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Analytical solutions for algebraic growth of disturbances in a stably stratified shear flow

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We investigate analytically the short-time response of disturbances in a density-varying Couette flow without viscous and diffusive effects. The complete inviscid problem is also solved as an initial value problem with a density perturbation. We show that the kinetic energy of the disturbances grows algebraically at early times, contrary to the well-known algebraic decay at time tending to infinity. This growth can persist for arbitrarily long times in response to sharp enough initial perturbations. The simplest in our three-stage study is a model problem forced by a buoyancy perturbation in the absence of background stratification. A linear growth with time is obtained in the vertical velocity component. This model provides an analogy between the transient mechanism of kinetic energy growth in a two-dimensional density-varying flow and the lift-up mechanism of the three-dimensional constant density flow. Next we consider weak stable background stratification. Interestingly, the lowest order solution here is the same as that of the model flow. Our final study shows that a strong background stratification results in a sub-linear growth with time of the perturbation. A framework is thus presented where two-dimensional streamwise disturbances can lead to large transient amplification, unlike in constant density flow where three dimensions are required.

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1. Introduction

A notable characteristic of the atmosphere and the ocean is that they are stably stratified, i.e. their density increases, for the most part, in the direction of gravity. A direct consequence of such density stratification is that, beyond a certain threshold stratification, internal gravity waves (IGWs) can be supported in the medium. If there is a shear flow in this environment, the interaction of stable stratification and background shear opens up a rich, and not entirely understood, range of phenomena. This interaction plays a crucial role in a wide range of situations of engineering and geophysical interest [1].

The stability of a stratified geophysical flow, in both unbounded and bounded model geometries, is a well-studied problem. Taylor [2] and Goldstein [3] independently modelled the stratified atmospheric flow and arrived at the now well-known Taylor–Goldstein equation.[1] Until the 1980s, the inviscid problem drew a lot of attention, primarily in unbounded flow, and the focus was on long-time behaviour [5–12]. An in-depth linear stability analysis of inviscid Couette flow, under both stable and unstable uniform stratification, was first done by Eliassen *et al.* [5]. Since their work predates the detailed studies of Case [13] and Dikii [6], they were probably also the first to introduce the notion of continuous spectra in hydrodynamic stability, except for a passing reference made by Rayleigh [14]. Splitting the disturbance flow obtained from a general initial value problem into the modal superposition of disturbance eigenfunctions, they were able to prove the completeness of their eigenfunction basis obtained from a linear stability study. Their analysis was incomplete for $\mathrm{Ri} > \frac{1}{4}$ (Ri being the Richardson number—a measure of stratification; to be defined in §2) and this was subsequently completed by Engevik [15]. The perturbations were found to be neutral without exponential growth or decay. Consequentially Eliassen *et al.* realized that the interactions between these neutral modes could produce a time-dependent behaviour of the perturbation field. They calculated the perturbation evolution in the long-time asymptotic limit, for sufficiently smooth initial conditions. The behaviour at long times for the streamfunction (and thus the perturbation velocity fields) exhibits a power-law dependence on time, which we denote as t^{β}. For the unstratified case, $\beta = -2$, and this value was obtained again by Case [13]. In the stratified case, there was a debate about the correct value of β, until Brown & Stewartson [12] settled it in favour of $\beta = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - \mathrm{Ri}}$, a result previously derived both by Eliassen *et al.* [5] and by Booker & Bretherton [10], but contradicted by Chimonas [11] and others.

As non-modal stability of parallel shear flows started to gain more prominence as a route to turbulence, the effects of stratification in such scenarios have been investigated [16–20]. Farrell & Ioannau [16] studied transient algebraic growth in the two-dimensional inviscid problem for both bounded and unbounded stratified Couette flow by giving optimal excitation to the flow. The full three-dimensional viscous unbounded problem was studied by Bakas *et al.* [17], again by the optimal excitation technique. They show that streamwise rolls are the ones that grow optimally in unbounded flow for specific values of Ri. In more recent times, Jerome *et al.* [21] obtained transient growth characteristics of stratified shear flow in both Couette and Poiseuille geometries, but in the context of a destabilizing temperature gradient. Mikhailenko *et al.* [22] also showed the non-modal effects to suppress the Rayleigh–Taylor instability while recovering the same asymptotic behaviour of Brown & Stewartson.

As can be surmised from above, short-time dynamics do play a significant role in the evolution of the flow. Here, the focus of our study is on the short-time evolution of disturbances in inviscid Couette flow via an initial value problem formulation. We deduce an analogy between the popular lift-up effect, a dominant mechanism for transient growth of three-dimensional disturbances in shear flows, and that of stratified shear flow. Using a sharp bump in the density profile as our initial condition, the entire problem is solved analytically. A sub-linear growth of the velocity perturbation, which will be cut off at later times depending on the smoothness of the initial profile, is obtained. We demonstrate that perturbation energy can grow to large

[1] Banks *et al.* [4] mentions that Haurwitz also independently derived the equation and thus the appropriate nomenclature should be Taylor–Goldstein–Haurwitz equation.

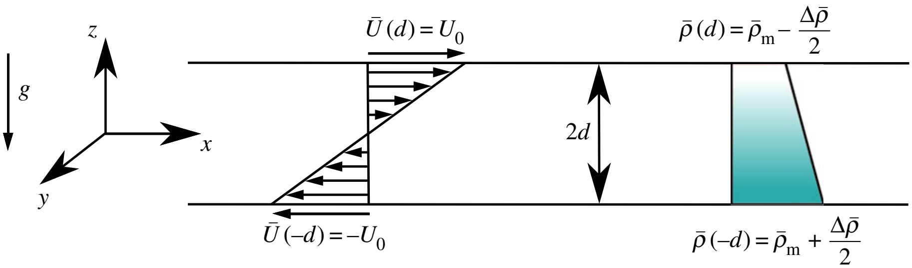

Figure 1. Flow schematic. The base state consists of a Couette flow velocity profile. A linear background density stratification in the stable scenario (increasing in the gravity direction) is prescribed. In §3a, the density stratification is set to zero, while in §3b we consider a weak stratification. In all cases, density perturbations are allowed. (Online version in colour.)

values even in two dimensions in the stratified case, unlike in constant density flow, where a three-dimensional nature of the disturbance is required (figure 1).

2. Problem formulation

We consider the evolution of perturbations in a stratified fluid while taking into account the Boussinesq approximation [23]. The mean state is composed of a parallel shear flow, $\bar{U}(z)$, in an ambient density field, $\bar{\rho}(z)$, with gravity acting along the negative z-direction.

For non-dimensionalization, we choose representative scales U_0 and d for velocity and length, respectively. On linearization about the mean flow quantities for a general parallel shear flow, we obtain the following linear inviscid system of equations describing the evolution of a three-dimensional velocity–density perturbation field:

$$\left[\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\nabla^2 - \bar{U}''\frac{\partial}{\partial x}\right] w = -\mathrm{Ri}_0 \nabla_H^2 \rho, \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\eta = -\bar{U}'\frac{\partial w}{\partial y} \tag{2.2}$$

and

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\rho = \frac{N^2}{N_0^2} w. \tag{2.3}$$

Here w and η are the normal component of the velocity and vorticity perturbations, respectively, and ρ is the density perturbation. $\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian. $N \equiv [-g\bar{\rho}'/\bar{\rho}_m]^{1/2}$ is the Brunt–Väisälä frequency, where $\bar{\rho}_m$ is chosen as the mean value of background density, and $N_0 \equiv \sqrt{g/d}$ is a reference frequency. $\mathrm{Ri}_0 = (N_0 d/U_0)^2 = gd/U_0^2$ is a reference Richardson number (or inverse squared Froude number Fr_0^{-2}), whereas it is the local Richardson number $\mathrm{Ri} = (Nd/U_0)^2$ which captures the local variation of density.

3. Inviscid algebraic instabilities

In constant density shear flow, it is known that three-dimensionality is required for perturbations to display the largest algebraic growth. We show in this section how 'stable' density stratification gives rise, in two dimensions, to large transient growth, which bears mathematical similarity to algebraic growth of three-dimensional disturbances in homogeneous (constant density) shear flows. Equations (2.1)–(2.3) reveal an analogy between three-dimensional homogeneous shear flow and two-dimensional stratified shear flow. Without loss of generality, let us consider uniform shear flow ($\bar{U} = z$), and a normal mode of the form $\mathrm{e}^{\mathrm{i}(k_x x + k_y y)}$ in the horizontal coordinates alone.

For three-dimensional inviscid stability of homogeneous uniform shear flow, the relevant linear equations are

$$\left(\frac{\partial}{\partial t} + \mathrm{i}k_x z\right)\nabla^2 w = 0 \tag{3.1}$$

and

$$\left(\frac{\partial}{\partial t} + \mathrm{i}k_x z\right)\eta = -\mathrm{i}k_y w. \tag{3.2}$$

We prescribe homogeneous boundary conditions for w. Equations (3.1) and (3.2) represent, in the inviscid limit, the classical Orr–Sommerfeld/Squire system which contains the celebrated 'lift-up' instability of Landahl [24]. Equation (3.2) yields

$$\eta = \eta_0(z)\,\mathrm{e}^{-\mathrm{i}k_x zt} - \mathrm{i}k_y \int_0^t w(z,t')\,\mathrm{e}^{-\mathrm{i}k_x z(t-t')}\,\mathrm{d}t', \tag{3.3}$$

the subscript 0 denoting an initial value. For streamwise independent perturbations ($k_x = 0$) $w(z,t) = w(z,0)$, and thus the normal vorticity increases linearly with time [25]. For $k_x \neq 0$, it can be shown that the initial transient will exhibit a linear growth before saturating.

In two dimensions, we have $\partial\xi/\partial x = -\nabla^2 w$, where ξ, the spanwise (y) component of perturbation vorticity, is aligned along the mean state vorticity. To explore an analogy with the three-dimensional 'lift-up' effect, we may write, for two-dimensional disturbances where density variations are possible,

$$\left(\frac{\partial}{\partial t} + \mathrm{i}k_x z\right)\rho = \frac{N^2}{N_0^2}w \tag{3.4}$$

and

$$\left(\frac{\partial}{\partial t} + \mathrm{i}k_x z\right)\xi = \mathrm{i}k_x\,\mathrm{Ri}_0\,\rho. \tag{3.5}$$

(a) Response to density perturbations with no background stratification

We study in this section how a flow which is initially at constant density everywhere will respond to a sharp localized density perturbation. For this, the background stratification, measured by N, is set to zero. However, we allow for density variations to affect dynamics, and to do so with ease we prescribe non-zero reference values of N_0, and therefore Ri_0. This study is equivalent to considering a regular perturbation series for small N^2 and finding the corresponding zeroth-order solution. We refer in the following text to this case, without background stratification, as our model problem. It is the simplest shear flow where density perturbations are allowed. Higher order corrections to this model, in the case where the background is stratified, will be addressed in the following subsection.

Now we have two different systems, the former three-dimensional and at constant density (equations (3.1) and (3.2)), and the latter two-dimensional and density stratified (equations (3.4) and (3.5)). With N set to zero, the similarity between the two sets of equations is immediately apparent, and we may draw a broad analogy between buoyancy perturbations ρ in the latter and the Laplacian of normal velocity $\nabla^2 w$ in the former. Both these quantities are advected by the flow, force the spanwise vorticity ξ in the latter problem, and the normal vorticity, η (via w), in the former. From (3.4) and (3.5), we have

$$\xi = \xi_0(z)\,\mathrm{e}^{-\mathrm{i}k_x zt} + \mathrm{i}k_x\rho_0(z)\,\mathrm{Ri}_0 t\,\mathrm{e}^{-\mathrm{i}k_x zt}. \tag{3.6}$$

Notice the linear growth in time of the spanwise vorticity ξ in response to an initial density perturbation which varies in z. Such linear growth was seen in equation (3.3) in the normal vorticity η for $k_x = 0$. We will hasten to point out to the reader that the analogy is incomplete, because in the three-dimensional unstratified case this growth in η translates to a growth in perturbation kinetic energy, whereas in the two-dimensional density stratified case an algebraic growth (of ξ) does not automatically imply algebraically growing perturbation kinetic energy. Since the velocity field emerges out of a spatially smoothening operation on the vorticity field, it can exhibit decay. In particular, the long-time asymptotics for this model problem shows decay of the form $w \sim t^{-1}$ for smooth initial conditions, with $\rho_0 \neq 0$ [12]. There is growth however at short times, as we shall soon see. Secondly, the physics of this density-initiated growth is different from that of 'lift-up'. The latter happens due to vertical transport of a streamwise momentum guided by a mean flow momentum gradient ($\bar{U}' \neq 0$). The former happens due to the density field advecting with the flow uninhibited by the normal velocity field, w, but possessing the ability to force, via buoyancy, the normal velocity perturbations. This growth could be viewed as an impulsive excitation of the system (a mass source forcing at $t = 0$). Consequently, we see that such linear growth would happen even in a stratified system with no background flow. This can be seen from the short-time expansion of Green's function for disturbances in a stratified medium [26].

To solve the model problem analytically, we choose as our initial condition a Gaussian density perturbation centred around a location z_0:

$$\rho_0(z) = \frac{\mathcal{C}}{2\sqrt{\pi\sigma}}\, \mathrm{e}^{-(z-z_0)^2/4\sigma} \tag{3.7}$$

along with $\xi_0(z) = 0$. Note that as $\sigma \to 0$ we approach a density-sheet initial condition. The solution of (3.6) can then be written down as

$$w = k_x \,\mathrm{Ri}_0 \mathcal{C} \frac{\sinh k_x(1-z)\mathcal{G}_1^0 + \sinh k_x(1+z)\mathcal{G}_2^0}{4\sinh 2k_x}\, t\, \mathrm{e}^{-\mathrm{i}k_x z_0 t}, \tag{3.8}$$

where

$$\begin{aligned}\mathcal{G}_1^0 = {} & \mathrm{e}^{-k_x^2(\mathrm{i}+t)^2\sigma + k_x(1+z_0)} \left(\mathrm{Erf}\left[\frac{1+z_0-2\mathrm{i}k_x\sigma(\mathrm{i}+t)}{2\sqrt{\sigma}}\right] + \mathrm{Erf}\left[\frac{z-z_0+2\mathrm{i}k_x\sigma(\mathrm{i}+t)}{2\sqrt{\sigma}}\right]\right)\\ & - \mathrm{e}^{-k_x^2(\mathrm{i}-t)^2\sigma - k_x(1+z_0)} \left(\mathrm{Erf}\left[\frac{1+z_0+2\mathrm{i}k_x\sigma(\mathrm{i}-t)}{2\sqrt{\sigma}}\right] + \mathrm{Erf}\left[\frac{z-z_0-2\mathrm{i}k_x\sigma(\mathrm{i}-t)}{2\sqrt{\sigma}}\right]\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_2^0 = {} & \mathrm{e}^{-k_x^2(\mathrm{i}-t)^2\sigma + k_x(1-z_0)} \left(\mathrm{Erf}\left[\frac{1-z_0-2\mathrm{i}k_x\sigma(\mathrm{i}-t)}{2\sqrt{\sigma}}\right] - \mathrm{Erf}\left[\frac{z-z_0-2\mathrm{i}k_x\sigma(\mathrm{i}-t)}{2\sqrt{\sigma}}\right]\right)\\ & - \mathrm{e}^{-k_x^2(\mathrm{i}+t)^2\sigma - k_x(1-z_0)} \left(\mathrm{Erf}\left[\frac{1-z_0+2\mathrm{i}k_x\sigma(\mathrm{i}+t)}{2\sqrt{\sigma}}\right] - \mathrm{Erf}\left[\frac{z-z_0+2\mathrm{i}k_x\sigma(\mathrm{i}+t)}{2\sqrt{\sigma}}\right]\right).\end{aligned}$$

For a sharp initial density profile having $\sigma \ll 1$, for $t < 1/\sqrt{\sigma}$ the solution has the asymptotic form

$$w \sim k_x^2 \,\mathrm{Ri}_0 \mathcal{C}\mathcal{G}(z, z_0) t\, \mathrm{e}^{-\mathrm{i}k_x z_0 t}. \tag{3.9}$$

Here $\mathcal{G}(z, z_0)$ is Green's function of the Rayleigh equation for a bounded simple shear flow, with homogeneous boundary conditions at $z = \pm 1$ with $z' = z_0$. Green's function has the following form [13]:

$$\mathcal{G}(z, z') = -\frac{\sinh k_x(1-z_>)\sinh k_x(1+z_<)}{k_x \sinh 2k_x}, \tag{3.10}$$

where $z_>$ and $z_<$ denote the greater or lesser of z and z', respectively. Incidentally, we report here that the factor $\sinh 2k_x$ in the denominator of equation (3.10) was incorrectly given as $\sinh k_x$ in the book of Schmid & Henningson [27]. The error arose due to not taking into account the change in domain from $z \in [0, 1]$ in the original work [13] to $z \in [-1, 1]$ in their setting.

On the other hand, for $t \gg \sigma^{-1/2}$, the solution (equation (3.8)) can be shown to decay as $O(t^{-1})$ in agreement with Brown & Stewartson [12]. We may now summarize our findings about this model problem as follows. A density sheet $\rho_0 \propto \delta(z - z_0)$, being uninfluenced by the de-phasing process of background shear, will display a growing velocity field and hence growing perturbation energy for all time. For a smooth initial condition, destructive interference due to shear will happen over a time scale of $O(\sigma^{-1/2})$. We thus anticipate a turnaround in the energy at this time, with a maximum value of $O(\sigma^{-1})$, given the initial linear growth in w.

To compare with these asymptotic limits, we now numerically solve (3.4) and (3.5) with $N^2 = 0$ for different levels of smoothness σ, with initial conditions given by (3.7). The kinetic energy of disturbance is the measure of interest here and is defined as follows:

$$\mathrm{E}(k_x; t) = \frac{1}{2}\int_{-1}^{1} |w(t)|^2 + |u(t)|^2\,\mathrm{d}z. \tag{3.11}$$

From numerical calculations, we can readily confirm the scaling obtained above. We have checked that over several orders of magnitude of change in σ the maximum kinetic energy $\mathrm{E}_{\max}$ scales linearly with $1/\sigma$. It must be noted that the energy values obtained are all proportional to the square of the reference Richardson number Ri_0. This scaling is described extremely well by $\mathrm{E}_{\max}\sigma = 0.03521\,\mathrm{Ri}_0^2$ when $k_x = 1$. The corresponding times ($t_{\max}$) at which this maximum of energy is attained, for each σ, scales linearly with $1/\sqrt{\sigma}$, and a good fit is given by $t\sqrt{\sigma} = 0.7180$.

Having explained the dynamics using a simple model the immediate question that arises is—how far will these features persist when we include background stratification? When $N^2 \neq 0$, we may anticipate that any vertical motion would encounter resistance from the mean stable density gradient, thus reducing the 'resonant' buoyancy forcing of itself that we saw above. In particular, would we obtain an initial algebraic growth? If yes, would the growth be linear, sub-linear or super-linear? The last scenario seems unlikely based on the inhibitive effect of stable background stratification but answers to the above questions demand an analysis.

(b) Weak background stratification

The model problem considered above has given us a means by which energy amplification occurs when forcing due to buoyancy perturbations are allowed. In a system where the density stratification is weak, i.e. if we have $N^2/N_0^2 = -\bar{\rho}'(z)/\bar{\rho}_{\mathrm{m}} \ll 1$, the neglect of the background stratification is justified, and the solutions obtained above for the model problem are valid. In fact, the solutions of the model problem form the leading-order solution when the quantities w and ρ are expanded in terms of a regular perturbation series in $\epsilon \equiv N^2/N_0^2$ as follows:

$$\rho = \rho^{(0)} + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)} + \cdots, \quad w = w^{(0)} + \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \cdots. \tag{3.12}$$

To proceed, we must prescribe a form of the background stratification. To choose the simplest stratification, the unperturbed system is considered to have a linear, stable density profile. This implies that ϵ is constant across the channel. We wish to investigate how the first correction to the model problem evolves. Thus to $O(\epsilon)$, we have the following governing equations:

$$\left(\frac{\partial}{\partial t} + \mathrm{i}k_x z\right)\rho^{(1)} = w^{(0)} \tag{3.13}$$

and

$$\left(\frac{\partial}{\partial t} + \mathrm{i}k_x z\right)\nabla^2 w^{(1)} = \mathrm{i}k_x\,\mathrm{Ri}_0\,\rho^{(1)}, \tag{3.14}$$

where $w^{(0)}$ evolves in time as per equation (3.8). Starting with $w^{(1)}(z,0) = \rho^{(1)}(z,0) = 0$, the solutions of $\rho^{(1)}$ and $w^{(1)}$ are then given as follows:

$$\rho^{(1)}(z,t) = \mathrm{e}^{-\mathrm{i}k_x z t}\int_0^t \mathrm{d}t'\,\mathrm{e}^{\mathrm{i}k_x z t'} w^{(0)}(z,t') \tag{3.15}$$

and

$$w^{(1)}(z,t) = k_x^2\,\mathrm{Ri}_0\int_{-1}^{1}\mathrm{d}z'\,\mathcal{G}(z,z')\,\mathrm{e}^{-\mathrm{i}k_x z' t}\int_0^t \mathrm{d}t'\,\mathrm{e}^{\mathrm{i}k_x z' t'}\rho^{(1)}(z',t'). \tag{3.16}$$

We now examine different initial conditions in $\rho^{(0)}$ from which these transients can be observed. We first consider the case where the initial condition in $\rho^{(0)}$ is a delta function at $z = z_0$.

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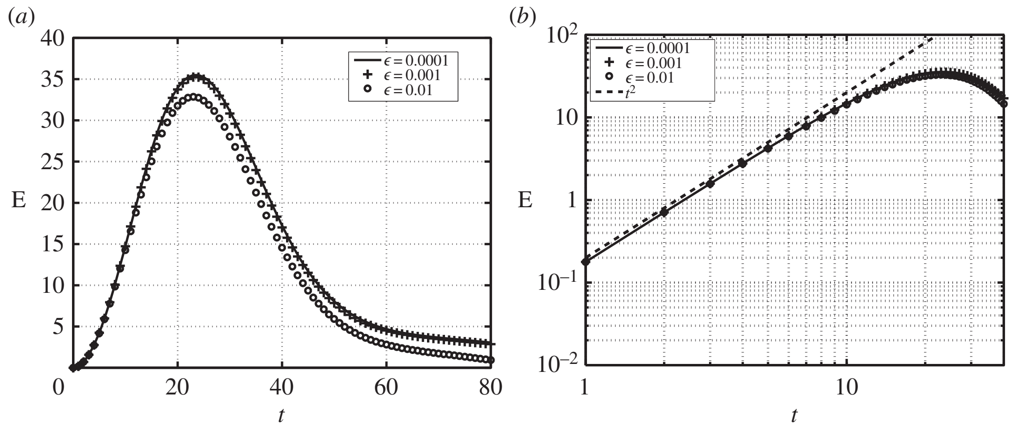

Figure 2. Gaussian initial conditions for buoyancy perturbations are used. $\mathrm{Ri}_0 = 1$, $k_x = 1$ and $\sigma = 0.001$. (*a*) Inviscid evolution of disturbance kinetic energy in a linear shear flow with weak background stratification. Background stratification causes a decrease in the kinetic energy attainable. (*b*) The early stages of the evolution shown in (*a*) are shown. The disturbance energy is seen to grow as t^2 before the background stratification starts to play a part in the evolution.

From the solution of the model problem, we already know that a delta function initial condition leads to a linear growth in the vertical component of the velocity field for all times, i.e.

$$\rho^{(0)} = \delta(z - z_0)\,\mathrm{e}^{-\mathrm{i}k_x z t} \tag{3.17}$$

and

$$w^{(0)} = k_x^2\,\mathrm{Ri}_0 \mathcal{G}(z, z_0) t\,\mathrm{e}^{-\mathrm{i}k_x z_0 t}. \tag{3.18}$$

As a consequence of w_0 growing for all t, the higher order quantities can be expected to grow faster than t at all times,

$$\rho^{(1)} = \mathrm{Ri}_0\,\mathrm{e}^{-\mathrm{i}k_x z t}\frac{\mathcal{G}(z, z_0)}{(z - z_0)^2}\{\mathrm{e}^{\mathrm{i}k_x(z - z_0)t}[1 - \mathrm{i}k_x(z - z_0)t] - 1\} \tag{3.19}$$

and

$$w^{(1)} = -\mathrm{i}k_x\,\mathrm{Ri}_0^2 \int_{-1}^{1} \mathrm{d}z' \frac{\mathcal{G}(z', z_0)\mathcal{G}(z, z')}{(z' - z_0)^3}\,\mathrm{e}^{-\mathrm{i}k_x z' t}\left\{\mathrm{e}^{\mathrm{i}k_x(z' - z_0)t}[2 - \mathrm{i}k_x(z' - z_0)t] - 2 - \mathrm{i}k_x(z' - z_0)t\right\}. \tag{3.20}$$

We now revert to the Gaussian initial condition for density perturbation that was employed for addressing the model problem, and ask how a weakly stratified system will evolve. The unstratified background yielded a leading order solution $w^{(0)}$ that grows up until a time proportional to $\sigma^{-1/2}$. The higher order quantities would then grow at a faster rate than $w^{(0)}$ at these times. However, this may not always translate to higher energy growth than in the model system.

In figure 2, the evolution of the disturbance kinetic energy for different ϵ is given. The energy plotted considers terms up to $O(\epsilon)$, and the initial growth is seen to be very close to the t^2 behaviour. For larger ϵ and/or large t, we have to consider higher order terms in the expansion series to get accurate results. As ϵ increases the maximum value of the energy decreases, suggesting that the stable background stratification is inhibiting the disturbance growth. As we reduce ϵ, the energy evolution curves collapse, as they should, to the zero background stratification case. Thus, it can be concluded that our simplest model system above, with no background density stratification, does indeed give us a reasonable picture of the evolving perturbation field when the background stratification is weak. We now need to demonstrate the persistence of algebraic growth of disturbances for the complete problem, where the assumption of weak background stratification is discarded.

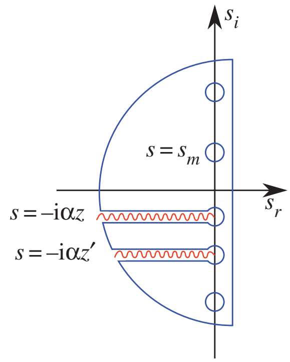

Figure 3. Bromwich contour showing the singularities. There are branch cuts at $s = -\mathrm{i}\alpha z$ and $-\mathrm{i}\alpha z'$, and poles at $s = s_m$ corresponding to mode m sheared gravity waves. The poles are absent when $\mathrm{Ri} < \frac{1}{4}$. (Online version in colour.)

(c) Strong background stratification

Now we shall consider the evolution of the disturbance when we take the stratification in the system to be strong. To simplify the algebra and be consistent with earlier work, we change our notation slightly in this section, to choose the reference frequency N_0 to be equal to the Brunt–Väisälä frequency N. This means that we now have $\mathrm{Ri} = \mathrm{Ri}_0$, with the density difference across the channel $\Delta\bar{\rho} = 2\bar{\rho}_\mathrm{m}$.

Laplace-transforming (3.4) and (3.5) and solving for w provides us with the following expression:

$$w(z,t) = \frac{1}{2\pi i}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st}\int_{-1}^{1}\left[\frac{\mathrm{i}\xi_0(z')}{(s+\mathrm{i}k_x z')} + \frac{k_x\,\mathrm{Ri}\,\rho_0(z')}{(s+\mathrm{i}k_x z')^2}\right]\mathcal{G}(z,z';s)\,\mathrm{d}z'\,\mathrm{d}s, \tag{3.21}$$

where

$$\left.\begin{aligned}
\mathcal{G}(z,z';s) &= \sqrt{(s+\mathrm{i}k_x z)(s+\mathrm{i}k_x z')}\\
&\quad\times\frac{\{K_n(\Lambda_>)I_n(\Lambda_{z'}) - I_n(\Lambda_>)K_n(\Lambda_{z'})\}\{K_n(\Lambda_<)I_n(\Lambda_z) - I_n(\Lambda_<)K_n(\Lambda_z)\}}{K_n(\Lambda_1)I_n(\Lambda_{-1}) - I_n(\Lambda_1)K_n(\Lambda_{-1})}\\
\Lambda_z &= -\mathrm{i}(s+\mathrm{i}k_x z),\quad \Lambda_{z'} = -\mathrm{i}(s+\mathrm{i}k_x z'),\quad \Lambda_1 = -\mathrm{i}(s+\mathrm{i}k_x),\quad \Lambda_{-1} = -\mathrm{i}(s-\mathrm{i}k_x)\\
\text{and}\quad \Lambda_> &= \Lambda_1\mathcal{H}(z-z') + \Lambda_{-1}\mathcal{H}(z'-z),\quad \Lambda_< = \Lambda_{-1}\mathcal{H}(z-z') + \Lambda_1\mathcal{H}(z'-z).
\end{aligned}\right\} \tag{3.22}$$

The quantity $n \equiv (\frac{1}{4} - \mathrm{Ri})^{1/2}$, which determines that no linear instability occurs above $\mathrm{Ri} = \frac{1}{4}$, will also be seen to determine the rate of initial growth. I_n, K_n are the modified Bessel's functions of order n and $\mathcal{H}(z)$ is the Heaviside function. γ in the above Bromwich integral is chosen so as to encompass all the singularities of the solution in the transform space.

Though a closed-form solution to the above inversion is not known, approximate forms of it have been obtained while working in the long-time limit. A large number of studies have focused on the unbounded domain setting (some erroneous). As mentioned before, Brown & Stewartson [12] reviewed several such attempts and firmly established the long-time behaviour to be algebraically decaying of the form t^β, with $\beta = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - \mathrm{Ri}}$. Again, our interest is in the short-time behaviour, and below we show that we are able to solve in this limit analytically. Secondly, ours is a bounded geometry.

Figure 3 depicts the singularities to be dealt with while performing the inverse Laplace transform. Green's function, $\mathcal{G}(z,z';s)$, has simple poles at $s = s_m$ corresponding to IGWs modified by shear. These poles are present only for $\mathrm{Ri} > \frac{1}{4}$. Figure 4 shows an example, at $\mathrm{Ri} = 5$ and $k_x = 1$, of the mode shapes corresponding to such poles, with and without shear. It is seen that the IGWs without shear are up–down symmetric, whereas in the sheared case this symmetry is broken. Returning to figure 3, we see that there are also algebraic branch-cuts at $s = -\mathrm{i}k_x z$ and $s = -\mathrm{i}k_x z'$,

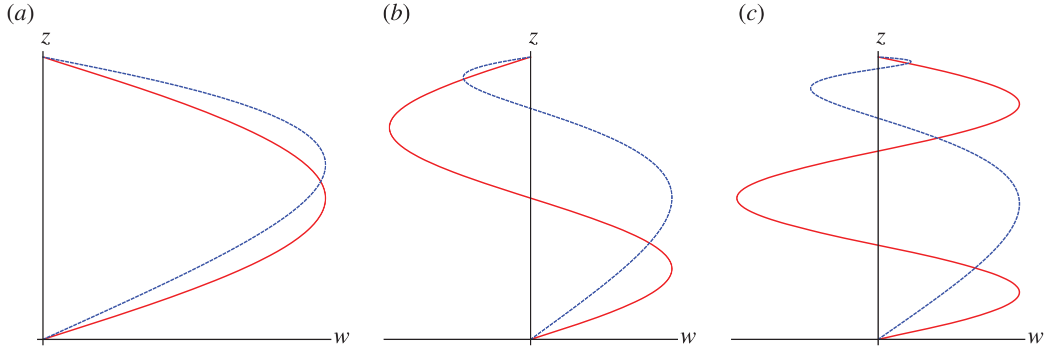

Figure 4. IGWs in the absence of shear (shown in red) compared with those subjected to uniform shear (shown by blue dashed lines). The modes shown correspond to poles of $\mathcal{G}(z, z'; s)$ with Ri = 5 and $k_x = 1$. It is evident that shear breaks the top–bottom symmetry of IGWs. (*a*) Mode 1, (*b*) mode 2 and (*c*) mode 3. (Online version in colour.)

whose location obviously depends on the wavenumber. An examination of the form of the initial condition which convolves with Green's function will convince the reader that $s = -\mathrm{i}k_x z'$ is the more severe initial condition and will therefore be primarily responsible for transients. In the vicinity of $s = -\mathrm{i}k_x z'$, we can approximate Green's function as

$$\mathcal{G}(z, z'; s) \sim (s + \mathrm{i}k_x z')^{-n+1/2} g(z, z'), \tag{3.23}$$

where

$$g(z, z') = -\frac{\mathrm{i}^n \Gamma(n)}{2^{1-n}} \frac{\sqrt{\mathrm{i}k_x(z - z')} I_n(\Lambda_>)\{K_n(\Lambda_<) I_n(k_x(z - z')) - I_n(\Lambda_<) K_n(k_x(z - z'))\}}{K_n(\Lambda_1) I_n(\Lambda_{-1}) - I_n(\Lambda_1) K_n(\Lambda_{-1})}.$$

Now for a density sheet initial condition, $\rho_z = \mathcal{C}\delta(z - z_0)$, the velocity field from (3.21) can be written down after performing the inverse Laplace transform as

$$w(z, t) = \frac{k_x \,\mathrm{Ri}\, \mathcal{C} g(z, z_0)}{\Gamma(3/2 + n)} \,\mathrm{e}^{-\mathrm{i}k z_0 t} t^{1/2+n} + O(t^{1/2-n}). \tag{3.24}$$

Thus we have shown that, at short times, with significant stable background density stratification, the perturbation velocity grows sub-linearly with time in response to an initial perturbation in the form of a density sheet. Note that, in the limit $\mathrm{Ri} \to 0$, we have $n \to \frac{1}{2}$, which recovers the linear growth of the unstratified case. For $\mathrm{Ri} \ll 1$, the perturbation would thus initially grow linearly as seen in §3b before transitioning to a sub-linear growth.

Similar to the unstratified background, and the weak background stratification cases, here too a smooth initial condition of width σ will exhibit kinetic energy growth at early times and decay at times later than $\sigma^{-1/2}$. In other words, the sharper the initial perturbation, the longer the growth phase can be sustained for. In the strong stratification case, the perturbation displays a sub-linear form $t^{1/2+n}$ before settling into the familiar $t^{-3/2+n}$ asymptotic decay of Booker & Bretherton [10]. Figure 5 shows a numerical evaluation[2] of (3.4) and (3.5) for a localized density initial condition. In figure 5*a*, the total energy exhibits algebraic growth in time, with growth being more pronounced for spatially sharper initial conditions. The contribution to kinetic energy from the vertical velocity component, which we term as $\mathrm{E}_\perp$, given by

$$\mathrm{E}_\perp(k_x; t) = \frac{1}{2} \int_{-1}^{1} |w(t)|^2 \,\mathrm{d}z, \tag{3.25}$$

is shown in figure 5*b*. There is an extremely short period, where $\mathrm{E}_\perp$ grows as t^2 before transitioning to the predicted t^{2n+1}. Our predicted behaviour is then visible for about a decade

[2]For the numerical calculation, the spatial discretization is done using the Chebyshev spectral collocation method [28] and the time integration is carried out using the Matlab software command ODE45, based on an explicit Runge–Kutta formula with adaptive step sizes.

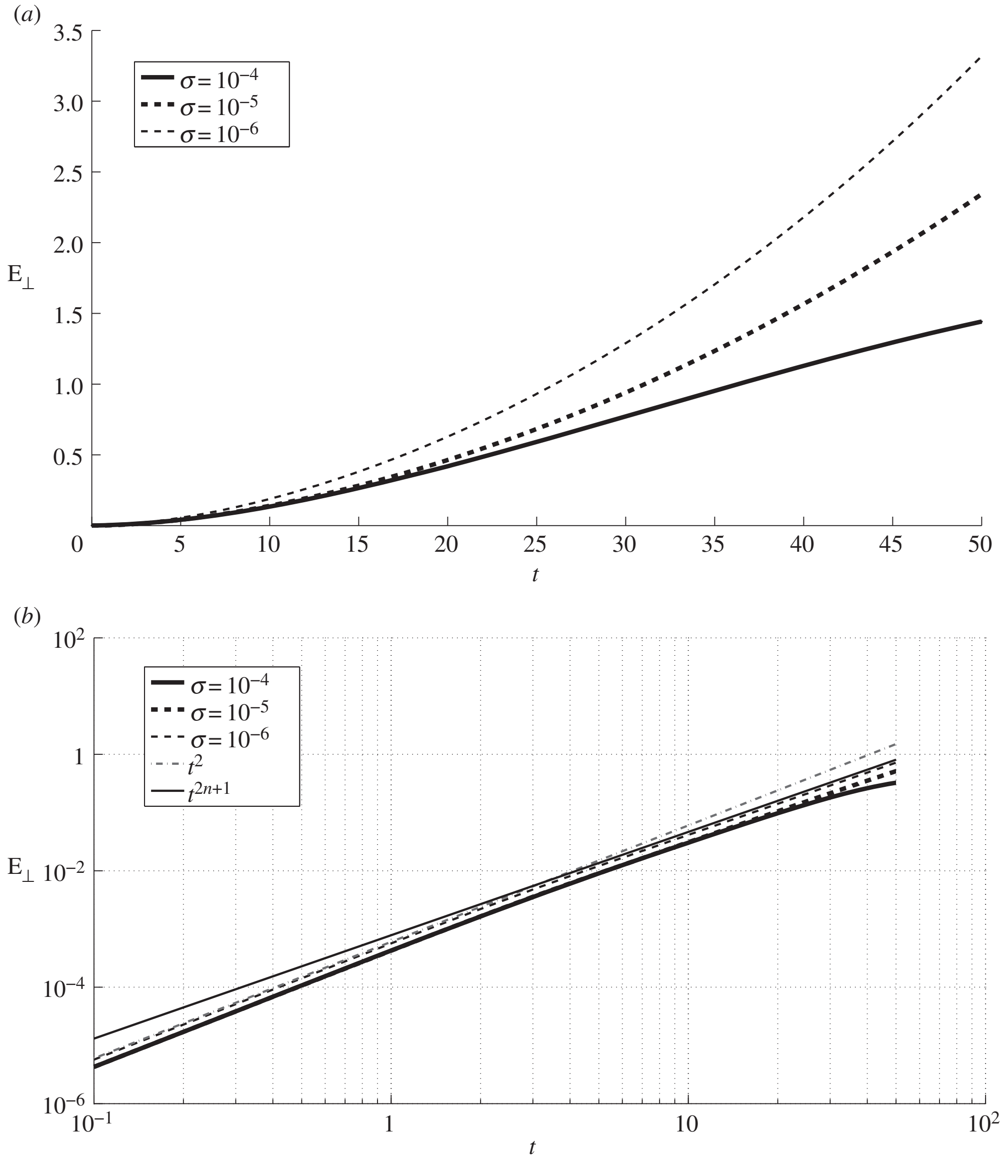

Figure 5. Kinetic energy evolution for initial conditions given by (3.7), and its comparison with the asymptotic estimate. Here $\mathrm{Ri} = 0.1$ ($N \neq 0$), $k_x = 1$. Sharper initial conditions give rise to larger kinetic energy growth. (*a*) Kinetic energy and (*b*) contribution to kinetic energy from the vertical velocity component.

in time. The wider initial condition ($\sigma = 10^{-4}$) then diverges first from this behaviour, as we expected, due to the destructive interference of shear. The reason for the initial t^2 behaviour is that, at times shorter than $O(1/\mathrm{Ri})$, the effects of background stratification have not yet come into play, and the behaviour is like that of an unstratified background flow.

4. Discussion and conclusion

In this paper, we have demonstrated algebraic instabilities in stably stratified shear flows by considering a sequence of scenarios—zero, weak and strong ($\mathrm{Ri} < 0.25$) background stratification. Conventional wisdom indicates that stable stratification and linear shear, both modally stable individually, would not give rise to any instabilities when coupled. The exception to this rule occurs when interfacial waves, riding on density discontinuities in a stably stratified scenario, resonantly interact via shear [29–31]. For a uniformly stratified shear flow, the scenario in this analysis, no such exponential instabilities exist.

We have shown how the perturbations in a uniformly stratified shear flow can grow in time due to buoyancy forcing acting on the momentum variables. For the simplified scenario of no background stratification, this is readily apparent, because here the density perturbations are not slaved to the velocity perturbations. When we consider both weak and strong background stratification, we need to resort to perturbation techniques and numerical simulations to identify the short-time algebraic growth. Our analytical calculations reveal that this growth can be made arbitrarily large, and the growth phase sustained for arbitrarily long time, by creating a density perturbation initial condition consisting of an increasingly sharp bump. Appealingly, the velocity perturbations are shown to grow sub-linearly, as $t^{1/2+n}$, with a decreasing exponent for increasing stratification. We have highlighted a very interesting analogy between three-dimensional unstratified flow and two-dimensional stratified flow, with stratification bringing in a source of non-normality in the eigenmodes, just as the Squire modes do in three dimensions.

This analysis occupies a relevant corner in the domain of non-modal stability of stratified shear flows, being dissimilar in approach to the popular optimal perturbation technique while maintaining the focus on identifying transient growth of disturbances. Optimal perturbation analysis of shear flows renders the entire problem to identifying the initial condition that maximizes an objective functional, perturbation kinetic energy being traditionally chosen. A density stratification suggests that the total perturbation energy, the sum of kinetic and potential energies, is the relevant measure [21,32]. We mention in passing that, in situations which do not allow for an evident measure of potential energy, other measures of optimality will need to be defined. One example is a multi-component flow where all components have the same density, but are vastly different in other properties. In such cases, a different approach to the treatment of the problem is perhaps more apt than optimal perturbation analysis. Our analysis reveals that local fluctuations in density field can lead to a transiently growing velocity field, which can have crucial implications for stirring up the flow although our choice may not be the optimal perturbation for the flow.

Data accessibility. This work does not report any data.

Authors' contributions. S.J. and A.R. contributed equally in carrying out calculations to obtain solutions in each of the cases described and in the preparation of the manuscript. R.B. was involved in the preliminary calculations. R.G. conceived the overall study and contributed to writing the manuscript.

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References

1. Turner JS. 1973 *Buoyancy effects in fluids*. Cambridge, UK: Cambridge University Press.
2. Taylor GI. 1931 Effect of variation in density on the stability of superposed streams of fluid. *Proc. R. Soc. Lond. A* **132**, 499–523. (doi:10.1098/rspa.1931.0115)
3. Goldstein S. 1931 On the stability of superposed streams of fluids of different densities. *Proc. R. Soc. Lond. A* **132**, 524–548. (doi:10.1098/rspa.1931.0116)
4. Banks WHH, Drazin PG, Zaturska MB. 1976 On the normal modes of parallel flow of inviscid stratified fluid. *J. Fluid Mech.* **75**, 149–171. (doi:10.1017/S0022112076000153)
5. Eliassen A, Høiland E, Riis E. 1953 *Two-dimensional perturbation of a flow with constant shear of a stratified fluid*, vol. 1. Oslo, Norway: Institute for Weather and Climate Research.
6. Dikii LA. 1960 On the stability of plane parallel flows of an inhomogeneous fluid. *J. Appl. Math. Mech.* **24**, 357–369. (doi:10.1016/0021-8928(60)90039-3)

7. Case KM. 1960 Stability of idealized atmosphere. I. Discussion of results. *Phys. Fluids* **3**, 149–154. (doi:10.1063/1.1706011)
8. Dyson FJ. 1960 Stability of idealized atmosphere. II. Zeros of the confluent hypergeometric function. *Phys. Fluids* **3**, 155–157. (doi:10.1063/1.1706012)
9. Kuo HL. 1963 Perturbations of plane Couette flow in stratified fluid and origin of cloud streets. *Phys. Fluids* **6**, 195–211. (doi:10.1063/1.1706719)
10. Booker JR, Bretherton FP. 1967 The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.* **27**, 513–539. (doi:10.1017/S0022112067000515)
11. Chimonas G. 1979 Algebraic disturbances in stratified shear flows. *J. Fluid Mech.* **90**, 1–19. (doi:10.1017/S0022112079002056)
12. Brown SN, Stewartson KD. 1980 On the algebraic decay of disturbances in a stratified linear shear flow. *J. Fluid Mech.* **90**, 811–816. (doi:10.1017/S0022112080001425)
13. Case KM. 1960 Stability of inviscid plane Couette flow. *Phys. Fluids* **3**, 143–148. (doi:10.1063/1.1706010)
14. Rayleigh JWS. 1945 *The theory of sound*. New York, NY: Dover Publications.
15. Engevik L. 1971 A note on a stability problem in hydrodynamics. *Acta Mech.* **12**, 143–153. (doi:10.1007/BF01178395)
16. Farrell BF, Ioannou PJ. 1993 Transient development of perturbations in stratified shear flow. *J. Atmos. Sci.* **50**, 2201–2214. (doi:10.1175/1520-0469(1993)050<2201:TDOPIS>2.0.CO;2)
17. Bakas NA, Ioannou PJ, Kefaliakos GE. 2001 The emergence of coherent structures in stratified shear flow. *J. Atmos. Sci.* **58**, 2790–2806. (doi:10.1175/1520-0469(2001)058<2790:TEOCSI>2.0.CO;2)
18. Biau D, Bottaro A. 2004 The effect of stable thermal stratification on shear flow stability. *Phys. Fluids* **16**, 4742–4745. (doi:10.1063/1.1810751)
19. Sameen A, Govindarajan R. 2007 The effect of wall heating on instability of channel flow. *J. Fluid Mech.* **577**, 417–442. (doi:10.1017/S0022112007004636)
20. Sameen A, Bale R, Govindarajan R. 2011 The effect of wall heating on instability of channel flow—corrigendum. *J. Fluid Mech.* **673**, 603–605. (doi:10.1017/S0022112011000206)
21. Jerome JJS, Chomaz JM, Huerre P. 2012 Transient growth in Rayleigh–Bènard–Poiseuille/Couette convection. *Phys. Fluids* **24**, 044103. (doi:10.1063/1.4704642)
22. Mikhailenko VS, Scime EE, Mikhailenko VV. 2005 Stability of stratified flow with inhomogeneous shear. *Phys. Rev. E* **71**, 026306. (doi:10.1103/PhysRevE.71.026306)
23. Spiegel EA, Veronis G. 1960 On the Boussinesq approximation for a compressible fluid. *Astrophys. J.* **131**, 441–447. (doi:10.1086/146849)
24. Landahl MT. 1980 A note on an algebraic instability of inviscid parallel shear flows. *J. Fluid Mech.* **98**, 243–251. (doi:10.1017/S0022112080000122)
25. Ellingsen T, Palm E. 1975 Stability of linear flow. *Phys. Fluids* **18**, 487–488. (doi:10.1063/1.861156)
26. Voisin B. 1991 Internal wave generation in uniformly stratified fluids. Part 1. Green's function and point source. *J. Fluid Mech.* **231**, 439–480. (doi:10.1017/S0022112091003464)
27. Schmid PJ, Henningson DS. 2001 *Stability and transition in shear flows*. New York, NY: Springer.
28. Trefethen LN. 2000 *Spectral methods in MATLAB*. Philadelphia, PA: SIAM.
29. Craik ADD. 1985 *Wave interactions and fluid flow*. Cambridge, UK: Cambridge University Press.
30. Dixit HN, Govindarajan R. 2011 Stability of a vortex in radial density stratification: role of wave interactions. *J. Fluid Mech.* **679**, 582–615. (doi:10.1017/jfm.2011.156)
31. Balmforth NJ, Roy A, Caulfield CP. 2012 Dynamics of vorticity defects in stratified shear flow. *J. Fluid Mech.* **694**, 292–331. (doi:10.1017/jfm.2011.548)
32. Kaminski AK, Caulfield CP, Taylor JR. 2014 Transient growth in strongly stratified shear layers. *J. Fluid Mech.* **758**, R4. (doi:10.1017/jfm.2014.552)