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We say that the series $\sum x_n$ converges to $s \in \mathbb{R}$ iff for each $\varepsilon > 0$, there exists an $m \in \mathbb{N}$ such that for each natural number $n > m$, we have $\left|\sum_{k=1}^{n} x_k - s\right| < \varepsilon$. In this case, we also say that the series sums to s, and write $\sum x_n = s$.

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A series that converges is called a convergent series.

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We say that a series $\sum x_n$ diverges to ∞ iff for each $r > 0$, there exists $m \in \mathbb{N}$ such that for each natural number $n > m$, $\sum_{k=1}^{n} x_k > r$. We write it as $\sum x_n = \infty$, and say that the series sums to ∞.

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(c) The series $1-1+1-1+1-1+\cdots$ diverges but it neither diverges to ∞ nor to $-\infty$.

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Similar to divergence of series, we can also define divergence of sequences.

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Theorem Let $m \in \mathbb{N}$. Let $f : [m, \infty) \to \mathbb{R}$ be a function. Let $\{x_n\}$ be a sequence such that $x_n = f(n)$ for all $n \geq m$. If $\lim_{x\to\infty} f(x) = \ell$, then $\lim_{n\to\infty} x_n = \ell$.

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Example 2: $\log x$ is defined on $[1, \infty)$. Using L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

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$$s_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1-(1/2)^n}{1-1/2} = 1 - \frac{1}{2^n} \ \to \ 1 \quad \text{since } 1/2^n \to 0.$$

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We generalize this example.

Geometric series

Let $a, r \in \mathbb{R}$ with $a \neq 0$. The nth partial sum of the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

is $s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$.

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(a) If $|r| < 1$, then $r^n \to 0$. The geometric series converges to $\lim_{n\to\infty} s_n = \dfrac{a}{1-r}$.

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(b) If $r = -1$ or $|r| > 1$, then r^n diverges. So, the geometric series $\sum ar^{n-1}$ diverges.
(c) And, for $r = 1$, the geometric series $a + a + \cdots$ diverges.

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Example 5: Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 4}{6^n}$. We can compute its sum:

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Sum of two divergent series can converge. For example, both $\sum(1/n)$ and $\sum(-1/n)$ diverge but their sum $\sum 0$ converges.

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That is, $x_{n+1} \to 0$.

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Define two sequences (s_n) and (t_n) by taking $s_n = x_1 + \cdots + x_n$ and $t_n = x_1 + \cdots + x_n + x_{n+1}$.
Now, $s_n \to s$ and $t_n \to s$. So, their difference $(t_n - s_n)$ converges to 0.
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The converse is not true, in general.
$1/2^n \to 0$ and the series $\sum(1/2^n)$ converges,
whereas $1/n \to 0$ but $\sum(1/n)$ does not converge. (We will see it.)

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Therefore, the series diverges to ∞.

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This is called re-indexing the series. As long as we preserve the order of the terms of the series, we can re-index without affecting its convergence and sum.

Helpful facts

- $r^n \to 0$ if $|r| < 1$, $\quad r^n \to \infty$ if $r > 1$, $\quad r^n$ diverges if $r \leq -1$.

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- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha}$ converges for $\alpha > 1$, and diverges to ∞ for $\alpha \leq 1$.

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We ask, which functions have power series expansion, and how to compute such a series?

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$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \quad \text{OR} \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

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If $R_n(x)$ converges to 0 for all x in an interval around the point $x = a$, then the ensuing series on the right hand side would converge and then the function can be written in the form of a series.

Taylor Series

When $f(x)$ is infinitely differentiable and the error $|R_n(x)| \to 0$ as $n \to \infty$, the function $f(x)$ is represented by a power series, called Taylor series at $x = a$:

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Example 6

Find the Taylor series expansion of the function $f(x) = 1/x$ at $x = 2$. In what interval around $x = 2$, the series converges?

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It is a geometric series with ratio $r = -(x-2)/2$. It converges absolutely for

$$|r| < 1, \quad \text{i.e.,} \quad |x-2| < 2 \quad \text{i.e.,} \quad 0 < x < 4.$$

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For any c, x in an interval around $x = 2$,

$$|R_n| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-2)^{n+1} \right| = \left| \frac{(x-2)^{n+1}}{c^{n+2}} \right|.$$

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Here, c lies between x and 2. If x is near 2, then $|R_n| \to 0$. Hence the Taylor series represents the function near $x = 2$.

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$$f(0) = 1, f'(0) = 1, \cdots, f^{(n)}(0) = 1, \cdots$$

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Hence, for each $x \in \mathbb{R}$,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

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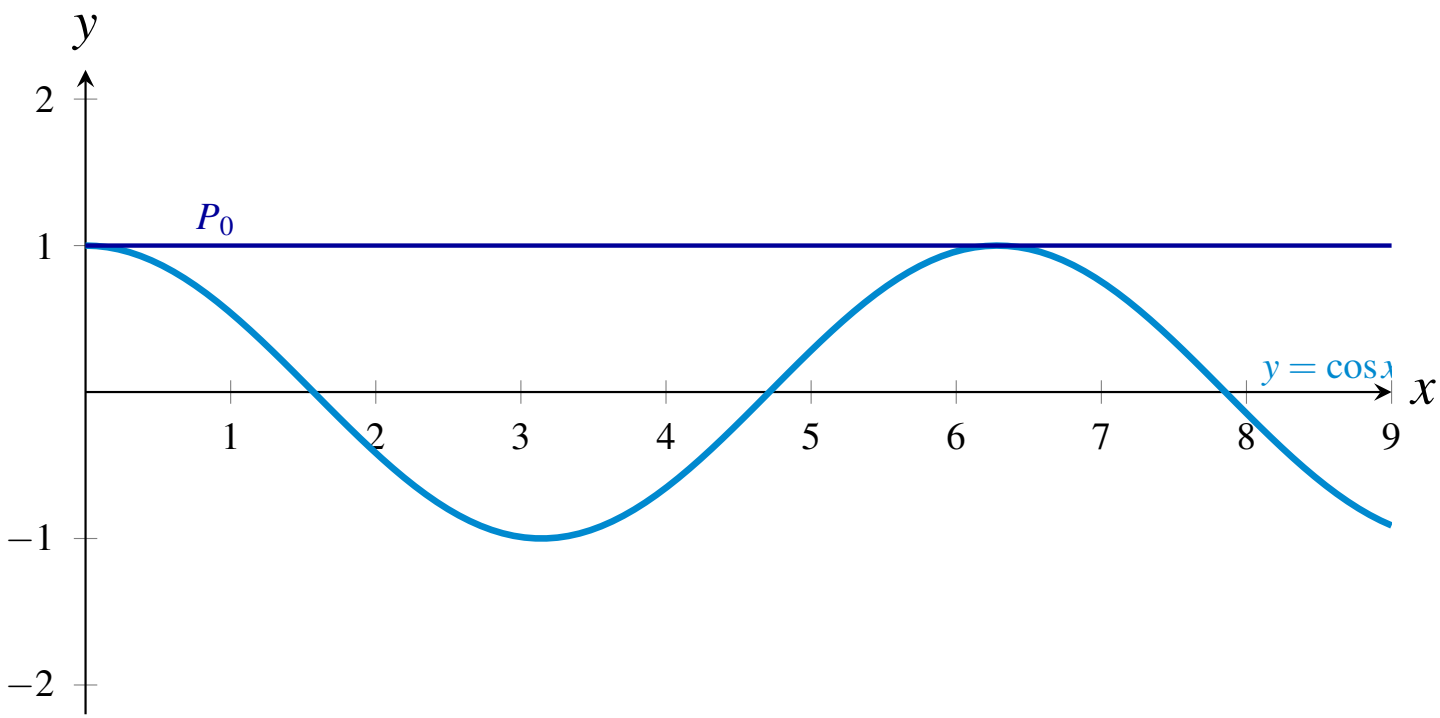
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$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{for each } x \in \mathbb{R}.$$

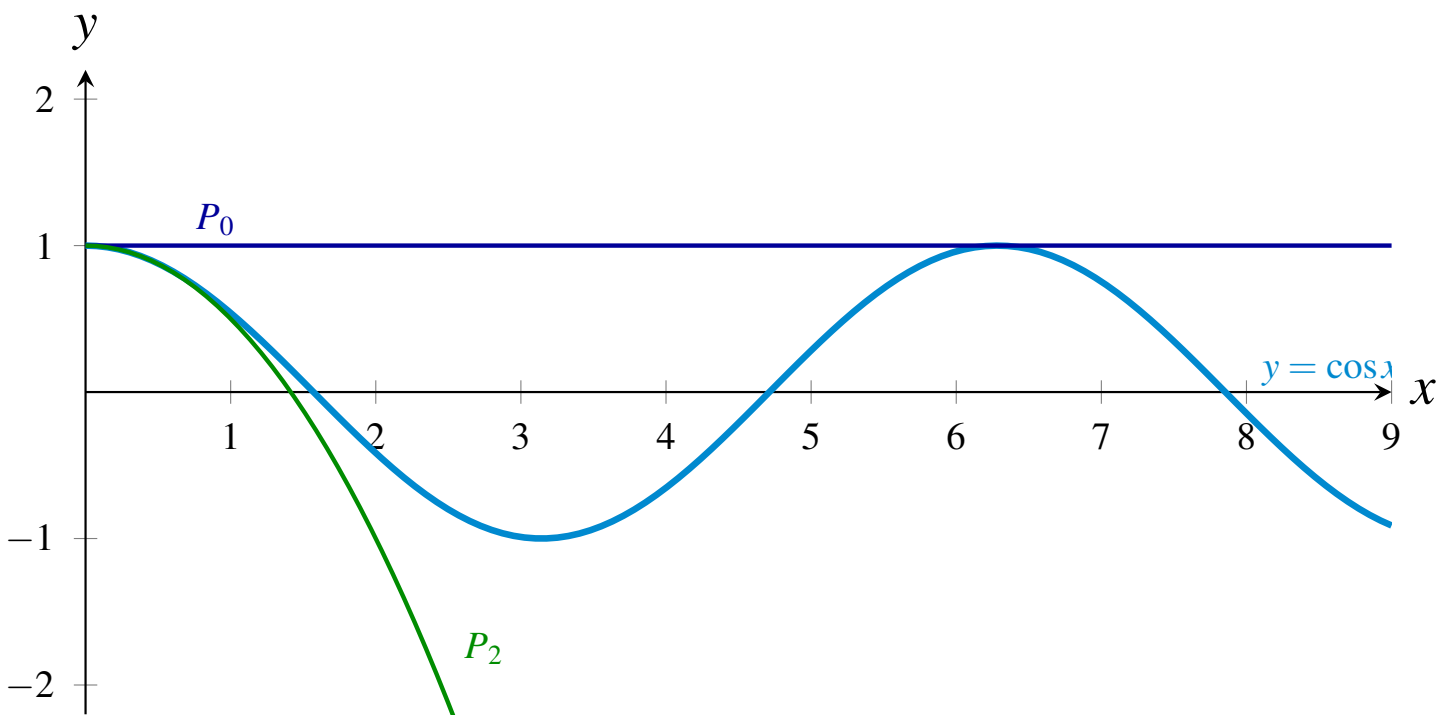
Example 8 (continued)

Taylor polynomials approximating $\cos x$ for $0 \leq x \leq 9$ are



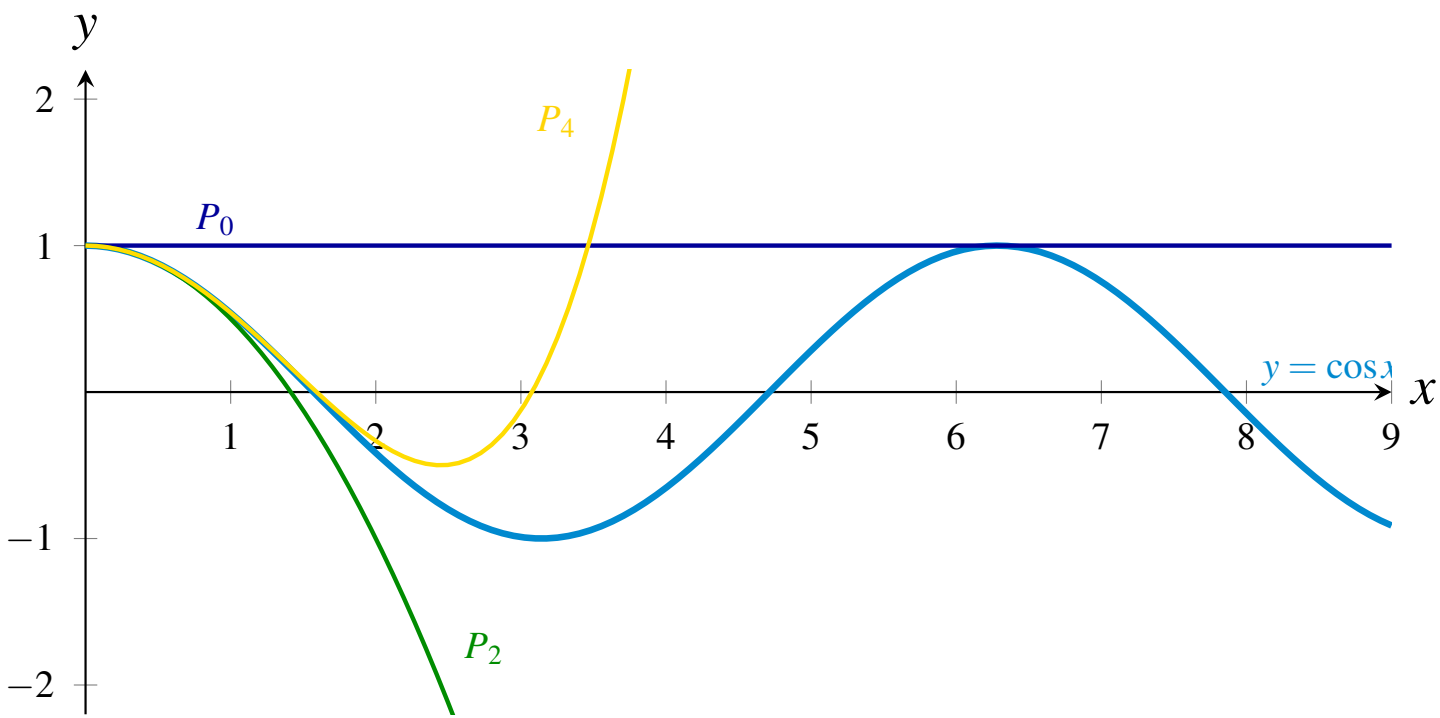
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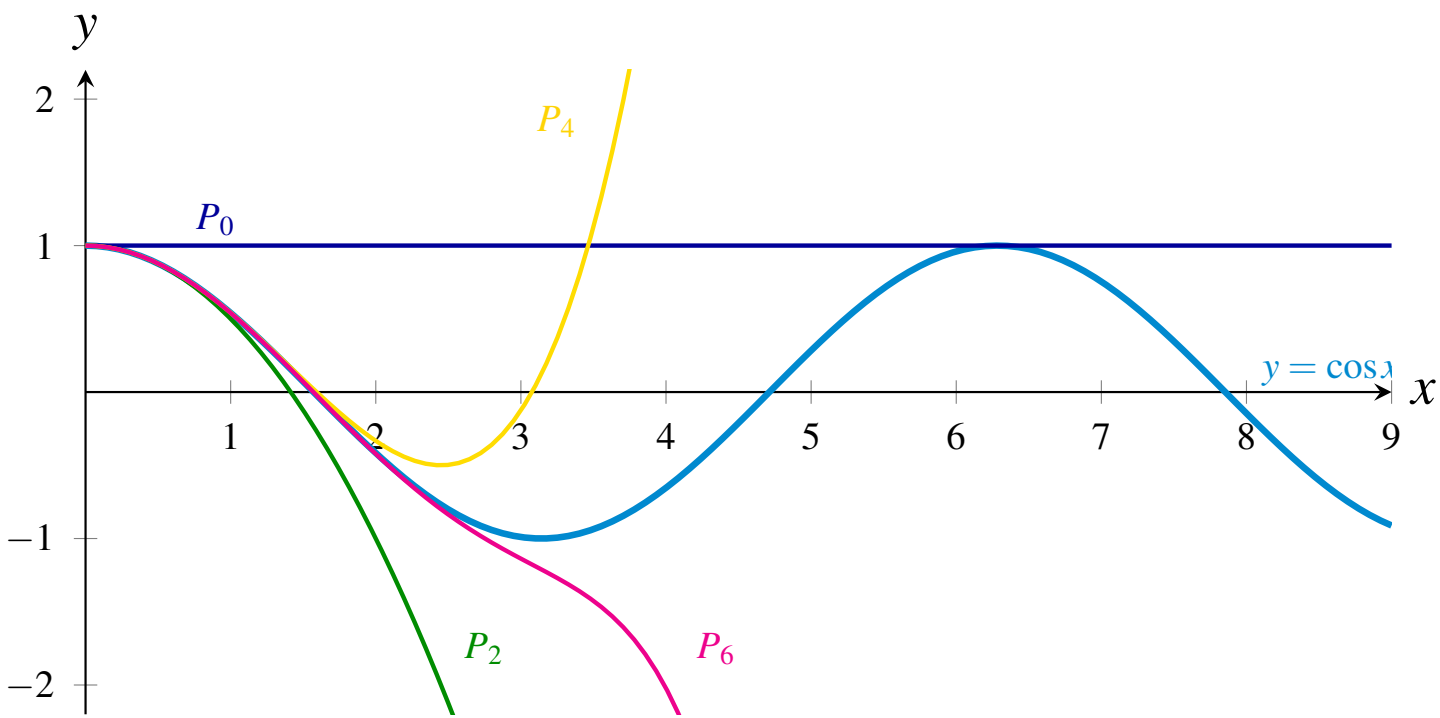
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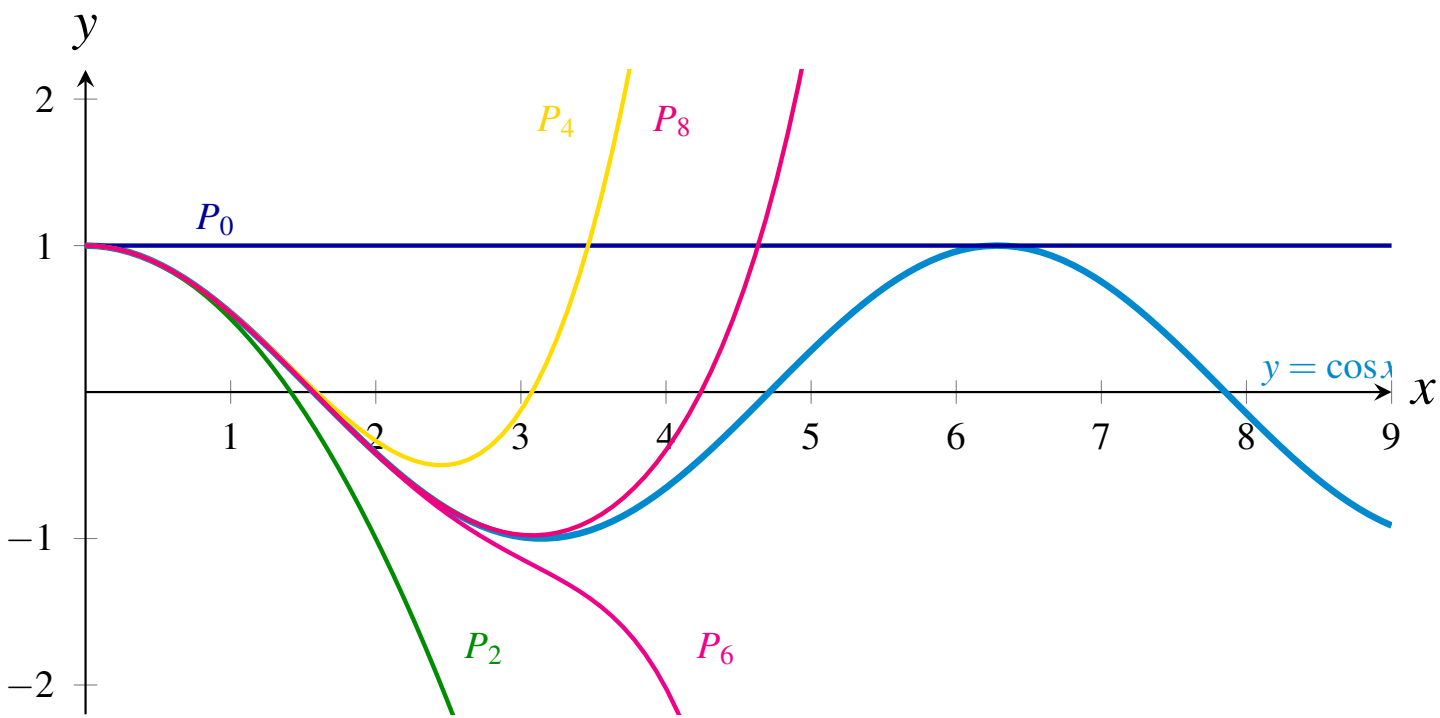
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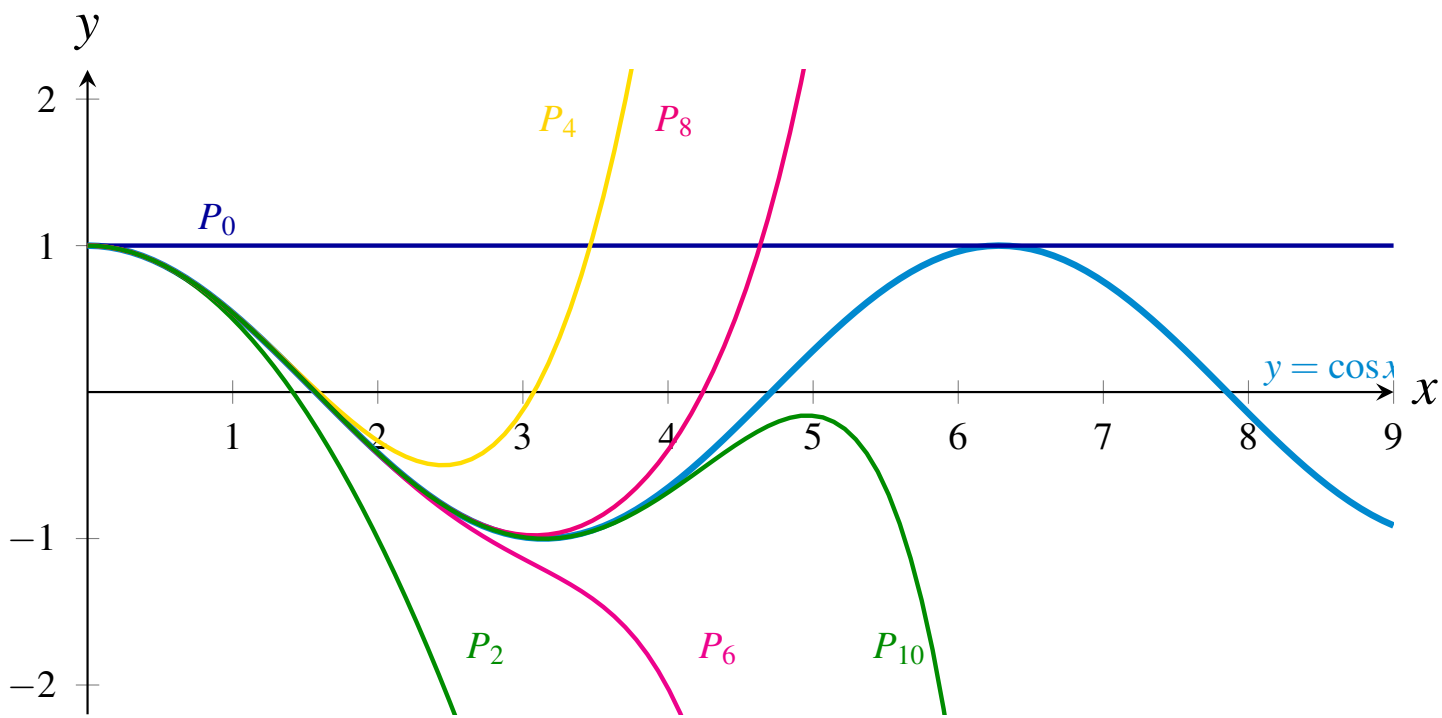
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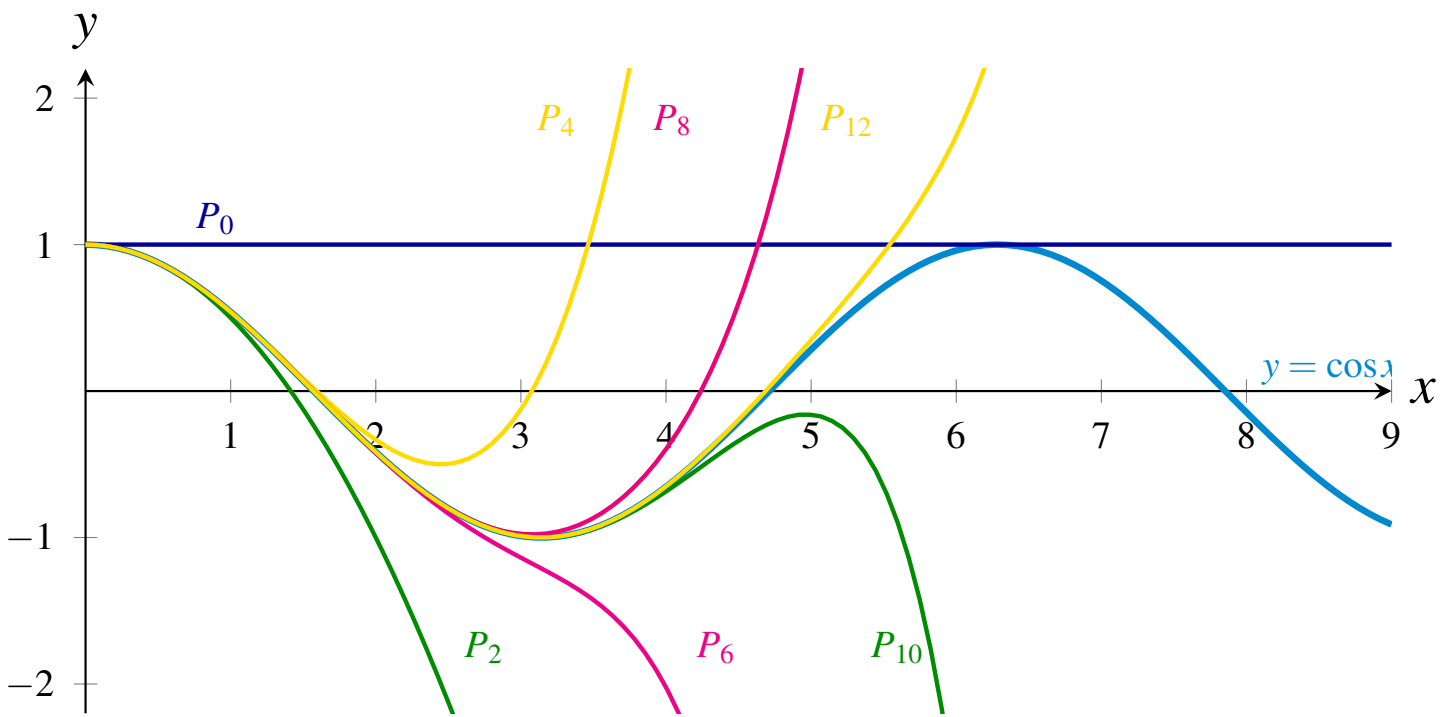
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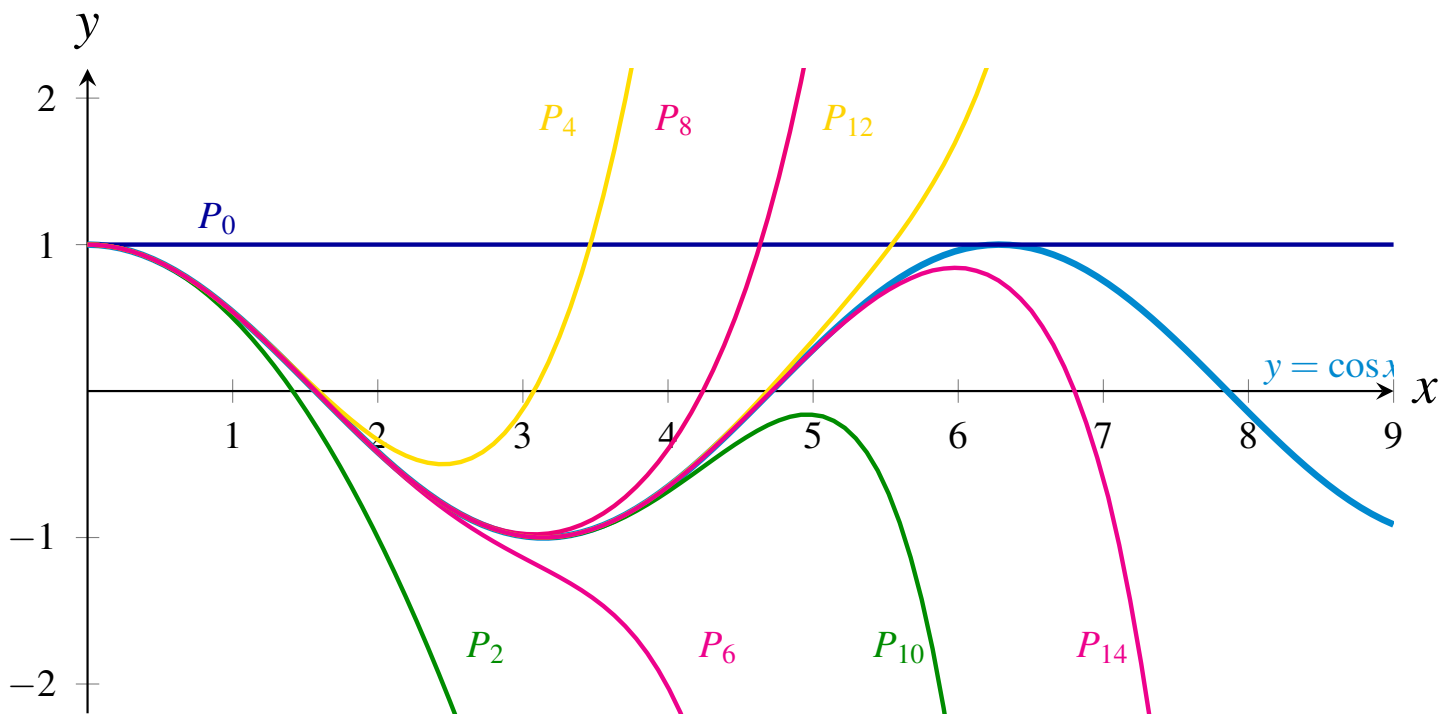
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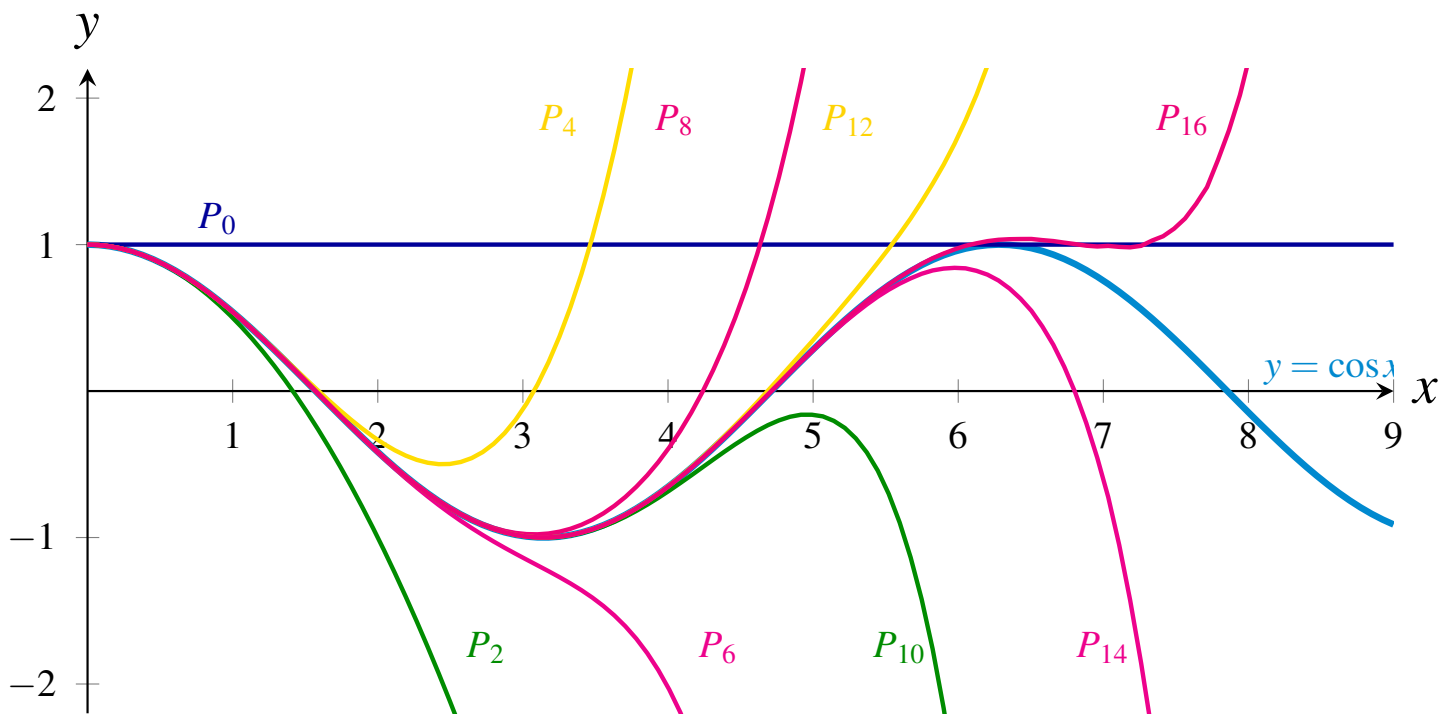
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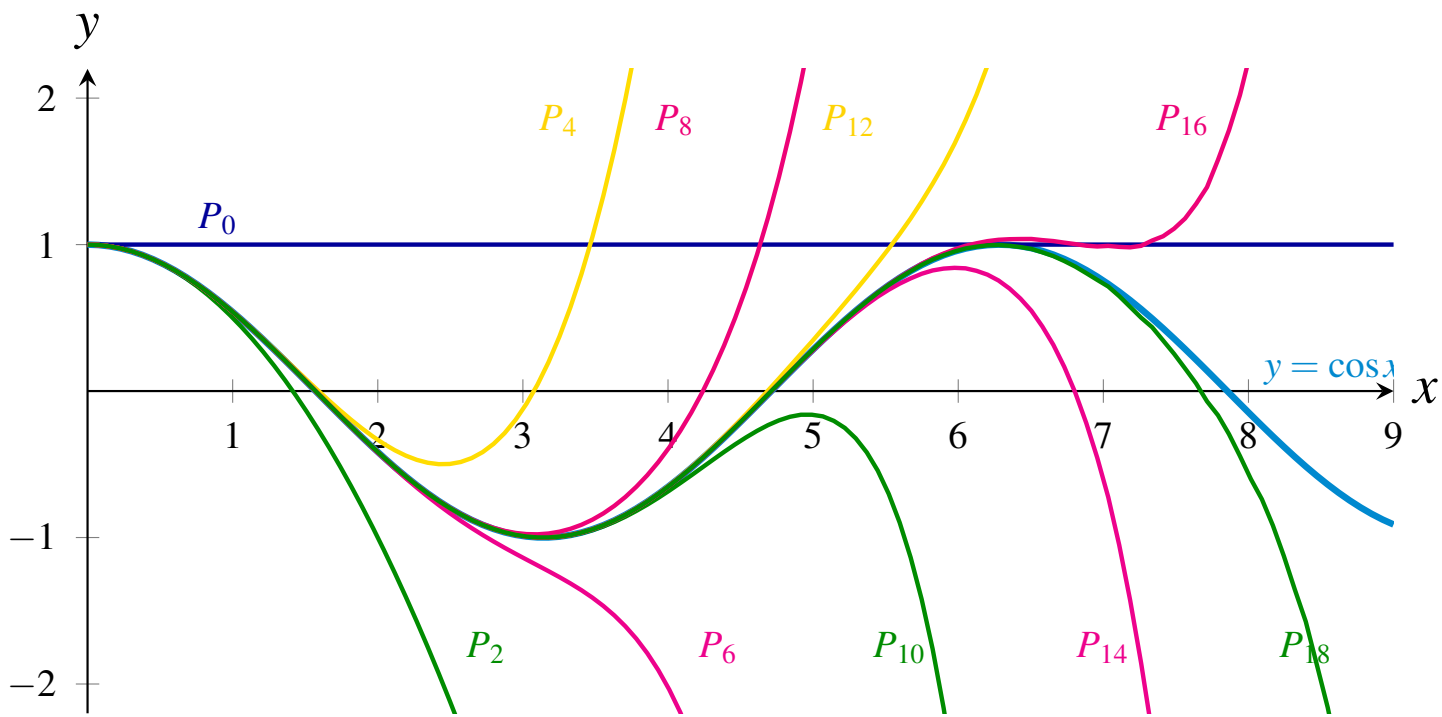
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Example 9: The binomial series

Let $m \in \mathbb{R}$. Consider the function $f(x) = (1+x)^m$. Its derivatives are

$$f^{(n)}(x) = m(m-1)\cdots(m-n+1)\,(1+x)^{m-n}, \qquad n = 1, 2, 3, \ldots$$

One can show that the Maclaurin series of f converges for $-1 < x < 1$, and that the remainder term in the Maclaurin expansion goes to 0 as $n \to \infty$. So, for $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n, \qquad \text{where} \quad \binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

The series so obtained is called the binomial series expansion of $(1+x)^m$.

For example, with $m = 1/2$, we have

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots \qquad \text{for } -1 < x < 1.$$

When $m \in \mathbb{N}$, the binomial series terminates to give a polynomial, and it represents $(1+x)^m$ for each $x \in \mathbb{R}$.

Trigonometric Series

In the power series $\sin x = x - \dfrac{x^3}{3!} + \cdots$, the periodicity of $\sin x$ is not obvious.

Recall: a function $f(x)$ is 2ℓ-periodic (for $\ell > 0$) if $f(x + 2\ell) = f(x)$ for all $x \in \mathbb{R}$. For the time being, we consider 2π-periodic functions.

A trigonometric series is a series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big).$$

If it converges to $f(x)$ for each x, then $f(x)$ is also 2π-periodic.

How to get the coefficients a_n and b_n? Multiply $f(t)$ by $\cos mt$ and integrate term by term (assuming this is permitted):

$$\int_{-\pi}^{\pi} f(t) \cos mt \, dt = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos mt \, dt + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nt \cos mt \, dt$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nt \cos mt \, dt.$$

Coefficients in a Trigonometric Series

For $m, n = 0, 1, 2, 3, \ldots,$

$$\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m > 0 \\ 2\pi & \text{if } n = m = 0 \end{cases} \quad \text{and} \quad \int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0.$$

Thus, we obtain

$$\int_{-\pi}^{\pi} f(t) \cos mt \, dt = \pi a_m \qquad \text{for all } m = 0, 1, 2, 3, \ldots$$

Similarly, multiplying $f(t)$ by $\sin mt$, integrating, and using the fact that

$$\int_{-\pi}^{\pi} \sin nt \sin mt \, dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m > 0 \\ 0 & \text{if } n = m = 0 \end{cases}$$

we obtain

$$\int_{-\pi}^{\pi} f(t) \sin mt \, dt = \pi b_m \qquad \text{for all } m = 1, 2, 3, \ldots$$

Fourier Series

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function which is integrable on $[-\pi, \pi]$.

Let

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \qquad \text{for } n = 0, 1, 2, 3, \ldots,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \qquad \text{for } n = 1, 2, 3, \ldots$$

Then the trigonometric series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the Fourier series of $f(x)$, and a_n, b_n are called the Fourier coefficients of f.

We will mention two types of functions for which the Fourier series converges.

Convergence of Fourier Series

Theorem

Let $f(x)$ be a 2π-periodic function on $\mathbb{R}$. Assume that at least one of the following conditions holds:

(i) $f(x)$ is bounded and piecewise monotonic on $[-\pi, \pi]$.

(ii) $f(x)$ is piecewise continuous on $[-\pi, \pi]$ and has left and right derivatives at each point.

Then the Fourier series of $f(x)$ converges at each $x \in \mathbb{R}$. Further, at any point $c \in \mathbb{R}$:

(1) If $f(x)$ is continuous at c, then the Fourier series sums to $f(c)$.

(2) If $f(x)$ is not continuous at c, then the Fourier series sums to $\frac{1}{2}\big[f(c+) + f(c-)\big]$.

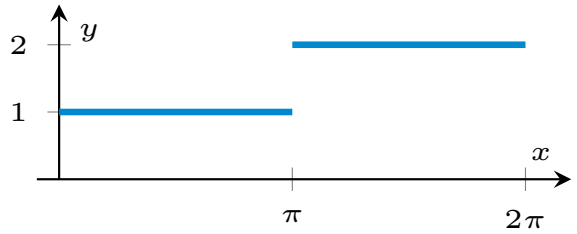
Here $f(c+) = \lim\limits_{x \to c+} f(x)$ and $f(c-) = \lim\limits_{x \to c-} f(x)$.

Fourier series can represent functions which cannot be represented by a Taylor series, or a conventional power series; for example, a step function.

Example 10

Consider the function $f(x)$ given below, extended to $\mathbb{R}$ with period 2π:

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \pi \\ 2 & \text{if } \pi \le x < 2\pi \end{cases}$$



By periodicity, the function $f(x)$ on $[-\pi, \pi)$ is

$$f(x) = \begin{cases} 2 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x < \pi. \end{cases}$$

The coefficients of the Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 2\,dt + \frac{1}{\pi}\int_{0}^{\pi} 1\,dt = 3,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 2\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \cos nt\,dt = 0 \qquad (n \ge 1),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 2\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin nt\,dt = \frac{(-1)^n - 1}{n\pi}.$$

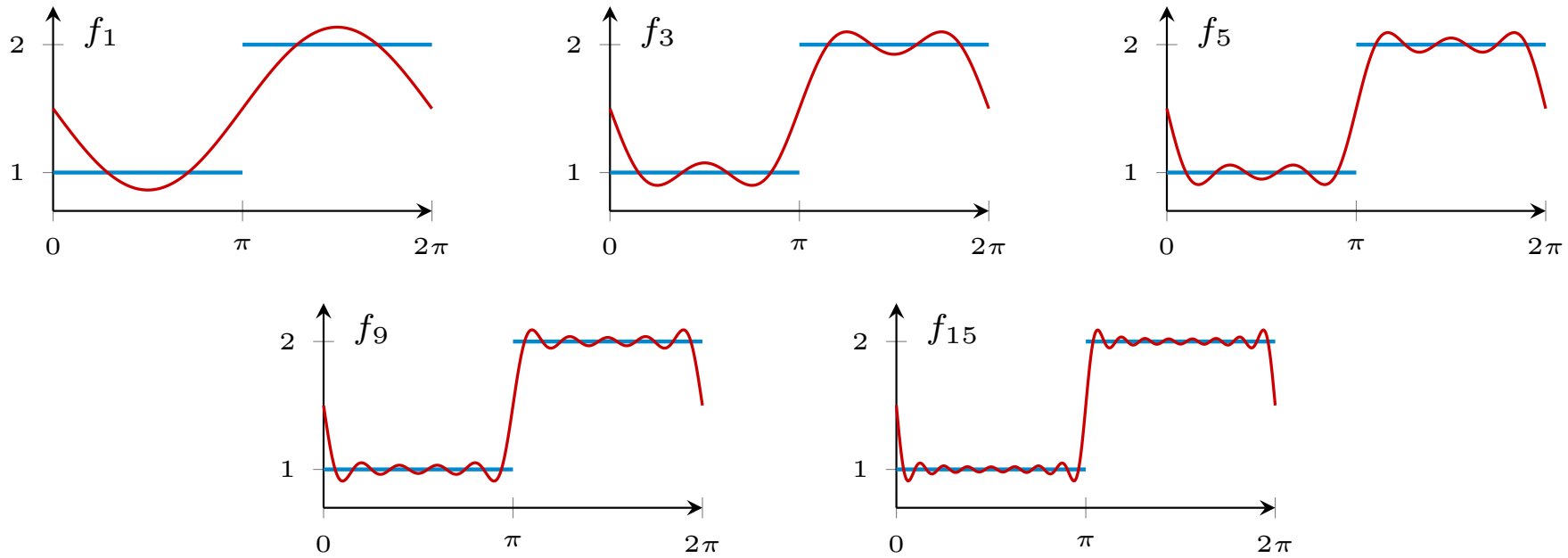
Example 10 Contd.

Here $b_1 = -\dfrac{2}{\pi},\ b_2 = 0,\ b_3 = -\dfrac{2}{3\pi},\ b_4 = 0,\ \ldots$ Therefore, for all $x \in (-\pi, \pi),\ x \neq 0,$

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

At the points of discontinuity $x = 0, \pm\pi$, the Fourier series sums to $\frac{1}{2}(1+2) = \frac{3}{2}$.

Let $f_m(x) = \frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx)$ be the mth partial sum. The approximations $f_1,\ f_3,\ f_5,\ f_9,\ f_{15}$ to f are:



Example 11

Consider the function $f(x) = x^2$ defined on $[0, 2\pi)$ and extended to $\mathbb{R}$ with period 2π. Here $f(2\pi) = f(0) = 0$. Then

$$f(-\pi) = f(-\pi + 2\pi) = f(\pi) = \pi^2, \qquad f(-\pi/2) = f(3\pi/2) = (3\pi/2)^2.$$

Thus the function $f(x)$ on $[-\pi, \pi)$ is given by

$$f(x) = \begin{cases} (x + 2\pi)^2 & \text{if } -\pi \le x < 0 \\ x^2 & \text{if } 0 \le x < \pi. \end{cases}$$

Notice that $f(x)$ is neither odd nor even. Since the integrands are 2π-periodic, we may integrate over $[0, 2\pi]$ instead of $[-\pi, \pi]$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{4}{n^2}, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = -\frac{4\pi}{n} \quad (n \ge 1).$$

Example 11 Contd.

Hence, for $x \neq 2k\pi$ $(k \in \mathbb{Z})$,

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right).$$

Due to the periodic extension, in the interval $\big(2k\pi,\, 2(k+1)\pi\big)$ we have $f(x) = (x - 2k\pi)^2$.

It has discontinuities at the points $x = 0,\ \pm 2\pi,\ \pm 4\pi,\ \ldots$

At such a point $x = 2k\pi$, the series converges to the average of the left and right limits:

$$\frac{1}{2}\Big[\lim_{x \to 2k\pi -} f(x) + \lim_{x \to 2k\pi +} f(x)\Big]$$

$$= \frac{1}{2}\Big[\lim_{x \to 2k\pi -} \big(x - 2(k-1)\pi\big)^2 + \lim_{x \to 2k\pi +} (x - 2k\pi)^2\Big] = \frac{1}{2}\big[4\pi^2 + 0\big] = 2\pi^2.$$

(Check: at $x = 0$ the series is $\frac{4\pi^2}{3} + 4\sum \frac{1}{n^2} = \frac{4\pi^2}{3} + \frac{4\pi^2}{6} = 2\pi^2$.)

Since $f(x)$ is extended by periodicity, whether we take the basic interval as $[-\pi, \pi]$ or as $[0, 2\pi]$ does not matter in the calculation of the coefficients.

Odd and Even Functions

Let $f(x)$ be 2π-periodic and satisfy the hypotheses of the convergence theorem.

Odd function: if $f(-x) = -f(x)$, then $f(t)\cos nt$ is odd and $f(t)\sin nt$ is even, so

$$a_n = 0 \quad (n \geq 0), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin nt\,dt,$$

and the Fourier series is a sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Even function: if $f(-x) = f(x)$, then $b_n = 0$ for all $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt \quad (n \geq 0), \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

and the Fourier series is a cosine series. These equalities hold at every point x where f is continuous.

Example 12

Consider $f(x) = x^2$ for $-\pi \le x \le \pi$. Its periodic extension to $\mathbb{R}$ is not the function x^2. For example, in the interval $[\pi, 3\pi]$, the extension is $f(x) = (x - 2\pi)^2$.

Notice that $f(x)$ is an even function, so $b_n = 0$. The coefficients of the cosine series are:

$$a_0 = \frac{2}{\pi}\int_0^\pi t^2\,dt = \frac{2}{3}\pi^2, \qquad a_n = \frac{2}{\pi}\int_0^\pi t^2\cos nt\,dt = \frac{4}{n^2}(-1)^n \quad (n \ge 1).$$

The extended function is continuous on $\mathbb{R}$. Therefore,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2} \qquad \text{for all } x \in [-\pi, \pi].$$

In particular, taking $x = 0$ and $x = \pi$, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 13

Show that for $0 < x < 2\pi$, $\displaystyle \frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

Let $f(x) = x$ for $0 \le x < 2\pi$. Extend $f(x)$ to $\mathbb{R}$ with period 2π (so that $f(2\pi) = f(0) = 0$).

Here, $f(x)$ is not an odd function:
$f(-\pi/2) = f(3\pi/2) = 3\pi/2 \neq -\pi/2 = -f(\pi/2)$.

The coefficients of the Fourier series for $f(x)$ are as follows:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi, \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = 0 \quad (n \ge 1),$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{1}{\pi}\Big[-\frac{t\cos nt}{n}\Big]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi} \cos nt\,dt = -\frac{2}{n}.$$

Since f is continuous on $(0, 2\pi)$, the convergence theorem gives

$$x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n} \qquad \text{for } 0 < x < 2\pi,$$

which yields the required result.

Half Range Expansions

Suppose a function $f : [0, \pi] \to \mathbb{R}$ is given. To find its Fourier series, we need to extend it to $\mathbb{R}$ so that the extended function is 2π-periodic. Such an extension can be done in many ways.

1. Even Extension:

First, extend $f(x)$ to $[-\pi, \pi]$ by requiring that $f(x)$ is an even function, i.e., $f(-x) = f(x)$ for each $x \in [-\pi, \pi]$.

Next, extend this $f(x)$, which is now defined on $[-\pi, \pi]$, to $\mathbb{R}$ with period 2π.

The Fourier series expansion of this extended $f(x)$ is a cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt, \quad n = 0, 1, 2, \ldots$$

In this case, we say that the Fourier series is a half-range cosine series expansion of $f(x)$.

Half Range Expansions Contd.

2. Odd Extension:

First, extend $f(x)$ from $(0, \pi)$ to $(-\pi, \pi)$ by requiring that $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$. In particular, this forces $f(0) = 0$.

Next, extend this $f(x)$ to $\mathbb{R}$ with period 2π. The extended function satisfies $f(-\pi) = -f(\pi)$ (odd) and $f(-\pi) = f(\pi)$ (periodic), so we must take $f(-\pi) = f(\pi) = 0$.

The Fourier series expansion of this extended $f(x)$ is a sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt, \quad n = 1, 2, 3, \ldots$$

We say that this Fourier series is a half-range sine series expansion of $f(x)$.

The sine series represents the given function at each point of $(0, \pi)$ where it is continuous. At $x = 0$ and $x = \pi$ the sine series sums to 0; so it represents $f(x)$ on all of $[0, \pi]$ only when $f(0) = 0 = f(\pi)$.

Example 14

Find the half-range Fourier series for $f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 \le x \le \pi. \end{cases}$

1. With an even extension, the Fourier coefficients are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} t\cos nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - t)\cos nt\,dt.$$

For $n = 0$ this gives $a_0 = \dfrac{\pi}{2}$, and for $n \ge 1$,

$$a_n = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] = \begin{cases} -\dfrac{8}{\pi n^2} & \text{if } n = 4k+2,\ k \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, writing $n = 2(2k+1)$,

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right) \qquad \text{for } x \in [0, \pi].$$

Example 14 Contd.

2. With an odd extension, the Fourier coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - t)\sin nt\,dt$$

$$= \frac{2}{\pi}\left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_0^{\pi/2} + \frac{2}{\pi}\left[\frac{t-\pi}{n}\cos nt - \frac{1}{n^2}\sin nt\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right) + \frac{2}{\pi}\left(\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right)$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} \frac{4}{\pi n^2}(-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Thus

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right) \qquad \text{for } x \in [0, \pi].$$

Sine Expansion of Cosine in $[0, \pi]$

Find the Fourier sine expansion of $\cos x$ in $[0, \pi]$.

We work with the odd extension of $\cos x$ to $\mathbb{R}$ with period 2π. The Fourier coefficients a_n are 0, and b_n are given by $b_1 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \cos t \sin t \, dt = 0$ and, for $n \geq 2$,

$$b_n = \frac{2}{\pi}\int_0^\pi \cos t \sin nt \, dt = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{4n}{\pi(n^2-1)} & \text{for } n \text{ even.} \end{cases}$$

Therefore, replacing n by $2n$,

$$\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n \sin(2nx)}{4n^2 - 1} \qquad \text{for } 0 < x < \pi.$$

At $x = 0$ and $x = \pi$ the sine series sums to 0, not to $\cos x$. Similarly, one finds the cosine expansion of $\sin x$ in $[0, \pi]$:

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1} \qquad \text{for } x \in [0, \pi].$$

Fourier Series on $(-\ell, \ell)$

In case our function f is defined on $(-\ell, \ell)$ and has period 2ℓ, we consider $g(x) = f(\ell x/\pi)$. Now g is 2π-periodic. It has the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big),$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\Big(\frac{\ell}{\pi} s\Big) \cos ns \, ds, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\Big(\frac{\ell}{\pi} s\Big) \sin ns \, ds.$$

Substituting $t = \dfrac{\ell}{\pi} s$, $\; ds = \dfrac{\pi}{\ell} \, dt$, we have

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(t) \cos\Big(\frac{n\pi t}{\ell}\Big) \, dt, \qquad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(t) \sin\Big(\frac{n\pi t}{\ell}\Big) \, dt.$$

The Fourier series for $f(x)$ is then obtained by replacing x with $\pi x/\ell$ in the above Fourier series. It is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}\Big).$$

Example 15

Construct the Fourier series for $f(x) = |x|$ for $x \in [-\ell, \ell]$, for some $\ell > 0$.

We extend the given function to $f : \mathbb{R} \to \mathbb{R}$ with period 2ℓ. Here, $f(x)$ is not $|x|$ on $\mathbb{R}$; it is $|x|$ on $[-\ell, \ell]$. Due to its period 2ℓ, it is $|x - 2\ell|$ on $[\ell, 3\ell]$, etc.

It is an even function. Thus all b_n are 0. The Fourier coefficients a_n are

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} |s| \, ds = \frac{2}{\ell} \int_0^{\ell} s \, ds = \ell,$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} s \cos\Big(\frac{n\pi s}{\ell}\Big) \, ds = \frac{2\ell}{n^2\pi^2} \big[(-1)^n - 1\big] = \begin{cases} 0 & \text{if } n \text{ even} \\ -\dfrac{4\ell}{n^2\pi^2} & \text{if } n \text{ odd.} \end{cases}$$

Since the extended function is continuous, for $x \in [-\ell, \ell]$,

$$|x| = \frac{\ell}{2} - \frac{4\ell}{\pi^2} \Big[\frac{\cos(\pi x/\ell)}{1^2} + \frac{\cos(3\pi x/\ell)}{3^2} + \cdots + \frac{\cos\big((2n+1)\pi x/\ell\big)}{(2n+1)^2} + \cdots \Big].$$

Half Range Expansions on $(0, \ell)$

Suppose a real valued function $f(x)$ is only defined on an interval $(0, \ell)$.

- Take an even extension of $f(x)$ to $(-\ell, \ell)$ and extend it to $\mathbb{R}$ with period 2ℓ. The Fourier series of this extended function is the *half-range cosine expansion*:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}, \qquad a_n = \frac{2}{\ell} \int_0^{\ell} f(t) \cos\left(\frac{n\pi t}{\ell}\right) dt.$$

- Take an odd extension of $f(x)$ to $(-\ell, \ell)$ and extend it to $\mathbb{R}$ with period 2ℓ. The Fourier series of this extended function is the *half-range sine expansion*:

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}, \qquad b_n = \frac{2}{\ell} \int_0^{\ell} f(t) \sin\left(\frac{n\pi t}{\ell}\right) dt.$$

Equivalently, we may first scale $(-\ell, \ell)$ to $(-\pi, \pi)$ and proceed as earlier; this gives the same coefficients.

Example 16

Find the half-range cosine series for $f(x) = 2x - 1$ for $0 < x < 1$ and show that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$.

Here $\ell = 1$, and the half-range cosine series is $\dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos(n\pi x)$, where

$a_0 = 2\displaystyle\int_0^1 (2x-1)\,dx = 0$ and

$$a_n = 2\int_0^1 (2x-1)\cos(n\pi x)\,dx = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{for } n \text{ even} \\ -\dfrac{8}{n^2\pi^2} & \text{for } n \text{ odd.} \end{cases}$$

The even periodic extension of f is continuous on $\mathbb{R}$. Hence

$$2x - 1 = -\frac{8}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right] \qquad \text{for } 0 < x < 1,$$

and at $x = 0$ the Fourier series sums to $\displaystyle\lim_{x \to 0+} (2x - 1) = -1$. Hence,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Example 17

Find the half-range sine series for $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x < 2. \end{cases}$

Here, $\ell = 2$. Thus, the Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{2}\Big),$$

where

$$b_n = \frac{2}{2}\int_0^2 f(t)\sin\Big(\frac{n\pi t}{2}\Big)\,dt = \int_0^1 \sin\Big(\frac{n\pi t}{2}\Big)\,dt = \frac{2}{n\pi}\Big[1 - \cos\Big(\frac{n\pi}{2}\Big)\Big].$$

So $b_1 = \dfrac{2}{\pi},\ b_2 = \dfrac{2}{\pi},\ b_3 = \dfrac{2}{3\pi},\ b_4 = 0,\ \ldots,$ and the series is

$$\frac{2}{\pi}\Big[\sin\frac{\pi x}{2} + \sin \pi x + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \frac{1}{3}\sin 3\pi x + \cdots\Big].$$

It sums to $f(x)$ for $0 < x < 2$, $x \neq 1$, and to $\dfrac{1}{2}$ at $x = 1$.

Fourier Series on (a, b)

Consider determining Fourier series for functions $f : (a, b) \to \mathbb{R}$. We have the following three approaches to the problem.

1. In the *first approach*, we define the continuous bijection $g : (0, \pi) \to (a, b)$ given by

$$g(y) = a + \frac{b-a}{\pi}\, y \qquad \text{for } y \in (0, \pi).$$

The composition $f \circ g$ is a function from $(0, \pi)$ to $\mathbb{R}$.

Next, we take an even extension of $f \circ g$ with period 2π, and call this extended function h.

We then construct the Fourier series for $h(y) = f(g(y))$.

Finally, we substitute

$$y = g^{-1}(x) = \frac{\pi(x-a)}{b-a} \qquad \text{for } x \in (a, b)$$

in the Fourier series for $h(y)$. This gives the half-range cosine series for $f(x)$ on (a, b).

Second Approach

2. In the *second approach*, we use the same continuous bijection $g : (0, \pi) \to (a, b)$ given by

$$g(y) = a + \frac{b-a}{\pi}\, y \qquad \text{for } y \in (0, \pi).$$

The composition map $f \circ g$ is a function from $(0, \pi)$ to $\mathbb{R}$. Next, we take an odd extension of $f \circ g$ with period 2π, and call this extended function h.

We then construct the Fourier series for $h(y)$. Finally, we substitute

$$y = g^{-1}(x) = \frac{\pi(x-a)}{b-a} \qquad \text{for } x \in (a, b)$$

in the Fourier series for $h(y)$.

This gives the half-range sine series for $f(x)$ on (a, b).

Third Approach

3. In the *third approach*, we define the continuous bijection $g : (-\pi, \pi) \to (a, b)$ given by

$$g(y) = \frac{a+b}{2} + \frac{b-a}{2\pi}\, y \qquad \text{for } y \in (-\pi, \pi).$$

The composition $f \circ g$ is a function from $(-\pi, \pi)$ to $\mathbb{R}$. Next, we extend $f \circ g$ with period 2π, and call this extended function h.

We then construct the Fourier series for $h(y)$.

Finally, in the Fourier series for $h(y)$, we substitute

$$y = g^{-1}(x) = \frac{\pi}{b-a}\,(2x - a - b)$$

to obtain the Fourier series for $f(x)$ on (a, b).

Third Approach Contd.

If a function f has domain $(0, \pi)$, then we can scale it to length 2π by using the third approach.

That is, we take the function $g : (-\pi, \pi) \to (0, \pi)$ as

$$g(y) = \frac{\pi + y}{2} \qquad \text{for } y \in (-\pi, \pi).$$

Then, we find the Fourier series for $h(y) = f(g(y))$.

Finally, in the Fourier series for $h(y)$, we substitute $y = g^{-1}(x) = 2x - \pi$ to obtain the Fourier series for $f(x)$.

In general, such a Fourier series need neither be a sine series nor a cosine series. It may involve both sine and cosine terms. Sometimes, this series is called a full range Fourier series.

Example 18

Consider $f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 < x \le \pi. \end{cases}$

Here, $f(x)$ is also given at the end-points, and we use the same formula for g there. That is, we define $g : [-\pi, \pi] \to [0, \pi]$ by

$$x = g(y) = \frac{\pi + y}{2}, \qquad h(y) = f\Big(\frac{\pi + y}{2}\Big) = \begin{cases} (\pi + y)/2 & \text{if } -\pi \le y \le 0 \\ (\pi - y)/2 & \text{if } 0 < y \le \pi. \end{cases}$$

Notice that $h(y)$ happens to be an even function with $h(-\pi) = h(\pi) = 0$. Then the Fourier coefficients are given by $b_n = 0$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} \frac{\pi + t}{2}\, dt + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi - t}{2}\, dt = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \frac{\pi + t}{2}\cos nt\, dt + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi - t}{2}\cos nt\, dt = \begin{cases} \dfrac{2}{\pi n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Example 18 Contd.

The Fourier series for $h(y)$ is given by

$$\frac{\pi}{4} + \sum_{n \text{ odd}} \frac{2}{\pi n^2} \cos ny.$$

Using $y = g^{-1}(x) = 2x - \pi$, we have the Fourier series for $f(x)$. Since the periodic extension of h is continuous on $\mathbb{R}$, the series represents $f(x)$ also at $x = 0$ and $x = \pi$. Therefore,

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\big((2n+1)(2x-\pi)\big) \qquad \text{for } x \in [0, \pi].$$

Remark. Since $\cos\big((2n+1)(2x-\pi)\big) = -\cos\big(2(2n+1)x\big)$, this is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\Big(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\Big),$$

which is exactly the half-range cosine series obtained in Example 14.