series with a convergence radius equal to infinity [24].

Supplemented by the assumption that the convergence is fast. But for all practical purposes, this series converges only locally because it is based on a Taylor series expansion about an initial function which is often monotonically increasing. This observation will be substantiated in the present work through application of the conventional ADM to dynamical system. This lack of global convergence is overcome here by recursively applying ADM over successive time intervals, thereby solving a sequence of IVPs, each valid over a given time interval. However, it is found that this discretization technique fails to provide accurate solutions when there are forcing terms and in several other cases where the system's nonlinearity is of a non-polynomial form. The reason behind is that it is too complicated to compute series terms other than first few in such cases using the conventional ADM. So, the forcing function is also expanded in a Taylor series and appropriately distributed in all the series terms. Finally, terms with higher powers of increments of the independent variable are removed. This provides a systematic algorithm for the analytical computation of series terms. Indeed the removal of higher power helps considerably in simplifying the analytical derivation of series terms. These techniques will be discussed in Section 3. Computation of local error and its global propagation is investigated theoretically as well as with help of examples and compared with existing numerical techniques. Special attention is given to applicability of this method to oscillators responding in the chaotic regime. The present method demands the usage of extensive symbolic computation in obtaining the series terms. But this method has certain advantages as follows,

(1) Conventional ADM gives only locally convergent results (see Section 2.1). It is shown that the present method successfully overcomes such limitations and has applicability to a large class of nonlinear dynamical systems of engineering interest.
(2) Like other numerical techniques this method is also versatile and can solve strongly nonlinear and chaotic systems – something not possible through purely analytical techniques.
(3) The present method can provide a piecewise functional form of the solution within each discrete interval. The approximated solution may thus be made as smooth as desired, which is not possible with any purely numerical technique like the Runge–Kutta method as the latter merely provides a discretized approximation of the true solution.
(4) The order of accuracy is quite high and may be further increased by simply generating more series terms without decreasing the time-step size.

(5) Compared to most existing numerical methods, the proposed method can work with relatively much larger time-step sizes.

In the next section (Section 2) a brief description of ADM is provided. A brief description of the limitations of conventional ADM, which led to the development of the present method, is also outlined. In Section 3, the method developed in the present work is discussed in detail in the context of multi-dimensional systems. To emphasize the higher accuracy of the present method, existing proof of convergence of ADM is discussed and the order of local error for the present method is derived in Section 4. In order to elucidate the implementation of the present schemes, numerical illustrations are provided on a range of strongly nonlinear oscillators in Section 5. Different programming techniques have been adopted for the betterment of this method, and they will be discussed in this section too.

2. Brief description of adomian decomposition method

Consider a dynamical system $F(u) = g(t)$ where F is a nonlinear ordinary differential operator with linear and nonlinear terms. The linear term is written as $Lu + Ru$ where L is chosen as the highest-ordered derivative. Thus the whole equation may be written as $Lu + Ru + Nu = g(t)$, where Nu denotes the nonlinear terms. u lies in a Banach space G. The equation is now recast as:

$$\begin{aligned} Lu &= g - Ru - Nu, \\ L^{-1}Lu &= L^{-1}g - L^{-1}Ru - L^{-1}Nu. \end{aligned} \tag{1}$$

If this corresponds to an IVP and L is a differential operator then L^{-1} may be regarded as multiple definite integrations from 0 to t. For example when L is $\mathrm{d}^2/\mathrm{d}t^2$, the last equation becomes

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{2}$$

where A and B are integration constants. For IVPs, they need to be found from initial conditions and may be identified as $u(0)$ and $\frac{\mathrm{d}u(0)}{\mathrm{d}t}$, respectively. Here, N is assumed to be a contracting nonlinear analytical operator and decomposed as an infinite sum of functions,

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots u_n), \tag{3}$$

where the A_n are the Adomian polynomials valid only for the specific Nu. Adomian polynomials A_n depend on u_i for $i = 0, \ldots, n-1$ and form a rapidly convergent series [17]. Now, let the solution u of Eq. (1) be found as a series of functions u_n i.e.,

$$u = \sum_{n=0}^{\infty} u_n. \tag{4}$$

Furthermore, this series is taken to be absolutely convergent, i.e., $\sum_{n=0}^{\infty}|u_n| < \infty$, where, u_0 is identified as the function $A + Bt + L^{-1}g$. This reduces Eq. (2) to

$$\sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{5}$$

Consequently, one may write

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \\ &\vdots \end{aligned} \tag{6}$$

With the preceeding assumptions on u and N, the Adomian series Eqs. (4) and (6) is a solution of Eq. (1) [25,22].

Next it is required to find Adomian's polynomials of Eq. (3), which are needed to derive series solution of Eq. (6). With the solution already written in the form $u = \sum_{n=0}^{\infty} u_n$, one may introduce the following power series in terms of an arbitrary scalar parameter λ

$$\hat{u}(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n, \tag{7}$$

where the series has a convergence radius ρ. Since both $\hat{u}$ and N are analytic over OD(0, ρ) (i.e., an open disc in G with centre at 0 and whose radius is ρ), $N\hat{u}$ is analytic over OD(0,ρ). Therefore, there exists operators A_k so that

$$N\hat{u}(\lambda) = \sum_{k=0}^{\infty} A_{\mathrm{k}} \lambda^n. \tag{8}$$

When the convergence radius is 1, $\hat{u}(1)$ converges to u. By Abel's theorem [26] one can arrive at $\lim_{\lambda \to 1^-} \hat{u}(\lambda) = u$, ($\lambda$ being a real number). Therefore

$$\lim_{\lambda \to 1^-} N\hat{u}(\lambda) = Nu. \tag{9}$$

Eq. (8) can be considered as a Taylor–Maclaurin series expansion of the function $N\hat{u}$. Then from Eqs. (7)–(9) one gets

$$A_{\mathrm{k}} = \frac{1}{\mathrm{k}!}\left(\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} f\left(\sum_{n=0}^{\infty} u_n \lambda^n\right)\right)_{\lambda=0}, \tag{10}$$

where $f(u)$ is the functional form of nonlinear term Nu. For more detailed derivation of Adomian polynomials one may refer [22].

The generation of Adomian polynomials may also be done by simply rearranging the Taylor series expansion of $f(u)$ with respect to a function u_0 as described in [27]. The Taylor series expansion of $f(u)$ about u_0 is as follows

$$
\begin{aligned}
f(u) &= f(u_0) + (u-u_0)f'(u_0) + \frac{1}{2!}(u-u_0)^2 f''(u_0) \\
&\quad + \frac{1}{3!}(u-u_0)^3 f'''(u_0) + \cdots \\
&= f(u_0) + (u_1+u_2+u_3+\cdots)f'(u_0) \\
&\quad + \frac{1}{2!}(u_1+u_2+u_3+\cdots)^2 f''(u_0) + \cdots
\end{aligned}
$$

This can be rearranged as

$$
\begin{aligned}
f(u) = f(u_0) &+ \left[\left(\frac{1}{1!}\sum_{i_1=1}^{1}\delta_{1,i_1}u_{i_1}\right)f'(u_0)\right] \\
&+ \left[\left(\frac{1}{1!}\sum_{i_1=1}^{2}\delta_{2,i_1}u_{i_1}\right)f'(u_0) + \left(\frac{1}{2!}\sum_{i_1,i_2=1}^{1}\delta_{2,i_1+i_2}u_{i_1}u_{i_2}\right)f''(u_0)\right] \\
&+ \left[\left(\frac{1}{1!}\sum_{i_1=1}^{3}\delta_{3,i_1}u_{i_1}\right)f'(u_0) + \left(\frac{1}{2!}\sum_{i_1,i_2=1}^{2}\delta_{3,i_1+i_2}u_{i_1}u_{i_2}\right)f''(u_0)\right. \\
&\left. + \left(\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^{1}\delta_{3,i_1+i_2+i_3}u_{i_1}u_{i_2}u_{i_3}\right)f'''(u_0)\right] + \cdots
\end{aligned} \tag{11}
$$

As a consequence, the nonlinear function becomes

$$
f(u) = f(u_0) + \sum_{n=1}^{\infty}\left[\sum_{v=1}^{n}\frac{1}{v!}\left(\sum_{i_1,i_2,\cdots,i_v=1}^{n+1-v}u_{i_1}u_{i_2}\cdots u_{i_v}\right)f^v u_0\right], \tag{12}
$$

where $\delta_{i,j}$ is the usual Kroneker delta function and f^v denotes vth derivative of f with respect to u_0. Now replacing

$$
A_n = \left[\sum_{v=1}^{n}\frac{1}{v!}\left(\sum_{i_1,i_2,\cdots,i_v=1}^{n+1-v}u_{i_1}u_{i_2}\cdots u_{i_v}\right)f^v u_0\right] \tag{13}
$$

one can write the nonlinear term as

$$
f(u) = \sum_{n=0}^{\infty}A_n. \tag{14}
$$

From this

$$
\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= u_1 f'(u_0), \\
A_2 &= u_2 f'(u_0) + \frac{u_1^2}{2!}f''(u_0), \\
A_3 &= u_3 f'(u_0) + u_1u_2 f''(u_0) + \frac{u_1^3}{3!}f'''(u_0), \\
&\vdots
\end{aligned} \tag{15}
$$

where $(.)'$ denotes derivative with respect to u_0. From the above discussion it is evident that polynomials A_n are not unique and that $\sum_{n=0}^{\infty}A_n$ is a generalized Taylor series expansion of Nu about function $u_0(t)$, and the series terms approach zero as $\frac{1}{(mn)!}$ where m is the order of highest differential operator [17].

2.1. Limitations of ADM

Though ADM has been applied to a wide class of problems ranging from Navier–Stokes, Schrödinger, Elliptic, Korteweg–de Vries, Lotka–Volterra to Fractional Differential Equations [17,28], it has some serious drawbacks. Jiao et al. [29] has observed that although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method. Venkatarangan and Rajalakshmi [30] while studying nonlinear oscillatory systems (Duffing, van der Pol and Rayleigh equation) with ADM have observed that the solution is not periodic. They proposed an alternative technique, where they have applied Laplace transformation to the series obtained by ADM, and then converted the transformed series into a meromorphic function by forming its Padé approximant, and then inverted the approximant, which yields a better solution that is periodic. But they have not shown any detailed error analysis and their plots show conceivable errors in the solutions.

Cherrault [19] pointed out that the decomposition method does not assure, on its own, existence and uniqueness of solutions. In fact it can only be safely applied when a fixed point theorem holds. While, approximate analytical methods work with similar accuracy over the entire time axis, the known form of ADM lacks this property. Here examples of Duffing's oscillator, van der Pol oscillator and mathematical pendulum are used to demonstrate the lack of global validity of ADM (Figs. 1–3) with a finite number of terms in the series solution.

The governing equations of motion of free undamped Duffing's oscillator, van der Pol oscillator and mathematical pendulum considered here are

$$
\ddot{x} + x + x^3 = 0, \quad x(0) = a,\ \dot{x}(0) = 0, \tag{16}
$$

$$
\ddot{x} + \mu(1-x^2)\dot{x} = 0, \quad x(0) = a,\ \dot{x}(0) = b, \tag{17}
$$

$$
\ddot{\theta} + k^2\sin\theta = 0, \quad \theta(0) = a,\ \dot{\theta}(0) = b. \tag{18}
$$

In integrating Duffing's oscillator, a six term ADM approximant is taken. Fig. 1 shows that even a six term

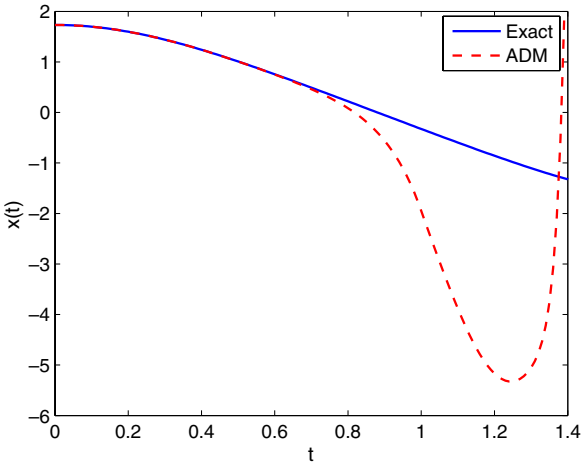

Fig. 1. Duffing's oscillator of Eq. (16), exact solution and solution by ADM with first six series terms.

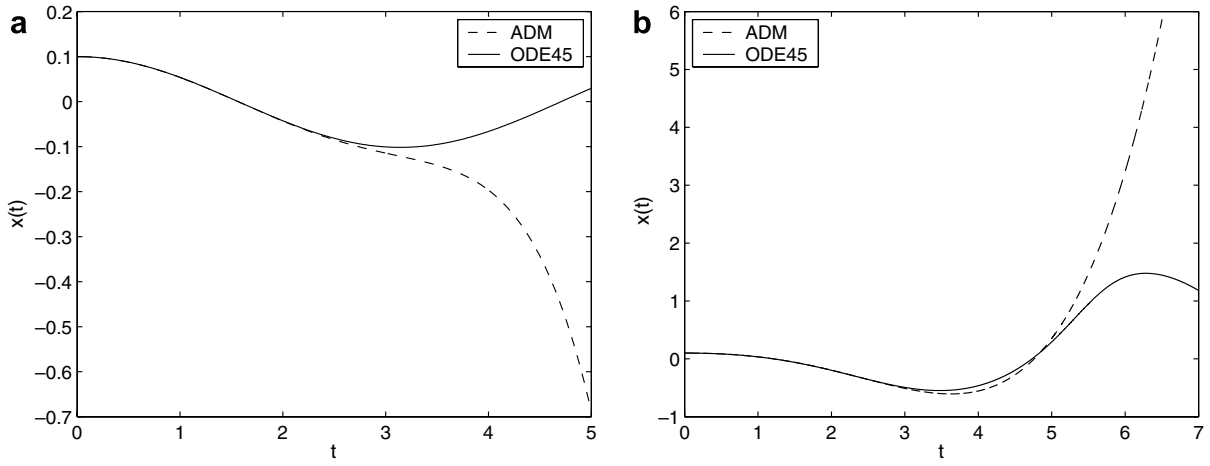

Fig. 2. Solution of the van der Pol oscillator (weakly nonlinear $\mu = 0.01$, see Eq. (17)) using ODE45 and ADM: (a) ADM series up to four terms, (b) ADM series up to six terms.

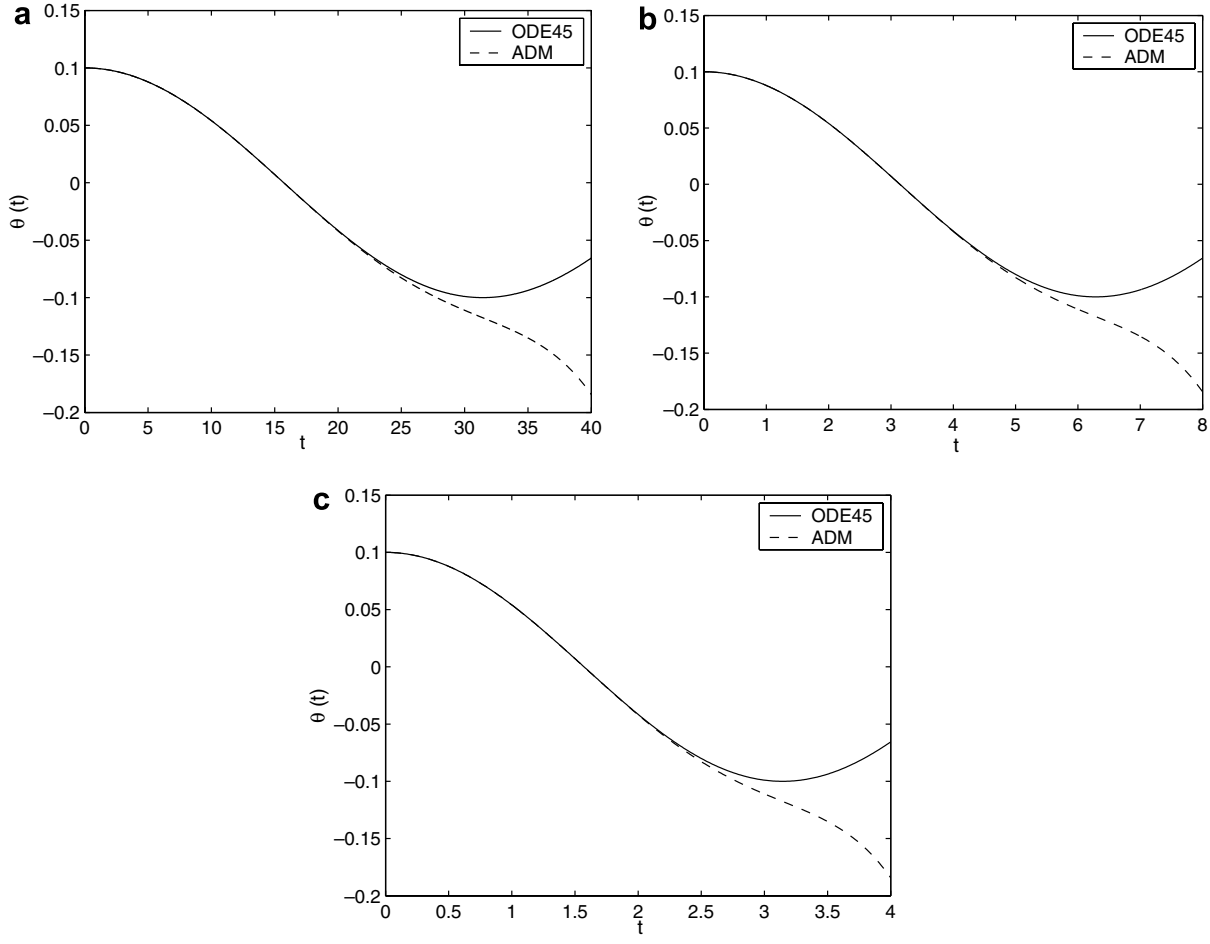

Fig. 3. Pendulum problem of Eq. (18): a comparison of solutions by ODE45 and ADM with first 10 series terms, (a) $k = 0.1$, (b) $k = 0.5$, (c) $k = 1$.

approximation is insufficient and ADM solution diverges rapidly from the exact solution after 0.7 s. Similar conclusions may be drawn for van der Pol and indeed, any other oscillator. Even though very weak nonlinearity is considered, ADM solution diverges. It may be observed that as the number of terms in the series increases, the time of divergence also increases. In a computer implementation, the maximum number of terms that may be included

in the series has to be limited, as there would be unacceptably high floating point errors trying to evaluate higher order terms. Solutions of a mathematical pendulum are shown in Fig. 3 for different value of k, and it is clear that ADM solutions diverge from actual solutions for all orders of nonlinearity. MATLAB software [31] command ODE45 is used and considered as accurate.[1] Here up to eleven terms in the approximation series are taken in ADM, and even this approximation becomes insufficient. As nonlinearity becomes more predominant, the time of divergence decreases. Therefore it is once more evident that the ADM solution is not globally valid, it diverges suddenly and sharply away from the true solution. This shows that a few terms approximant is inadequate to represent the solution of a dynamical system in the long times.

3. The present method

The purpose of this section is to outline the methodology, herein proposed, within a general setting. Let G be a Banach space. We are required to solve a dynamical system in the Banach space G of the functions $\mathbf{u} = \mathbf{u}(t)$, which are continuous and almost everywhere bounded. We will consider a general nonlinear dynamical system which is posed as an IVP and described as $\mathbf{DX} = \mathbf{F}$, where

$$\mathbf{X} = \mathbf{X}(t) : [0, T] \subset \mathbb{R} \to G$$

and the initial conditions (ICs) are assumed to have been prescribed. Here $\mathbf{X} \in G$, $\mathbf{D}$ is an operator which is a combination of differential and algebraic operators and contains nonlinearity. Moreover, $\mathbf{F} : [0, T] \subset \mathbb{R} \to G$ is a representation of the forcing terms. Now let the IVP be rewritten as

$$\mathbf{LX} + \mathbf{RX} + \mathbf{NX} = \mathbf{F}, \tag{19}$$

where $\mathbf{L}$ is the linear operator corresponding to the highest order derivative and $\mathbf{R}$ is the linear operator having derivatives of lower orders. $\mathbf{N}$ is the nonlinear operator. Applying $\mathbf{L}^{-1}$ on both sides and using the first fundamental theorem of calculus one gets

$$\mathbf{X} = \widehat{\mathbf{X}} - \mathbf{L}^{-1}\mathbf{RX} - \mathbf{L}^{-1}\mathbf{NX} + \mathbf{L}^{-1}\mathbf{F}, \tag{20}$$

where $\widehat{\mathbf{X}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m)^t$ is a vector function of initial conditions, to be described shortly; where, m is the length of the vector $\mathbf{X}$. The matrix form of Eqs. (19) and (20) is

$$\begin{bmatrix} L_{11} & & \\ & L_{22} & \\ & & \ddots \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} + \begin{bmatrix} R_{11} & R_{12} & \ldots \\ R_{21} & R_{22} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} N_1\mathbf{X} \\ N_2\mathbf{X} \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}, \tag{21}$$

where N_i is an analytic nonlinear operator, and a function of vector $\mathbf{X}$. Since only the highest order differential operator is taken, this gives a diagonal coefficient matrix on the left hand side. This leads to

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \end{pmatrix} - \begin{bmatrix} L_{11}^{-1} & & \\ & L_{22}^{-1} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & \ldots \\ R_{21} & R_{22} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} - \begin{bmatrix} L_{11}^{-1} & & \\ & L_{22}^{-1} & \\ & & \ddots \end{bmatrix} \begin{pmatrix} N_1\mathbf{X} \\ N_2\mathbf{X} \\ \vdots \end{pmatrix} + \begin{bmatrix} L_{11}^{-1} & & \\ & L_{22}^{-1} & \\ & & \ddots \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}. \tag{22}$$

One has, for the ith scaler component of $\widehat{\mathbf{X}}$, the following Taylor expansion

$$\hat{x}_i = x_i(0) + \frac{\mathrm{d}x_i(0)}{\mathrm{d}t} t + \frac{\mathrm{d}^2 x_i(0)}{\mathrm{d}t^2} \frac{t^2}{2!} + \ldots + \frac{\mathrm{d}^{n_{ii}} x(0)}{\mathrm{d}t^{n_{ii}}} \frac{t^{n_{ii}}}{n_{ii}!}, \quad i = 1, 2, \ldots, m, \tag{23}$$

where $n_{ii} + 1$ is equal to the order of differentiation in L_{ii}. Now, the solution $\mathbf{X}$ and nonlinear terms $\mathbf{N}$ are approximated as

$$\begin{pmatrix} N_1\mathbf{X} \\ N_2\mathbf{X} \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} A_{1n} \\ \sum_{n=0}^{\infty} A_{2n} \\ \vdots \end{pmatrix} \tag{24}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} x_{1n} \\ \sum_{n=0}^{\infty} x_{2n} \\ \vdots \end{pmatrix}. \tag{25}$$

[1] Throughout this text whenever exact analytical solution is not available the MATLAB software command ODE45 is used. ODE45 solves initial value problems for ordinary differential equations (ODEs) with adaptive step sizes. ODE45 is based on an explicit Runge–Kutta (4,5) formula, the Dormand–Prince pair [32]. It is a one-step solver – in computing $y(t_n)$, it needs only the solution at the immediately preceding time point, $y(t_n - 1)$. Throughout this text ODE45 is used with both absolute and relative tolerance equal to 2.2×10^{-14}. Thus this is considered as quite accurate and whenever the exact solution is not available, error is computed with respect to this.

Thus Eq. (21) reduces to

$$\begin{pmatrix} \sum_{n=0}^{\infty} x_{1n} \\ \sum_{n=0}^{\infty} x_{2n} \\ \vdots \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \end{pmatrix} - \begin{bmatrix} L_{11}^{-1} & & \\ & L_{22}^{-1} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & \dots \\ R_{21} & R_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \times \begin{pmatrix} \sum_{n=0}^{\infty} x_{1n} \\ \sum_{n=0}^{\infty} x_{2n} \\ \vdots \end{pmatrix} - \begin{bmatrix} L_{11}^{-1} & & \\ & L_{22}^{-1} & \\ & & \ddots \end{bmatrix} \begin{pmatrix} \sum_{n=0}^{\infty} A_{1n} \\ \sum_{n=0}^{\infty} A_{2n} \\ \vdots \end{pmatrix}, \tag{26}$$

where

$$\mathbf{X}_0 = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \end{pmatrix} = \begin{pmatrix} \hat{x}_1 + L_{11}^{-1} f_1 \\ \hat{x}_2 + L_{22}^{-1} f_2 \\ \vdots \end{pmatrix}. \tag{27}$$

Eq. (26) may now be written as

$$\begin{pmatrix} x_{1(n+1)} \\ x_{2(n+1)} \\ \vdots \end{pmatrix} = \begin{pmatrix} -L_{11}^{-1} \sum_{j=1}^{m} R_{1j} x_{jn} - L_{11}^{-1} A_{1n} \\ -L_{22}^{-1} \sum_{j=1}^{m} R_{2j} x_{jn} - L_{22}^{-1} A_{2n} \\ \vdots \end{pmatrix} \tag{28}$$

equivalently,

$$x_{i(n+1)} = -L_{ii}^{-1} \sum_{j=1}^{m} R_{ij} x_{jn} - L_{ii}^{-1} A_{in}, \quad n = 0, 1, 2, \dots, \infty, \tag{29}$$

where $i = 1, 2, \dots, m$.

Since ADM uses a functional form of Taylor series expansion of nonlinear terms about a function $\mathbf{u_0}$, then Adomian's polynomials represent the nonlinear function accurately near $\mathbf{u}_0$, if $\mathbf{u} - \mathbf{u}_0$ is sufficiently small. Additionally, for practical purposes, one has to take only a finite number of terms, which would yield wrong results in a global sense (refer to Section 2.1). To overcome this problem the present method discretizes the time axis just like any other numerical method and solves multiple initial value problems (IVPs). The ICs for the IVP valid over each interval are taken from the solutions of the last IVP. Let the subset of time axis $[0, T]$ be divided into ζ subintervals and so ordered that $0 = t_0 < t_1 < t_2 < \dots < t_\zeta = T$, and $h_i = t_i - t_{i-1}$, $i \in \mathbb{N}$.[2] Now, the ADM is applied as an algorithm for the analytical approximation of the dynamical response in that sequence of time intervals. Considering the ith time interval $[t_{i-1}, t_i]$, Eq. (19) may be written as:

$$\begin{aligned} \mathbf{X}(\tau) &= \widehat{\mathbf{X}}(\tau) - \mathbf{L}^{-1}\mathbf{R}\mathbf{X}(\tau) - \mathbf{L}^{-1}\mathbf{N}\mathbf{X}(\tau) + \mathbf{L}^{-1}\mathbf{F}(t_{i-1} + \tau) \\ &= \mathbf{X}_0(\tau) - \mathbf{L}^{-1}\mathbf{R}\mathbf{X}(\tau) - \mathbf{L}^{-1}\mathbf{N}\mathbf{X}(\tau) \\ \tau &= t - t_{i-1} \text{ and } \tau \in [0, h_i]. \end{aligned} \tag{30}$$

Here, $\mathbf{L}^{-1}$ is a diagonal matrix whose elements are definite integral operators from 0 to τ.

3.1. Expansion of the forcing term

Since x_{i0} appears in the recursive derivation of all Adomian polynomials (A_{in}), x_{i0} needs to be a simple function to work with (for $i = 1, 2, \dots, m$). Towards this, $\mathbf{L}^{-1}\mathbf{F}$ is expanded in a Taylor series and only the first term is taken into $\mathbf{X}_0$, and rest of the terms are distributed in x_i; $i > 0$. This helps in effective computation of Adomian polynomials and hence the series solution. Let $\mathbf{L}^{-1}\mathbf{F} = \widehat{\mathbf{F}} = (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_m)^t$. Taylor series expansion of its elements are

$$\hat{f}_i = \hat{f}_i(\tau = 0) + \frac{\mathrm{d}\hat{f}_i(\tau = 0)}{\mathrm{d}\tau}\tau + \frac{\mathrm{d}^2\hat{f}_i(\tau = 0)}{\mathrm{d}\tau^2}\frac{\tau^2}{2!} + \cdots, \tag{31}$$

where $i = 1, \dots, m$.

Thus Eq. (27) becomes

$$\mathbf{X}_0 = \widehat{\mathbf{X}} + \begin{pmatrix} \hat{f}_1(\tau = 0) \\ \hat{f}_2(\tau = 0) \\ \vdots \end{pmatrix} \tag{32}$$

and Eq. (29) becomes

$$x_{i(n+1)} = -L_{ii}^{-1} \sum_{j=1}^{m} R_{ij} x_{jn} - L_{ii}^{-1} A_{in} + \frac{\mathrm{d}^{n+1}\hat{f}_i(\tau = 0)}{\mathrm{d}\tau^{n+1}} \frac{\tau^{n+1}}{(n+1)!}, \quad n = 0, 1, \dots, \infty. \tag{33}$$

3.2. Removal of higher order terms

Even after implementing the schemes mentioned above along with conventional ADM, some times it is extremely difficult to derive analytical expressions of series terms of the solution. The reason is that L_{ii}^{-1} is an integrating operator, so higher powers of τ are generated and pose difficulties in the analytical computation of x_{in}'s. Thus one is restricted to retain only the first few terms of the series in Eq. (25). However, this restriction may be used to one's advantage if, terms containing higher powers of τ are judiciously removed. This is the crux of this new technique. Removing higher powers of τ results in an amazing simplification towards an approximate analytical derivation of series solutions. Though at a first glance it appears that removal of terms containing higher powers of τ (τ^n, say $n \geqslant 10$) may reduce the accuracy of the solution, but in turn it makes the method more accurate by avoiding a source of possible numerical inaccuracies, including underflows, for $\tau < 1$. Indeed, the order of accuracy of the

[2] Throughout this work uniform step size is adopted, i.e. $h_i = h$ for all i.

method is determined by the highest power of τ retained in the approximation. In other words, this technique helps in easy computation of more series terms, at the same time it enables more accurate derivation of coefficients of lower powers of τ.

4. Convergence of the series solution and error analysis

The objective of this section is to theoretically explore the convergence of the proposed approach and to obtain estimates on its order in terms of powers of a chosen step size. In the first part of this section, we will discuss the proof of convergence (to the true solution) of the the series solution, from the point of view of dynamical systems. In other words, we investigate the issue of convergence of the two series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} A_n$. In the second part, we will present an error analysis and, in the sequel, derive the order of accuracy of the present method. While we present the proofs in the context of one-dimensional systems, one may readily extend the arguments for higher dimensional systems.

4.1. Convergence of Adomian's series

Some earlier investigations on ADM used fixed point theorems for proving convergence [19,22]. Particularly the assumption that the nonlinear operator is contractive may not be valid for all practical cases. We need to avoid this hypothesis which lacks universatality and which are difficult to verify in physical problems. In fact, in the context of dynamical systems, a contraction mapping may be regarded as a stable solution, which may not be the case always. For instance, one may have a saddle node which is not contractive in nature. In the present work we attempt to show that the new method based on ADM may be extended to such cases and even to chaotic regimes. This proof mainly follows [24]. First note that Eq. (2) may be rewritten as another nonlinear functional equation

$$u - N(u) = f, \tag{34}$$

where N denotes another nonlinear operator. The solution of Eq. (34) is given by $u = \sum_{n=0}^{\infty} u_n$, where the terms in the series are:

$$\begin{aligned} &u_0 = f, \\ &u_1 = A_0(u_0), \\ &\vdots \\ &u_{n+1} = A_n(u_0, \ldots, u_n), \\ &\vdots \end{aligned} \tag{35}$$

In the above equations, A_i's are the Adomian's polynomials generated by the Mclaurin expansion of the nonlinear operator with respect to the artificial parameter λ, (see Eq. (10)). Cherrualut and Adomian [24] proved the convergence based on two assumptions on u and $N(u)$, and these are

(1) the solution u of Eq. (34) may be found as a series of functions u_i, i.e., $u = \sum_{n=0}^{\infty} u_n$. Furthermore, this series is supposed absolutely convergent, i.e., $\sum_{n=0}^{\infty} |u_n| < +\infty$;
(2) the nonlinear function $N(u)$ is developable in an entire series with a convergence radius equal to infinity. In other words, one may write

$$N(u) = \sum_{n=0}^{\infty} N^{(n)}(0) \frac{u^n}{n!}, \quad |u| < \infty. \tag{36}$$

Based on these two hypotheses, one may readily prove the following.

Theorem. *When hypotheses* 1 *and* 2 *are true and* u_i*'s satisfy Eq.* (35)*, the series* $u = \sum_{n=0}^{\infty} u_n$ *is a solution of Eq.* (34)*.*

Proof. Since the radius of convergence is infinity, by hypothesis 2 the series of Eq. (36) converges for any u. Moreover, by hypothesis 1 we know that $u = \sum_{n=0}^{\infty} u_n$ is absolutely convergent and therefore the series may be substituted in Eq. (36) to obtain:

$$N(u) = \sum_{n=0}^{\infty} \left[N^{(n)}(0) \frac{\left(\sum_{n=0}^{\infty} u_n\right)^n}{n!} \right], \quad |u| < \infty. \tag{37}$$

Owing to hypothesis 1, there exists absolute convergence of $\sum_{n=0}^{\infty} u_n = U < \infty$. Now, $N(u)$ needs to be rearranged. Rewriting u^n as

$$u^n = \left(\sum_{i=0}^{\infty} u_i \right)^n = \sum_{j=0}^{\infty} \gamma_{nj}(u_0, \ldots, u_j), \tag{38}$$

where γ_{nj} is a function, which depends only on $(u_0,\ldots,u_j)$, (this can be easily proved by following [33]). Thus series of Eq. (37) can be written as

$$\begin{aligned} N(u) &= \sum_{n=0}^{\infty} \left[\frac{N^{(n)}(0)}{n!} \sum_{j=0}^{\infty} \gamma_{nj}(u_0, \ldots, u_j) \right] \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{N^{(n)}(0)}{n!} \gamma_{nj}. \end{aligned} \tag{39}$$

Since, $\sum_{j=0}^{\infty} \gamma_{nj} \leqslant U^n$, one can write, taking the absolute value

$$|N(u)| \leqslant \sum_{n=0}^{\infty} \left| \frac{N^{(n)}(0)}{n!} \right| U^n. \tag{40}$$

The last series converges due to the hypothesis 2. Therefore, it is proved that series defining $N(u)$ converges absolutely, thus can be rearranged. It is already stated that Adomian series $\sum_{n=0}^{\infty} A_n$ is a generalization of Taylor series [19]. Now one has to prove that u_i satisfies Eq. (35). Now, substituting the last two series in Eq. (34) leads to

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f. \tag{41}$$

Eq. (41) is identically satisfied if we have the relationships $u_0 = f,\ u_1 = A_0, \ldots, u_n = A_{n-1}, \ldots$ This gives Adomian's relationships of Eq. (35), hence the theorem is proved. □

Thus ADM provides a solution which converges to the true solution as one takes more and more terms in the series. Since, none of the hypothesis is violated by the discretization technique the present method also can provide a series solution, which converges to the true solution. In fact with the help of discretization of time axis boundedness assumption of u, becomes more realistic.

4.2. Error estimates

We make an attempt to have a better understanding of the method developed in the present work through an estimation of local errors. It is evident from the last paragraph that the ADM based solution converges to the true solution only if one considers infinitely many terms in the series solution, since the radius of convergence is infinitely large as noted in hypothesis 2. Taking infinitely many terms in the series solution is not feasible for practical implementations. Moreover when the independent variable (time) is not bounded, the ADM based series solution will diverge from the true solution at high values of time, even if very large number of series terms are taken. This is where the discretization of time axis makes itself indispensable. With the best of authors' knowledge this concept of a relationship between convergence and discretization is novel and has been taken up for the first time in this paper. Thus discretization of time axis provides a bound on time intervals and hence on (increments of) u, which obviates the necessity of a boundedness assumption on u. In this section we will compute the error for the finite series solution by restricting attention of a single time step. Let first p terms are taken in series solution so that:

$$u \cong \sum_{i=0}^{p} u_i. \tag{42}$$

From Eq. (13), it is evident that calculation of A_n requires inclusion of terms up to $N^n(u_0)(u-u_0)^n \frac{1}{n!}$ in the Taylor series expansion of $N(u)$ with respect to u_0. So it is evident that error in calculation of A_n is of order $\mathcal{O}(u-u_0)^n$. Recall the discretization of time axis of Section 3, for ith time interval

$$\begin{aligned} u(\tau) - u_0 = u(t_{i-1}) &+ \frac{\mathrm{d}u}{\mathrm{d}\tau}(t_{i-1})\tau + \ldots + \frac{\tau^k}{k!}\frac{\mathrm{d}^k u}{\mathrm{d}\tau^k}(t_{i-1}) \\ &+ \frac{\tau^{k+1}}{(k+1)!}\frac{\mathrm{d}^{k+1}u}{\mathrm{d}\tau^{k+1}}(t_{i-1}) + \ldots \\ &- \left[u(t_{i-1}) + \frac{\mathrm{d}u}{\mathrm{d}\tau}(t_{i-1})\tau + \ldots + \frac{\tau^k}{k!}\frac{\mathrm{d}^k u}{\mathrm{d}\tau^k}(t_{i-1})\right] \\ &= \sum_{m=k+1}^{\infty} \frac{\tau^m}{m!}\frac{\mathrm{d}^m u}{\mathrm{d}\tau^m}(t_{i-1}) \quad \tau \in [t_{i-1}, t_i]. \end{aligned}$$

Therefore the maximum difference between u and u_0 is of the order h^{k+1}. Here, k is the order of the differential operator L of Eq. (1) and $h = t_i - t_{i-1}$.

Now, the error in the truncated series is:

$$E = \left| u - \sum_{i=0}^{p} u_i \right| = \left| \sum_{i=p+1}^{\infty} u_i \right|. \tag{43}$$

Referring to the Eq. (5) one gets

$$u_{i+1} = -L^{-1}Ru_i - L^{-1}A_i. \tag{44}$$

Therefore the order of error in u_{i+1} is equal to that in A_i, which is

$$\mathcal{O}(u-u_0)^i. \tag{45}$$

Thus, a estimate of local error over a particular time interval is given by:

$$E_{\mathrm{Local}} = \left| \sum_{i=p+1}^{\infty} u_i \right| \equiv \mathcal{O}(u-u_0)^{\mathrm{p}} \equiv \mathcal{O}(h^{\mathrm{p(k+1)}}). \tag{46}$$

The global error order is one integral order less than the corresponding local error order

$$E_{\mathrm{Global}} = \mathcal{O}(h^{\mathrm{p(k+1)-1}}). \tag{47}$$

Since the order of operator L cannot be controlled, one has to increase p to get better solutions. The advantage of using ADM which uses the Taylor series expansion with respect to a function u_0 is evident here- we achieve error $\mathcal{O}(h^{\mathrm{p(k+1)-1}})$ instead $\mathcal{O}(h^{\mathrm{p-1}})$. Thus it is clear that by increasing the number of series terms, one can achieve more accurate solution and a far higher rate of convergence (depending on the value of k, and, for mechanical oscillators, $k = 2$) than is possible with most of the available numerical integration techniques.

5. Illustrative examples and numerical results

5.1. Single degree of freedom system

In this section, a few non-chaotic single degree of freedom systems, are considered and applications of the method explored.

5.1.1. The duffing oscillator

The following Eq. (48) models a nonlinear spring problem (the Duffing oscillator). Present work takes only positive values of α.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x + \alpha x^3 = 0 \quad x(0) = a; \quad \dot{x}(0) = b. \tag{48}$$

For $\alpha > 0$ this equation represents a hard spring and for $\alpha < 0$ it represents a soft spring. It is sometimes used as an approximation for the pendulum by setting $\alpha = -1/6$. Here α is taken as 1.

5.1.1.1. Exact solution. The exact solution (refer [34]) in terms of Jacobian elliptic functions is

$$x(t) = \pm\frac{\sqrt{\Psi^2-1}}{\sqrt{\alpha}}\mathrm{cn}\left(\Psi t + \Delta; \frac{\Psi^2-1}{2\Psi^2}\right) \quad \alpha > 0, \tag{49}$$

where Ψ and Δ are two independent arbitrary constants of the general solution of the second order ODE. Since it is an autonomous ODE, one of the arbitrary constants will be the phase Δ. cn denotes the Jacobian elliptic CN function. Note that the modulus m (the second argument of the cn function) is fixed in terms of the arbitrary Ψ. Since $0 \leqslant m \leqslant 1$, the cn solution is valid when $\alpha > 0$, $\Psi^2 \geqslant 1$.

5.1.1.2. Solution using present method. Comparing the Duffing equation with the standard form (see Eq. (1)), the operators can be identified as $L = \frac{d^2}{dt^2}$, $R = 1$, $Nx = x^3$. The choice made here (i.e., $L = \frac{d^2}{dt^2}$, $R = 1$) yields the simplest Green's function for computation. In this case, L^{-1} is a two-fold definite integral from 0 to t. Generally, this choice for L having the highest ordered derivative is the most desirable one, because the integrations are the simplest. If we invert the entire linear operator, convergence is expected to be much faster as suggested in [16], but to achieve a simpler computational procedure, we split the linear operator and invert only L. Therefore, for the ith time interval, the solution becomes:

$$x = x(t_{i-1}) + \frac{dx(t_{i-1})}{d\tau}\tau - L^{-1}x - \alpha L^{-1}x^3. \tag{50}$$

The initial conditions for the IVP in the ith interval is taken from end value of solutions over the $(i-1)$th interval. That is, $x(\tau = 0)$ and $\dot{x}(\tau = 0)$ in the ith interval is equal to $x(t_{i-1})$ and $\dot{x}(t_{i-1})$, which are taken from ith interval. Thus,

$$\sum_{n=0}^{\infty} x_n = x_0 - L^{-1}\sum_{n=0}^{\infty} x_n - \alpha L^{-1}\sum_{n=0}^{\infty} A_n, \tag{51}$$

where A_n are the Adomian polynomials for $Nx = x^3$. Now, using Eq. (15), one gets:

$$\begin{aligned} A_0 &= x_0^3, \\ A_1 &= 3x_0^2x_1, \\ A_2 &= 3x_0x_1^2 + 3x_0^2x_2, \\ A_3 &= x_0^3 + 6x_0x_1x_2 + 3x_0^2x_3, \\ A_4 &= 3x_1^2x_2 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_0^2x_4, \\ &\vdots \end{aligned} \tag{52}$$

Therefore from Eq. (51) one gets series terms of the solution as:

$$\begin{aligned} x_0 &= a + b\tau, \\ x_1 &= -L^{-1}x_0 - \alpha L^{-1}A_0, \\ x_2 &= -L^{-1}x_1 - \alpha L^{-1}A_1, \\ &\vdots \end{aligned} \tag{53}$$

where $a = x(\tau = 0)$ and $b = \dot{x}(\tau = 0)$. By standard ADM the terms may be calculated as (see Eq. (6))

$$\begin{aligned} x_0 &= a + b\tau, \\ x_1 &= -1/20\alpha b^3\tau^5 - 1/4\alpha ab^2\tau^4 + (-1/6b - 1/2\alpha a^2 b)\tau^3 \\ &\quad + (-1/2a - 1/2\alpha a^3)\tau^2, \\ x_2 &= \frac{1}{480}\alpha^2 b^5\tau^9 + \frac{3}{160}\alpha^2 ab^4\tau^8 \\ &\quad + \left(\frac{1}{840}\alpha b^3 - 1/14\alpha(-1/6b - 1/2\alpha a^2 b)b^2 + \frac{11}{280}\alpha^2 a^2 b^3\right)\tau^7 \\ &\quad + \left(\frac{1}{120}\alpha ab^2 - 1/10\alpha(-1/2a - 1/2\alpha a^3)b^2\right. \\ &\quad \left. - 1/5\alpha(-1/6b - 1/2\alpha a^2 b)ab + 1/40\alpha^2 a^3 b^2\right)\tau^6 \\ &\quad + \left(-3/10\alpha(-1/2a - 1/2\alpha a^3)ab - \frac{3}{20}\alpha(-1/6b - 1/2\alpha a^2 b)a^2\right. \\ &\quad \left. + \frac{1}{120}b + 1/40\alpha a^2 b\right)\tau^5 \\ &\quad + (-1/4\alpha(-1/2a - 1/2\alpha a^3)a^2 + 1/24a + 1/24\alpha a^3)\tau^4, \\ &\vdots \end{aligned}$$

$$x = \sum_{n=0}^{\infty} x_n. \tag{54}$$

Here a four term approximation is taken, i.e., $x = \sum_{n=0}^{3} x_n$. In order to remain focussed on the central issues, a uniform step size has presently been chosen. For different step sizes we have plotted the solution by the present method and corresponding error. Fig. 4 shows that the present method gives an excellent result for this oscillator. Error comparison with RK4 shows that error through the present method is much less than that of RK4 for step size = 0.01. This method works with very high step sizes say 0.4 (see Fig. 5), where RK4 fails with a step size 0.3 (see Fig. 6). For this problem, nonlinearity is of a simple polynomial form and L^{-1} is the double integral. As mentioned before, terms of orders τ^{10} or more are removed. In this way, up to first 10 terms of the series solution are obtained yielding an accurate result. It is worth mentioning that method of perturbation fails for such a highly nonlinear problem ($\alpha = 1$).

5.1.2. The van der pol oscillator

The van der Pol oscillator is prototypical of systems describing self-sustaining oscillations in which energy is fed into small oscillations and removed from large oscillations. This equation arises in the study of circuits containing vacuum tubes and can be written along with the initial conditions as:

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0, \quad x(0) = a,\ \dot{x}(0) = b. \tag{55}$$

If $\mu = 0$ the equation reduces to that of a simple harmonic oscillator. Here cases with $\mu \neq 0$ will only be considered. One may see that the sign of the damping term, $-\mu(1-x^2)\frac{dx}{dt}$ switches, depending upon whether $|x|$ is larger or smaller than unity. Since $x^2\frac{dx}{dt} = \frac{1}{3}\frac{dx^3}{dt}$, one may rewrite Eq. (55) as:

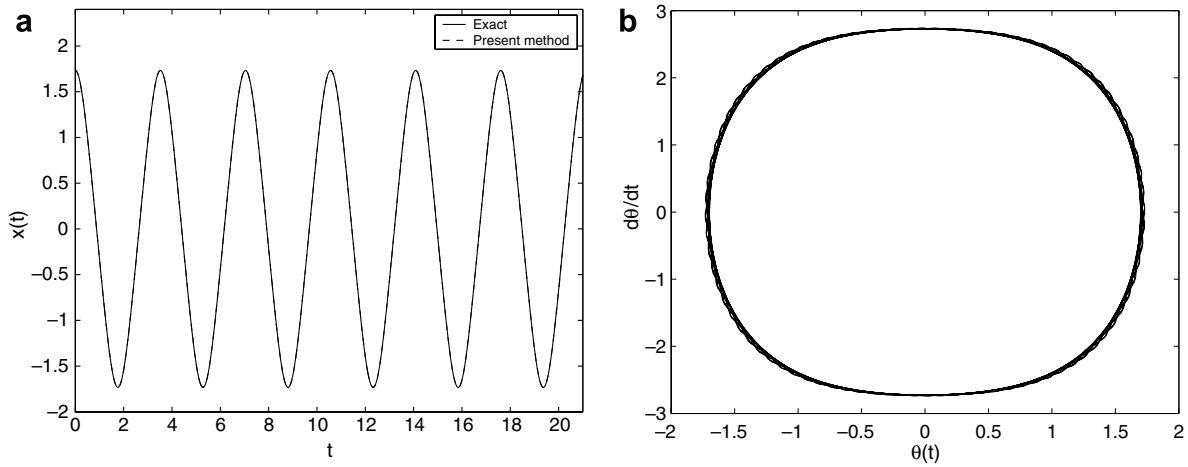

Fig. 4. Duffing oscillator: (a) a comparison of solutions by present method (with first six series term and a step size = 0.01) with exact, (b) phase portrait by present method.

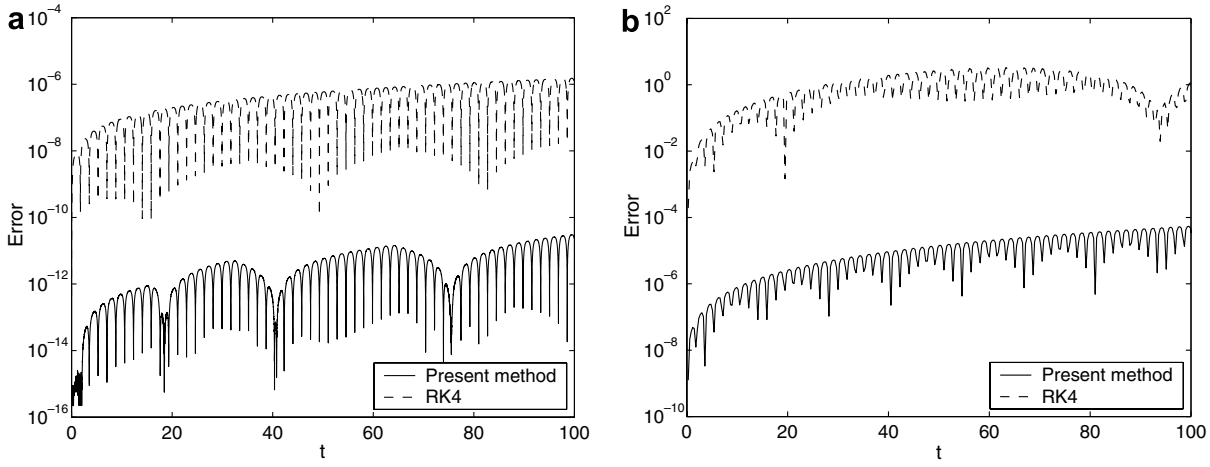

Fig. 5. Duffing oscillator: (a) error using RK4 and present method (with first six series terms) for step size = 0.01; (b) error in present method (with first 10 series terms) for step size = 0.3.

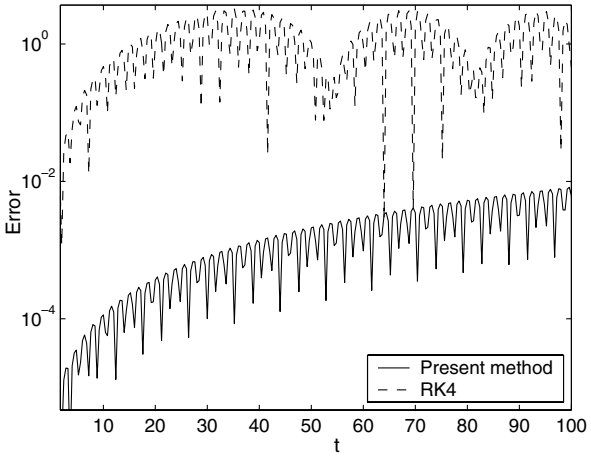

Fig. 6. Duffing oscillator: error using RK4 and present method (with first 10 series terms) for step size = 0.4.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\mu\frac{\mathrm{d}x}{\mathrm{d}t}+x+\frac{\mu}{3}\frac{\mathrm{d}x^3}{\mathrm{d}t}=0. \tag{56}$$

Thus the equation gets reduced to a more convenient standard form $Lu+Ru+Nu=g(t)$ where, $L=\frac{\mathrm{d}^2}{\mathrm{d}t^2}$, $R=-\mu\frac{\mathrm{d}}{\mathrm{d}t}+1$, $Nx=\frac{\mu}{3}\frac{\mathrm{d}}{\mathrm{d}t}Mx$, M is another nonlinear operator defined as $Mx=x^3$. This way of representation of the nonlinear term provides for an ease of calculating ADM polynomials since we know A_ns for M from the solution of Duffing oscillator (see Eq. (52)). This trick is followed from [16]. The choice for the operators here are, $L=\frac{\mathrm{d}^2}{\mathrm{d}t^2}$, $R=-\mu\frac{\mathrm{d}}{\mathrm{d}t}+1$ and this choice yields simpler Green's function for computations than $L=\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\mu\frac{\mathrm{d}}{\mathrm{d}t}+1$ and $R=0$. Thus, over the ith time interval, one has:

$$\begin{aligned} x&=x(t_{i-1})+\frac{\mathrm{d}x(t_{i-1})}{\mathrm{d}\tau}\tau+\mu L^{-1}\frac{\mathrm{d}x}{\mathrm{d}\tau}-L^{-1}x-\frac{\mu}{3}L^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}Mx,\\ \sum_{n=0}^{\infty}x_n&=x_0+\mu L^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=0}^{\infty}x_n-L^{-1}\sum_{n=0}^{\infty}x_n-\frac{\mu}{3}L^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=0}^{\infty}A_n. \end{aligned} \tag{57}$$

Note that the differential operator and summation commute and thus from the above equation one gets the series terms of the solution as

$$\begin{aligned} x_0 &= a + b\,\mathrm{d}\tau, \\ x_1 &= \mu L^{-1}\frac{\mathrm{d}x_0}{\mathrm{d}\tau} - L^{-1}x_0 - \frac{\mu}{3}L^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}A_0, \\ x_2 &= \mu L^{-1}\frac{\mathrm{d}x_1}{\mathrm{d}\tau} - L^{-1}x_1 - \frac{\mu}{3}L^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}A_1, \\ &\vdots \end{aligned} \tag{58}$$

These terms may be formed to be (see Eq. (6)):

$$\begin{aligned} x_0 &= a + b\tau, \\ x_1 &= -\frac{1}{12}\mu b^3\tau^4 + \frac{1}{3}\left(-\frac{1}{2}b - \mu ab^2\right)\tau^3 + \frac{1}{2}(\mu b - a - \mu a^2 b)\tau^2, \\ x_2 &= \frac{1}{84}\mu^2 b^5\tau^7 + \frac{1}{6}\left(\frac{1}{60}\mu b^3 - \frac{1}{15}\mu\left(3\left(-\frac{1}{2}b - \mu ab^2\right)b^2 - 2\mu b^4 a\right.\right. \\ &\quad \left.\left. + 6\left(\left(-\frac{1}{6}b - \frac{1}{3}\mu ab^2\right)b - \frac{1}{12}\mu b^3 a\right)b\right)\right)\tau^6 \\ &\quad + \frac{1}{5}\left(\frac{1}{24}b + \frac{1}{12}\mu ab^2 - \frac{1}{12}\mu^2 b^3 - \frac{1}{12}\mu\left(3(-\mu a^2 b + \mu b - a)b^2\right.\right. \\ &\quad + 6\left(-\frac{1}{2}b - \mu ab^2\right)ab - \mu b^3 a^2 \\ &\quad \left.\left. + 6\left(\left(-\frac{1}{2}\mu a^2 b + \frac{1}{2}\mu b - \frac{1}{2}a\right)b + \left(-\frac{1}{6}b - \frac{1}{3}\mu ab^2\right)a\right)b\right)\right)\tau^5 \\ &\quad + \frac{1}{4}\left(\frac{1}{3}\mu\left(-\frac{1}{2}b - \mu ab^2\right) - \frac{1}{9}\mu\left(6(-\mu a^2 b + \mu b - a)ab\right.\right. \\ &\quad + 3\left(-\frac{1}{2}b - \mu ab^2\right)a^2 + 6\left(-\frac{1}{2}\mu a^2 b + \frac{1}{2}\mu b - \frac{1}{2}a\right)ab\Big) \\ &\quad \left. + \frac{1}{6}\mu a^2 b - \frac{1}{6}\mu b + \frac{1}{6}a\right)\tau^4 \\ &\quad + \frac{1}{3}\left(\frac{1}{2}\mu(-\mu a^2 b + \mu b - a) - \frac{1}{2}\mu(-\mu a^2 b + \mu b - a)a^2\right)\tau^3 \\ &\vdots \end{aligned} \tag{59}$$

First five terms are taken in the the series solution. No exact analytical solution is available for this equation [35], so we resort to ODE45 of MATLAB to compare results from the present work. Fig. 7 shows solutions of the van der Pol oscillator through the present method and ODE45. When the higher power terms are not removed, we could compute only up to the first 5 terms of the series solution. However, with almost the same computational effort, we could compute up to the first 10 terms when terms of order higher than τ^{15} are removed. Whereas the removal of higher power terms can lead to such a high step size as 0.35, without this removal, the maximum step size may only be about 0.2, as shown in Figs. 7(a) and 8(a). It is noteworthy that the present method enables the treatment of strong nonlinearity, albeit with relatively smaller step sizes (see Fig. 7(a) and (b)).

5.1.3. The simple pendulum

The equation of motion for a simple pendulum is given by

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + k^2\sin(\theta) = 0,\ \theta(0) = a, \quad \frac{\mathrm{d}\theta(0)}{\mathrm{d}t} = b \tag{60}$$

where k^2 is a constant. The angle θ, is the angle from the vertically down position, with counter-clockwise rotation being positive. Depending on initial conditions, two kinds of motions are possible, one is the pendulum rotations and the other is the pendulum oscillations (when the pendulum oscillates, it does not have enough energy to trace a complete circle by crossing the separatrix at $\theta = \pi$). It is the latter case that will be studied through present method. In standard form, Eq. (60) becomes $L\theta + N\theta = 0$ where, $L = \frac{\mathrm{d}^2}{\mathrm{d}t^2}$, $N\theta = k^2\sin(\theta)$. In this case also for ith time interval, L^{-1} is the two-fold definite integral from 0 to τ, $\tau \in [t_{i-1}, t_i]$. Therefore, over ith interval, the solution becomes

$$\theta = \theta(t_{i-1}) + \frac{\mathrm{d}\theta(t_{i-1})}{\mathrm{d}\tau}\tau - k^2 L^{-1}\sin(\theta) \tag{61}$$

or

$$\sum_{n=0}^{\infty}\theta_n = \theta_0 - k^2 L^{-1}\sum_{n=0}^{\infty}A_n, \tag{62}$$

where the A_ns are the appropriate polynomials for $N\theta = sin(\theta)$. Here, θ_0 is given by $\theta(0) + \frac{\mathrm{d}\theta(0)}{\mathrm{d}\tau}\tau$. Now, using Eq. (15), one gets:

$$\begin{aligned} A_0 &= \sin(\theta_0), \\ A_1 &= \theta_1\cos(\theta_0), \\ A_2 &= \theta_2\cos(\theta_0) - \frac{1}{2}\theta_1^2\sin(\theta_0), \\ A_3 &= \theta_3\cos(\theta_0) - \theta_1\theta_2\sin(\theta_0) - \frac{1}{6}\theta_1^3\cos(\theta_0), \\ &\vdots \end{aligned} \tag{63}$$

Assuming $\theta(0) = a$ and $\dot{\theta}(0) = b$, the terms in the series are:

$$\begin{aligned} \theta_0 &= a + b\tau, \\ \theta_1 &= -k^2 L^{-1}A_0, \\ \theta_2 &= -k^2 L^{-1}A_1, \\ \theta_3 &= -k^2 L^{-1}A_2, \\ &\vdots \end{aligned} \tag{64}$$

The series terms of the solutions by ADM as derived here are

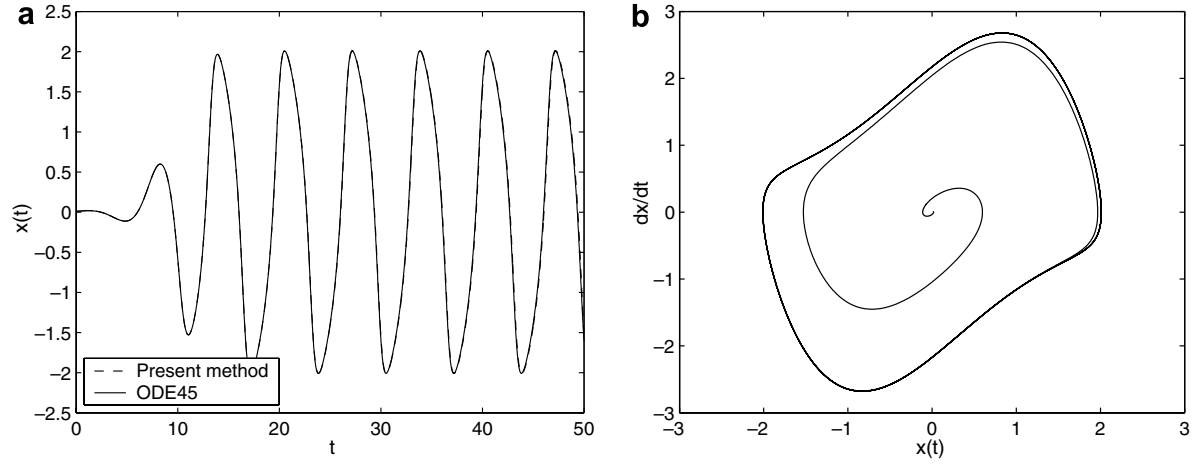

Fig. 7. van der Pol oscillator for $a = 0.01$, $b = 0.01$, $\mu = 1$: (a) comparison of solutions by ODE45 and present method (step size = 0.2, with first five series terms); (b) phase portrait by present method.

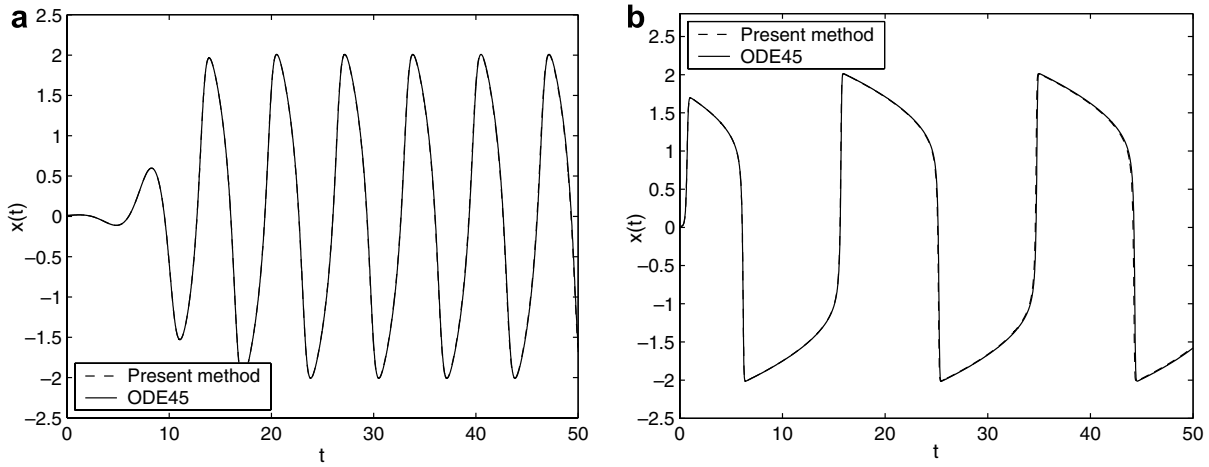

Fig. 8. van der Pol oscillator: comparison of solution of by ODE45 and present method (after removal of terms of power higher than τ^{15} for first 10 terms of series solution): (a) step size = 0.35, $\mu = 1$, $a = 0.01$, $b = 0.01$; (b) step size = 0.05, $\mu = 10$, $a = 0.01$, $b = 0.01$.

$$
\begin{aligned}
\theta_0 &= a + b\tau, \\
\theta_1 &= -\frac{k^2(-\sin(a+b\tau) + \cos(a)\tau b + \sin(a))}{b^2}, \\
\theta_2 &= -\frac{1}{4}\frac{k^4}{b^4}(-\cos(a+b\tau)\sin(a+b\tau) - b\tau - 8\cos(a)\sin(a+b\tau) \\
&\quad + 4\cos(a)\cos(a+b\tau)b\tau + 4\sin(a)\cos(a+b\tau) \\
&\quad + 6(\cos(a))^2\tau b + 4(\sin(a))^2\tau b + 5\cos(a)\sin(a)) \\
&\vdots
\end{aligned}
\tag{65}
$$

Fig. 9(a) shows that the comparison of present method and ODE45 is quite good for a typically periodic orbit of a simple pendulum. Fig. 9(b) shows corresponding phase portrait. Present method works for a step size as high as 1, where the time period is approximately 2π, see Fig. 10(a). Error comparisons with RK4 show that the present method results in significantly less error (Fig. 10(b)). As already stated, one advantage of this method over a numerical technique is that, being a numerical-analytical technique, it can provide a functional form of the solution over a time interval, i.e. solutions are obtainable as sufficiently smooth functions (see Fig. 10(a)). It has to be noted that the very high powers of b in the denominator of Eq. (65) cause numerical problems and render the algorithm inapplicable to this problem. Here, b denotes the initial angular velocity in a particular interval. Whenever, the extremities of an time interval come very close with the maxima or minima of the solution, then one may get wrong solution. One way out is to use adaptive programming which can control step sizes. Another way is to multiply the solution with b^n, during the whole calculation and substitute it at the end. Where, maximum power of b in the denominator is n. These techniques along with others to tackle such problems will be addressed in the next paper.

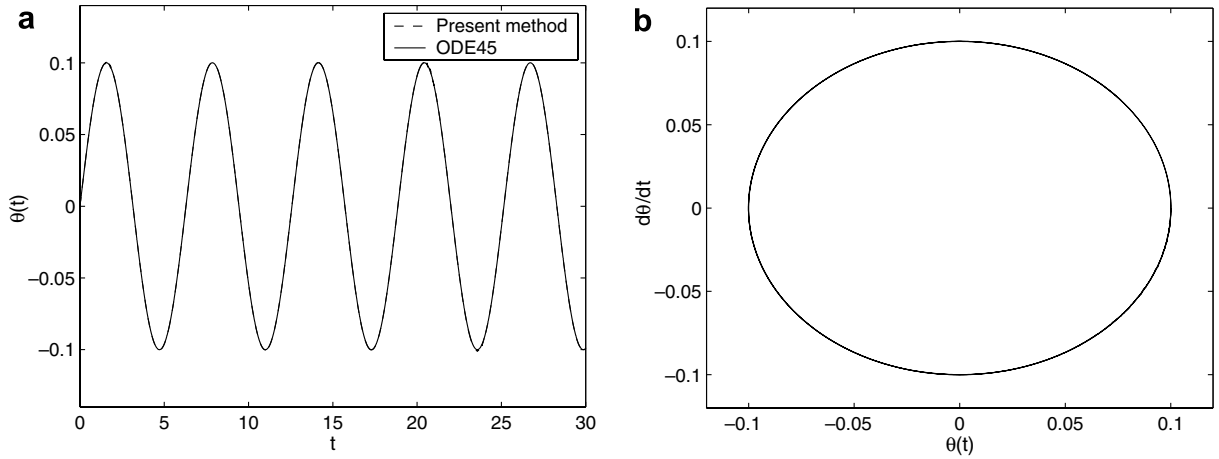

Fig. 9. (a) Solution of a simple pendulum by ODE45 and present method (step size = 0.5, with first 10 series terms); (b) phase portrait: symmetric periodic oval shaped orbit by present method.

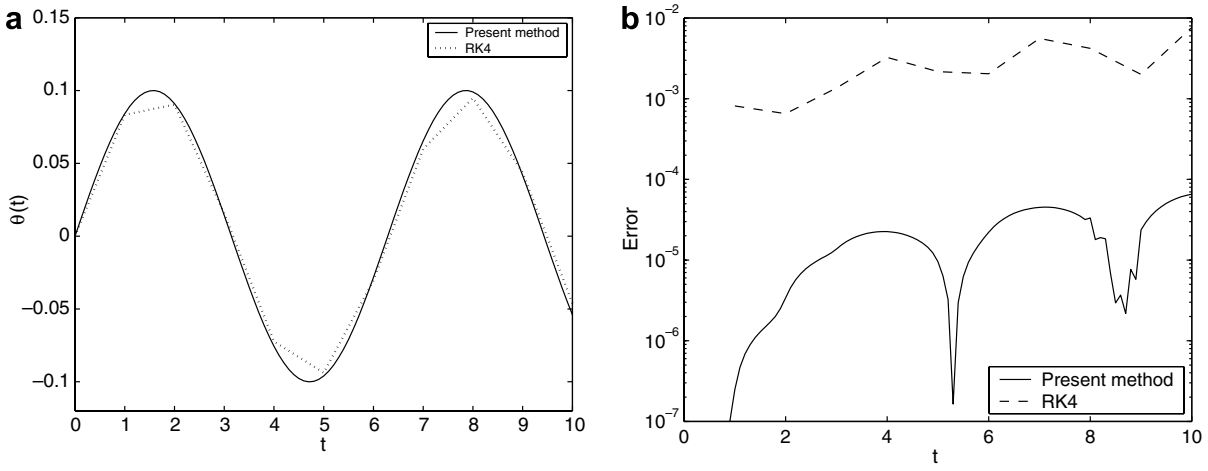

Fig. 10. (a) A comparison of solutions of simple pendulum by present method and RK4 for step size = 1; (b) comparison of error by present method with respect to ODE45.

5.2. Forced vibrations and chaotic systems

In this section, the decomposition technique described thus far is numerically illustrated for solutions of the Duffing oscillator under sinusoidal excitations. The governing equation is:

$$\ddot{x} + c\dot{x} + k_1 x + k_2 x^3 = F\cos(\omega t), \quad x(0) = a,\ \dot{x}(0) = b. \tag{66}$$

The above equation will be referred to as Duffing–Holmes's oscillator, when the following three parameters written as $c = 2\pi\epsilon_1$, $k_1 = -4\pi^2\epsilon_2$, $k_2 = 4\pi^2\epsilon_2$, $F = 4\pi^2\epsilon_3$, $\omega = 2\pi$ in terms of other three parameters ϵ_1, ϵ_2, ϵ_3, which serves as a one-mode approximation to lateral vibration of a buckled beam.

The forcing term is now expanded in Taylor series as described in Section 3.1.

$$\begin{aligned} &F\cos(t_{i-1}+\tau) \\ &= \frac{1}{2}F\cos(\omega t_{i-1})\tau^2 - \frac{1}{6}F\sin(\omega t_{i-1})\omega\tau^3 \\ &\quad - \frac{1}{24}F\cos(\omega t_{i-1})\omega^2\tau^4 + \frac{1}{120}F\sin(\omega t_{i-1})\omega^3\tau^5 \\ &\quad + \frac{1}{720}F\cos(\omega t_{i-1})\omega^4\tau^6 - \frac{1}{5040}F\sin(\omega t_{i-1})\omega^5\tau^7 \\ &\quad - \frac{1}{40320}F\cos(\omega t_{i-1})\omega^6\tau^8 + \frac{1}{362880}F\sin(\omega t_{i-1})\omega^7\tau^9 \\ &\quad + \frac{1}{3628800}F\cos(\omega t_{i-1})\omega^8\tau^{10} + O(\tau^{11}). \end{aligned} \tag{67}$$

Thus the series terms become

$$\begin{aligned} x_0(\tau) &= a + b\tau + 1/2F\cos(\omega t_{i-1})\tau^2, \\ x_1(\tau) &= -L^{-1}x_0 - k_1L^{-1}x_0 - k_2L^{-1}A_0 - 1/6F\sin(\omega t_{i-1})\omega\tau^3, \\ x_2(\tau) &= -L^{-1}x_1 - k_1L^{-1}x_1 - k_2L^{-1}A_1 - 1/24F\cos(\omega t_{i-1})\omega^2\tau^4, \\ &\vdots \end{aligned} \tag{68}$$

The last equation is valid over the ith time interval $[t_{i-1}, t_i]$, with $\tau \in [t_{i-1}, t_i]$. Comparisons of accuracy of this method vis-a-vis the RK4 are again made with reference to the solutions through the adaptive solver ODE45; see Fig. 11. First six terms of series solution have been included and terms with higher powers than τ^{15} are removed from solution. It is found that for forced vibration this method works extremely well. For both the time history and phase portrait of a typically 3-periodic orbit of Duffing–Holmes' oscillator, the solution via the present method is indistinguishable from that of ODE45. The corresponding error comparison with RK4 shows that present method is far superior in terms of accuracy. The same conclusion may be drawn for other types of periodic orbits, such as the 1-periodic orbit of the Duffing oscillator shown in Fig. 12. To check the applicability of the method for simulating chaotic solutions, comparisons have been made with RK4 with ODE45 as the reference solver. Given the exponentially diverging nature of two nearby chaotic trajectories, due to the positivity of the maximal Liapunov exponent, the comparison is good (Figs. 13 and 14). The phase portrait (Fig. 13(a)) shows a remarkably good match between solutions via ODE45 and present method; they are visually indistinguishable. The observation is further emphasized in Fig. 13(b) showing error comparison between present and RK4 methods with respect to ODE45 for the same step size. Both accumulation of error and time of divergence of two trajectories, starting from the same initial condition and being integrated via two different integration schemes, are function of the step size used. Fig. 14 displays the time of divergence for different step sizes. This shows that for step size (h) equal to 0.1 trajectories via both present and RK4 methods remain reasonably close to the trajectory via ODE45 without diverging up to $t = 100$ s. But the time of divergence from the ODE45 based trajectory for $h = 0.5$ and 0.75 using RK4 are respectively 26 and 24 s. On the other hand, by using present method, these times are respectively 66 and 43 s. This shows that, in terms of accuracy, the present method is considerably better than RK4 with same step size (Fig. 15 shows the Poincare section of the chaotic orbit).

It must be noted that the present method, when applied to Eq. (66), without the first two Taylor expansion of the forcing and removal of higher order terms, may only allow construction of the first two terms in the series. Hence such a scheme would show up large errors and is essentially inapplicable to the present problem, even when applied recursively. The two main reasons behind this failure may be summed up follows. First, the derivation of Adomian's polynomials becomes very difficult if u_0 is not a function with a very

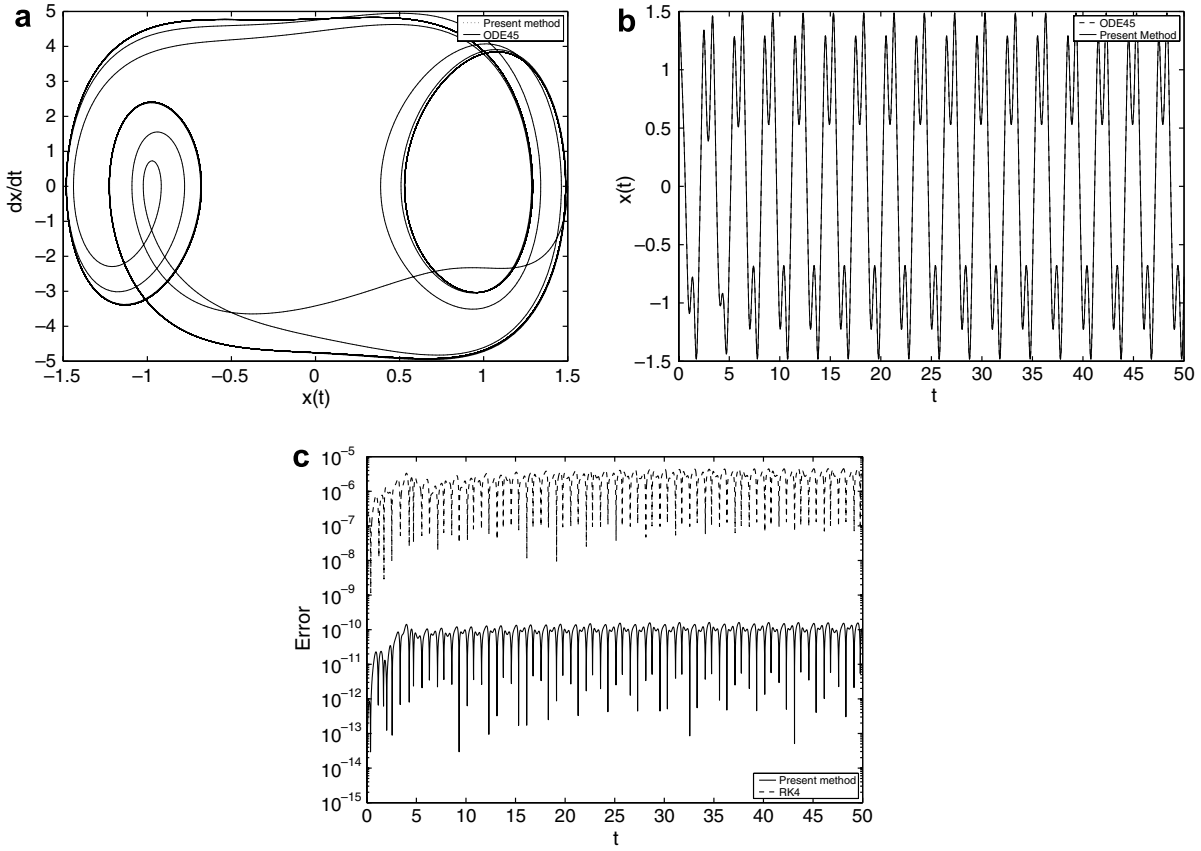

Fig. 11. (a) A typically 3-periodic orbit of Duffing–Holmes' oscillator using ODE45 and present method (with first six series terms; step size = 0.01), for $\epsilon_1 = 0.25$, $\epsilon_1 = 0.5$, $\epsilon_1 = 0.4$. (b) Time-history of the 3-periodic orbit. (c) Errors by present method and RK4 with respect to ODE45.

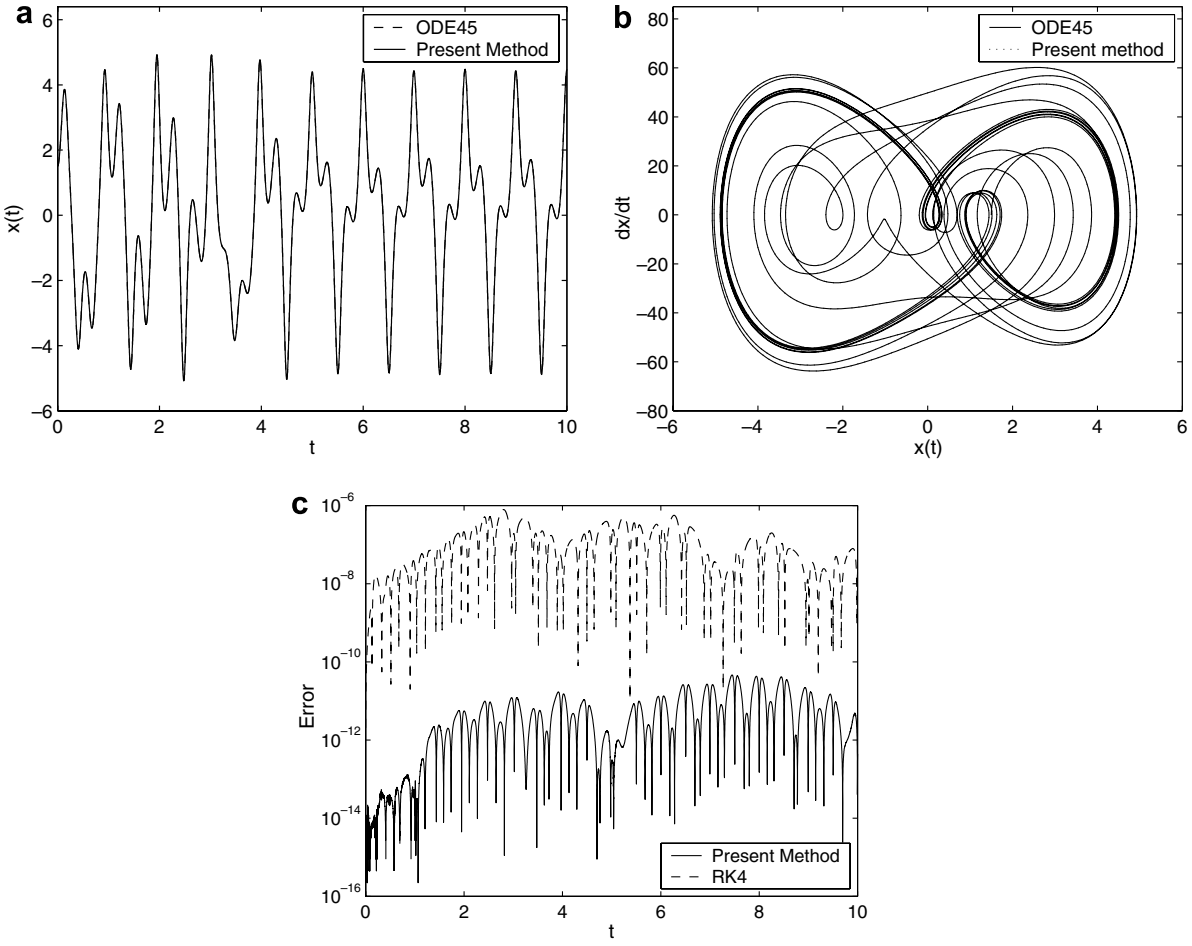

Fig. 12. (a) Time-history plots of the Duffing oscillator (Eq. (66) with $c = 2\pi\epsilon_1$, $k_1 = 4\pi^2\epsilon_2$, $k_2 = 4\pi^2\epsilon_2$, $F = 4\pi^2\epsilon_3$, $\omega = 2\pi$) using ODE45 and present method (with first six series terms; step size 0.001); $\epsilon_1 = 0.25$, $\epsilon_2 = 0.5$, $\epsilon_3 = 15.5$. (b) Evolutions of one periodic orbits on the phase-plane using present method and ODE45. (c) Errors using present method and RK4 with respect to ODE45.

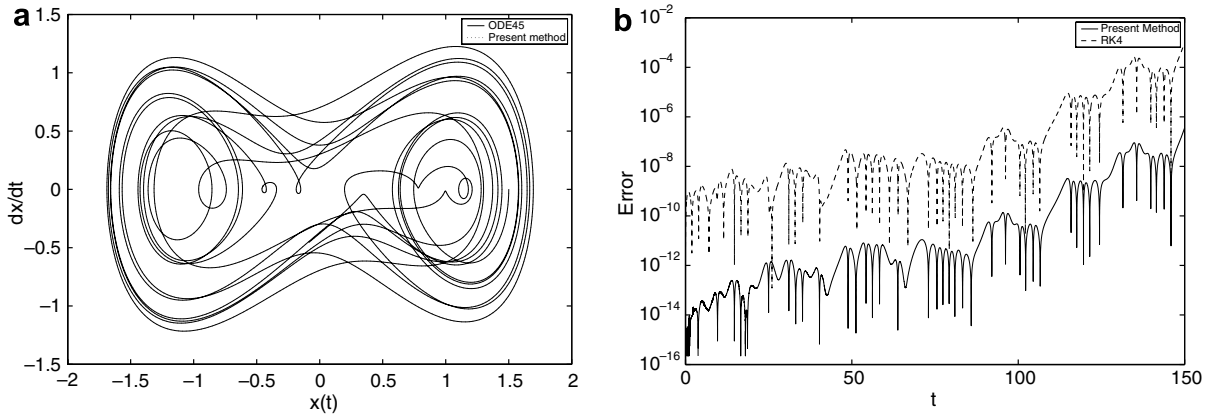

Fig. 13. (a) Phase portrait of a chaotic orbit (Eq. (66) for $c = 0.15$, $k_1 = -1$, $k_2 = 1$, $F = 0.41$ and $\omega = 0.4$) using ODE45 and present method (with first five series terms; step size = 0.01). (b) Errors of present method and RK4 with respect to ODE45.

simple form. Second, due to repeated integrations required during both derivation of Adomian's polynomials and series terms, lots of terms with higher powers of τ get generated (for deriving each series term the highest power of τ increases by 4), and this makes removal of higher power mandatory. Thus the above two algorithmic tools are seen to have an utmost importance, when applied along with a discretization of the time axis, to a nonlinear dynamical system.

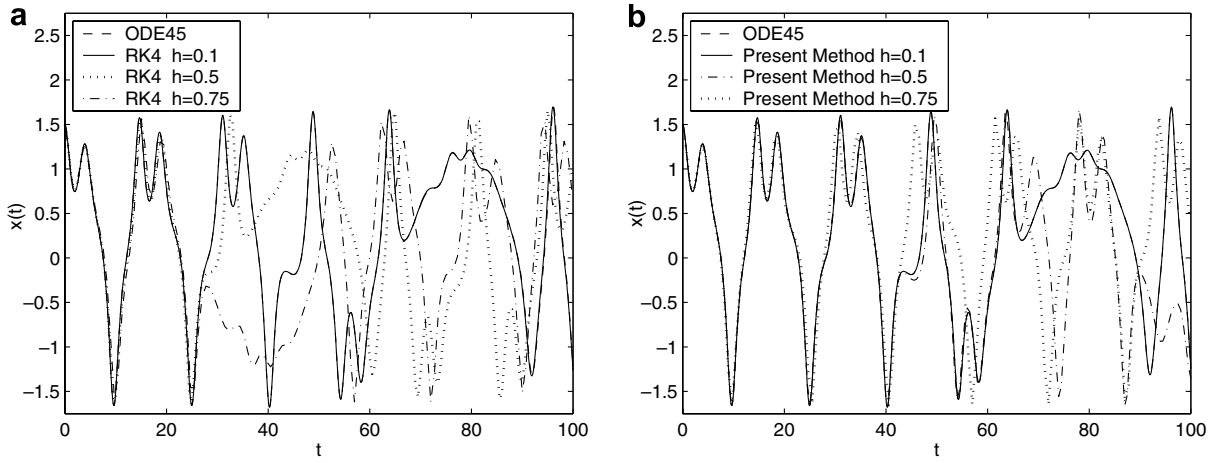

Fig. 14. Time-history plots of a chaotic orbit (of Fig. 13) for different step sizes, and time of divergence from ODE45 trajectory through (a) RK4, (b) present method.

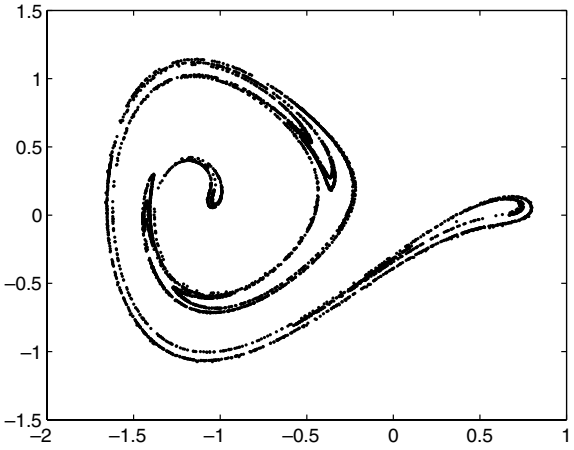

Fig. 15. Poncaré section for a typical strange attractor of Fig. 13 through the present method.

5.3. Higher dimensional nonlinear oscillator

The methodology developed in this paper in Section 3 is quite general and may thus be applied to any multi-dimensional, nonlinear dynamical system with a sufficiently differentiable vector field. Two illustrative examples will now be discussed.

5.3.1. The Lorenz system

The Lorenz oscillator, introduced by Edward Lorenz in 1963, is a nonlinear three-dimensional dynamical system derived from the simplified equations of convection rolls arising in the dynamical equations of the atmosphere. For a certain set of parameters, the system exhibits chaotic behavior and displays a strange attractor. Lorenz took a few *Navier–Stokes* equations, from the physics field of fluid dynamics, simplified them and got as a result the following three-dimensional system.

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy. \end{aligned} \tag{69}$$

Here x is the rate of convective *overturning* and y and z the horizontal and vertical temperature variations respectively. The parameter σ represents the Prandtl number, which is a ratio of kinematic viscosity to thermal conductivity; r is called a Rayleigh number and is proportional to the temperature difference between the upper and lower surfaces of the fluid; and b is a geometric factor. The only nonlinear terms are the two quadratic ones. The system also arises in lasers, dynamos, and specific waterwheels. Using the standard ADM form, Eq. (69) becomes:

$$\begin{aligned} Lx &= \sigma(y-x), \\ Ly &= rx - y - xz, \\ Lz &= -bz + xy, \end{aligned} \tag{70}$$

where $L = \frac{\mathrm{d}}{\mathrm{d}x}$. In matrix from this may be written as

$$\mathbf{LX} = \mathbf{RX} + \mathbf{N} + \mathbf{F}, \tag{71}$$

where $\mathbf{L} = \mathrm{diag}[L, L, L]$, $\mathbf{X} = [x, y, z]^{\mathrm{t}}$, $\mathbf{R} = [-\sigma, \sigma, 0; r, -1, 0; 0, 0, -b]$, $\mathbf{N} = [0, -xz, xy]^{\mathrm{t}}$ and $\mathbf{F} = [0,0,0]^{\mathrm{t}}$. Now, the series expansion of $\mathbf{X}$ and $\mathbf{N}$ are

$$\mathbf{X} = \begin{pmatrix} \sum_{n=0}^{\infty} x_n \\ \sum_{n=0}^{\infty} y_n \\ \sum_{n=0}^{\infty} z_n \end{pmatrix} \quad \text{and} \quad \mathbf{N} = \begin{pmatrix} 0 \\ \sum_{n=0}^{\infty} A_{2n} \\ \sum_{n=0}^{\infty} A_{3n} \end{pmatrix}. \tag{72}$$

Thus one can write

$$\begin{aligned}
&A_{20} = -x_0 z_0, \quad A_{30} = -x_0 y_0,\\
&A_{21} = -(x_1 z_0 + x_0 z_1), \quad A_{31} = -(x_1 y_0 + x_0 y_1),\\
&A_{22} = -(x_2 z_0 + x_1 z_1 + x_0 z_2), \quad A_{32} = -(x_2 y_0 + x_1 y_1 + x_0 y_2),\\
&\vdots
\end{aligned} \tag{73}$$

Following regular ADM technique, Eq. (71) may be reduced to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} + \begin{bmatrix} L^{-1} & & \\ & L^{-1} & \\ & & L^{-1} \end{bmatrix} \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & b \end{bmatrix} \times \begin{pmatrix} \sum_{n=0}^{\infty} x_n \\ \sum_{n=0}^{\infty} y_n \\ \sum_{n=0}^{\infty} z_n \end{pmatrix} + \begin{bmatrix} L^{-1} & & \\ & L^{-1} & \\ & & L^{-1} \end{bmatrix} \begin{pmatrix} 0 \\ \sum_{n=0}^{\infty} A_{2n} \\ \sum_{n=0}^{\infty} A_{3n} \end{pmatrix}, \tag{74}$$

where $(\hat{x}, \hat{y}, \hat{z}) = (x(0), y(0), z(0))$. One may write

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \hat{x} + L^{-1}\sigma\left(\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} x_n\right) \\ \hat{y} + L^{-1}\left(r\sum_{n=0}^{\infty} x_n - \sum_{n=0}^{\infty} y_n\right) + L^{-1}\sum_{n=0}^{\infty} A_{2n} \\ \hat{z} + L^{-1} b \sum_{n=0}^{\infty} z_n + L^{-1}\sum_{n=0}^{\infty} A_{3n} \end{bmatrix}. \tag{75}$$

The series terms are recursively calculated as:

$$\begin{aligned}
x_{n+1} &= L^{-1}\sigma(y_n - x_n),\\
y_{n+1} &= L^{-1}(r x_n - y_n) + L^{-1} A_{2n},\\
z_{n+1} &= L^{-1}(b z_n) + L^{-1} A_{3n}.
\end{aligned} \tag{76}$$

Presently, while computing the series solution, up to eleven terms are included. No removal of higher order terms has been done due to the simplicity of the nonlinear terms. The parameter values are as $\sigma = 10$, $b = 8/3$, $r = 28$. The step size in both RK4 and present method is taken as 0.001. The Lorenz attractor weaves in and out of itself and it is plotted on a three-dimensional phase space in Fig. 17(a). Fig. 16 shows time histories of the state variables in Lorenz equations by ODE45 and present method. Solutions by the two methods clearly match very well. Figs. 17(b) and 18(a), show the history plots of the state variables starting from the same IC-s in the chaotic regime and integrated using ODE45, RK4 and the present method. While solutions through RK4 diverge from the more correct ODE45 solution at $t = 32$, those via the present method diverge far away at $t = 40$.

5.3.2. The Rössler system

Otto Rössler [36] observed chaotic dynamics in a small reaction diffusion system, which arose from work in chemical kinetics.

As with other chaotic systems, the Rössler system is sensitive to ICs, and hence two initial states, no matter how close they are, will diverge-usually sooner rather than later. While the equations look simple enough, they lead to counter intuitively complex trajectories. The equations in the state space are:

$$\begin{aligned}
\dot{x} &= -y - z,\\
\dot{y} &= x + a y,\\
\dot{z} &= b + xz - cz,
\end{aligned} \tag{77}$$

where a, b, and c are constant parameters. In the present study, we have used $a = 0.25$, $b = 1$, $c = 5.5$. Rössler's attractor displays a type of banding, which suggests that perhaps it is related to the Cantor set. Another interesting fact about Rössler's attractor is that it has a half-twist in it, which makes it look somewhat like a Möbius strip (Fig. 19). Following the same principle as described for Lorenz attractor in Section 5.3.1, results have been obtainded the for Rössler's attractor. Up to eleven terms are taken in the series solution. No removal of higher order terms have been done due to the simple forms of the non-

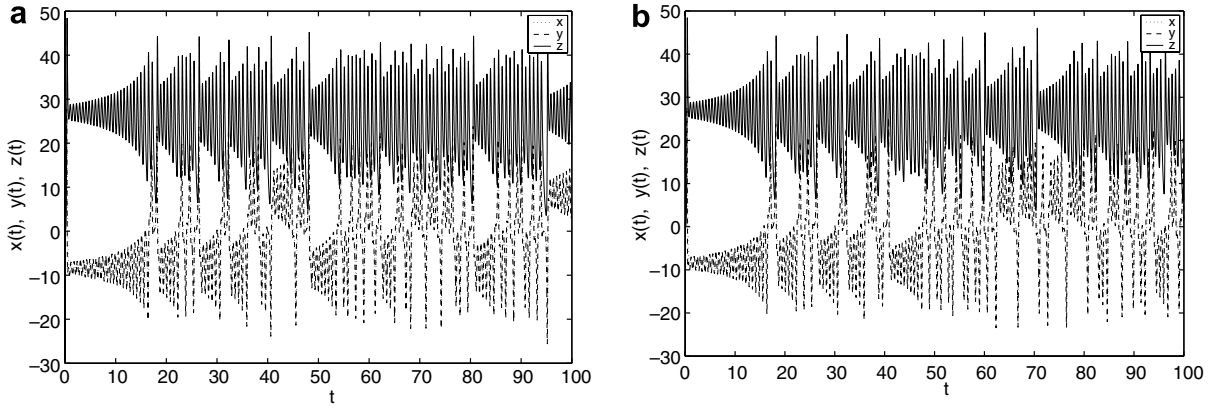

Fig. 16. Solutions of the Lorenz system by (a) ODE45, (b) present method.

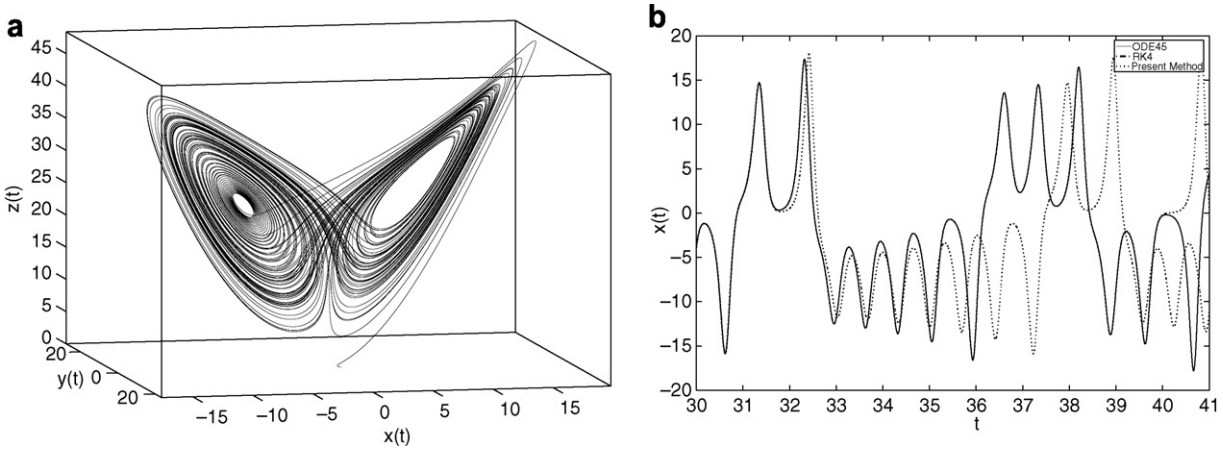

Fig. 17. (a) Lorenz attractor by present method; (b) $x(t)$ of Lorenz system by present method, RK4 and ODE45.

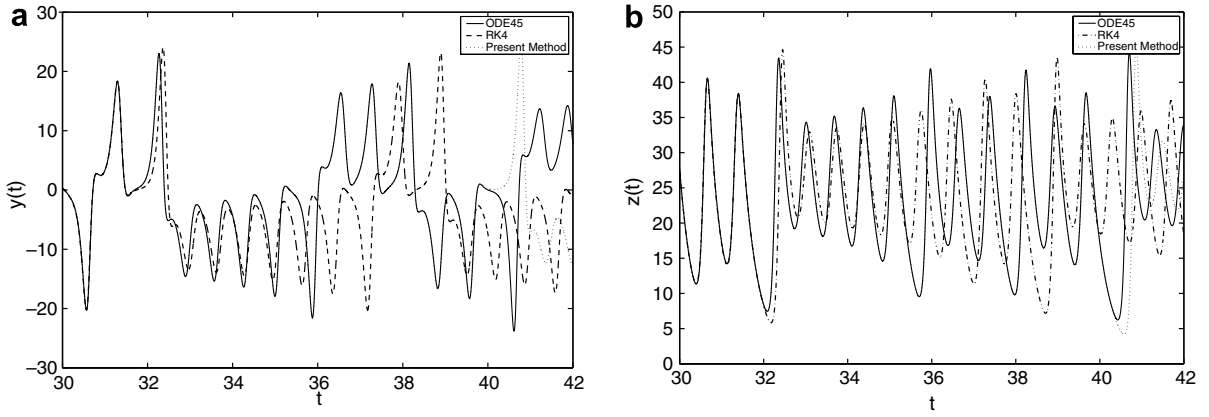

Fig. 18. $y(t)$ and $z(t)$ of Lorenz system by present method, RK4 and ODE45.

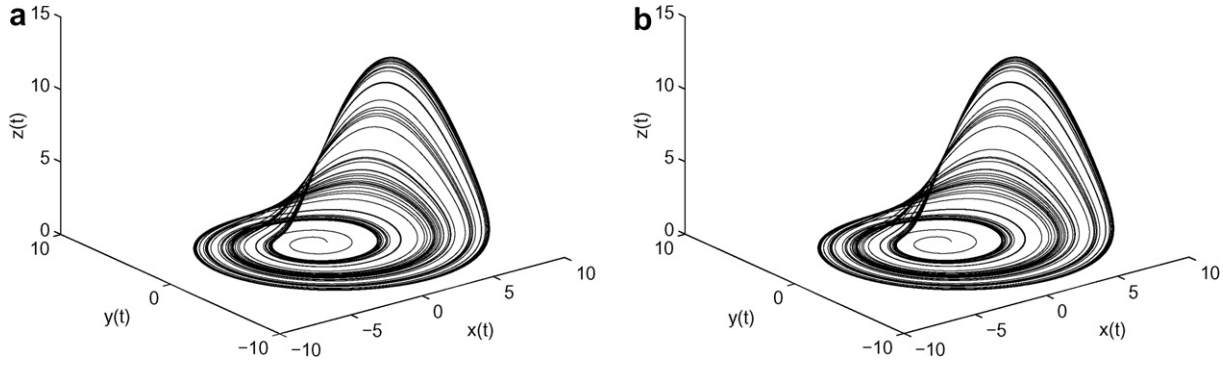

Fig. 19. Rössler's attractor by (a) ODE45, (b) present method.

linear terms. Fig. 19 shows that results via the present method are very close to those via ODE45. Step size in the decomposition method is taken as 0.05, whereas for purposes of comparison, step size in RK4 is taken as 0.01. Time histories are plotted for the three state variables (Fig. 20) and compared with ODE45 and RK4. While, first visible errors in RK4 appear at time $t = 107, 104, 105$, these errors through the present method appear at time $t = 180$, 177, 183, for x, y and z respectively. Thus the present scheme works much better than the RK4 even with a step size that is five time higher. Though analytical derivation of series terms requires some extra work, the present method certainly has less error. It might be noted that present method involves some extra work compared to purely numerical methods; this is required to maintain an adequately accurate functional form of the solution.

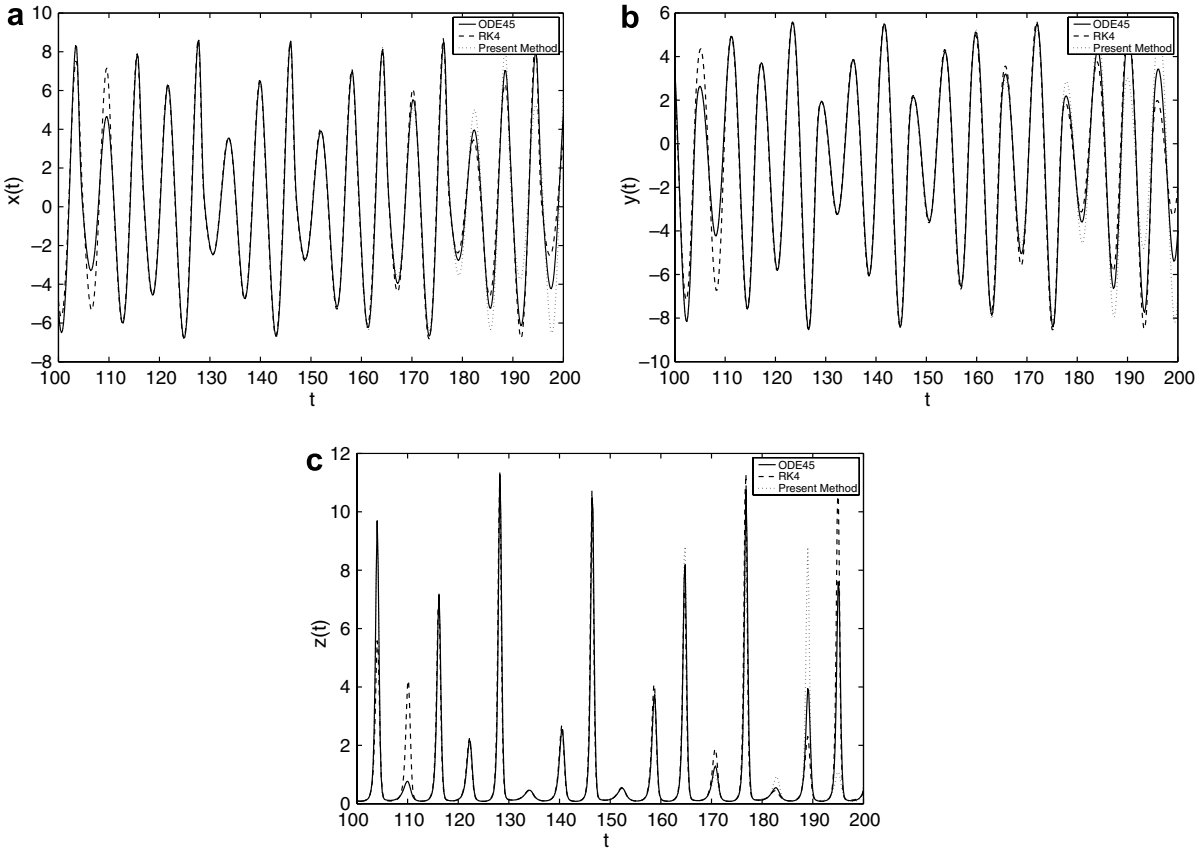

Fig. 20. Histories of $x(t)$, $y(t)$ and $z(t)$ of Rössler's system by present method, RK4 and ODE45.

6. Discussion and conclusions

A recursive and modified form of the Adomian decomposition is proposed for numeric-analytic integrations of nonlinear oscillators. A specific advantage of this method over any purely numerical method is that it offers a smooth, functional form of the solution over a time step. In the process, one may readily obtain a piecewise smooth solution of the nonlinear dynamical system over any time interval of interest. Moreover, the present procedure offers an explicit time-marching algorithm that works accurately over such high step sizes for which most of the available integration schemes will fail to be accurate. A comparative study of the solutions through the present method and a fourth order Runge–Kutta method, applied to several nonlinear oscillators, vis-a-vis exact solutions (whenever available) or those obtained through the ODE45 integrator of MATLAB (which works with adaptive step sizes) clearly brings out this point. The method developed here is also readily applicable to a general enough nonlinear dynamical system with an arbitrarily high dimension.

The functional form of the solution obtainable through the modified ADM enables one to do a stability analysis of periodic orbits (through computations of Floquet exponents). It also allows one to accurately compute the maximal Liapunov exponent (for a detection of chaos) by suitably exploiting the functional form. One may apply the procedure for solving a class of nonlinear boundary value problems, which may be conditionally posed as initial value problems. The algorithm may be extended to solve nonlinear stochastic differential equations of engineering interest. Finally, the authors are presently working towards the development of an Adomian-like finite element (spatial) discretization scheme for nonlinear partial differential equations.

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