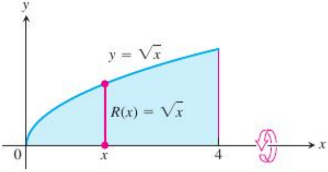
MA 1101

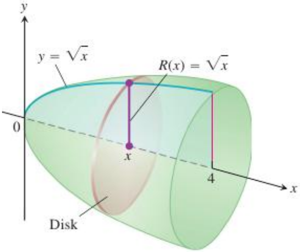
Functions of Several Variables

Multiple Integrals

Solids of revolution

The solid obtained by rotating a plane region about a straight line in the same plane is called a solid of revolution. The line is called the axis of revolution





Suppose the region is bounded above by the curve $y = f(x)$ and below by the x-axis, where $a \leq x \leq b$.
To find the volume of the solid, divide the interval $[a, b]$ into n equal parts:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Volume of revolution

On the ith subinterval, the slice of the solid is a portion of a cylinder whose cross section with a plane vertical to its axis is a circle.
So, an approximation of the slice is $\pi[f(x_i^*)]^2(x_i - x_{i-1})$ for a point $x_i^* \in [x_{i-1}, x_i]$.
Then the volume of the solid of revolution is approximated by the sum

$$\sum_{i=1}^{n} \pi[f(x_i^*)]^2(x_i - x_{i-1}).$$

The volume of the solid of revolution is the limit of this sum where $n \longrightarrow \infty$.
Also, the cross sectional area for $x \in [a, b]$ is $A(x) = \pi(f(x))^2$.
Assuming that $A(x)$ is a continuous function of x, the volume is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx.$$

If the axis of revolution is a straight line other than the x-axis, similar formulas can be obtained for the volume.

Examples

Example 1: The region between the curve $y = \sqrt{x},\ 0 \leq x \leq 4$ and the x-axis is revolved around x-axis. Find the volume of the solid of revolution.

As shown in the above figure, the required volume is

$$V = \int_0^4 \pi(\sqrt{x})^2\,dx = \int_0^4 \pi\,x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi.$$

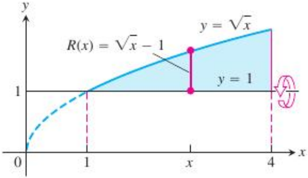
Example 2: Find the volume of the sphere $x^2 + y^2 + z^2 = a^2,\ a > 0$.

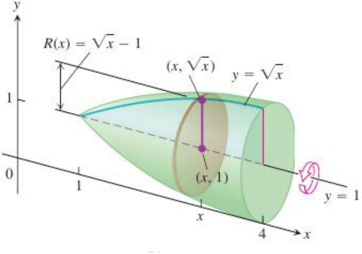
Think of the sphere as the solid of revolution of the region bounded by the upper semi-circle $x^2 + y^2 = a^2,\ y \geq 0$. Here, $-a \leq x \leq a$. The curve is thus $y = \sqrt{a^2 - x^2}$. Then the volume of the sphere is

$$V = \int_{-a}^{a} \pi(\sqrt{a^2 - x^2})^2\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

Example 3

Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1,\ x = 4$ about the line $y = 1$.



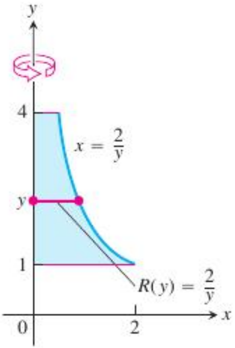


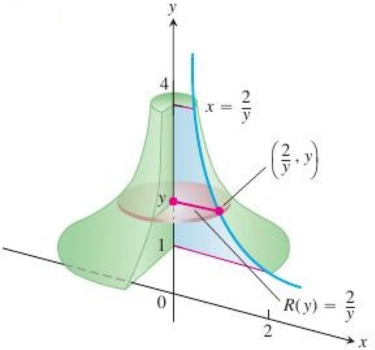
The required volume is

$$V = \int_1^4 \pi[R(x)]^2\,dx = \int_1^4 \pi(\sqrt{x}-1)^2\,dx = \int_1^4 \pi(x-2\sqrt{x}+1)\,dx = \frac{7\pi}{6}.$$

Example 4

Find the volume of the solid generated by revolving the region between the y-axis and the curve $xy = 2$, $1 \le y \le 4$, about the y-axis.



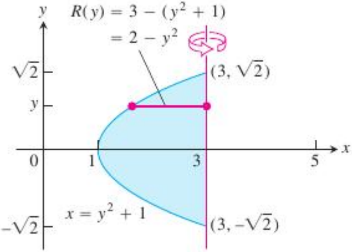


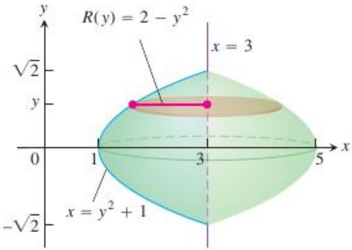
The volume is

$$V = \int_1^4 \pi [R(y)]^2 \, dy = \pi \int_1^4 \frac{4}{y^2} \, dy = 3\pi.$$

Example 5

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.





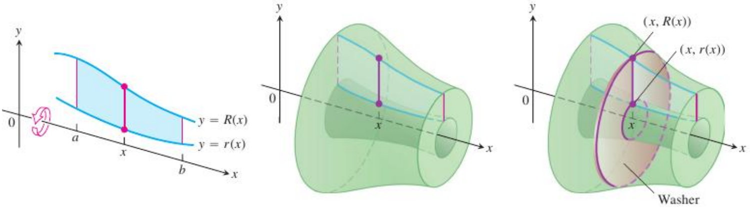
Notice that the cross sections are perpendicular to the axis of revolution: $x = 3$.

The volume is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 dy = \frac{64\pi\sqrt{2}}{15}.$$

With holes

If the region which revolves does not border the axis of revolution, then there are holes in the solid.

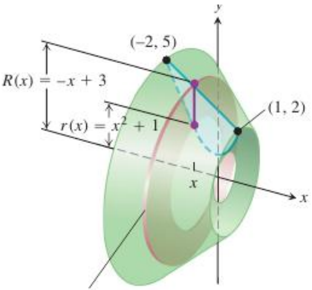


In this case, we subtract the volume of the hole to obtain the volume of the solid of revolution.
The volume of the the solid of revolution is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left[(R(x))^2 - (r(x))^2\right] dx.$$

Example 6

The region bounded by the curve $y = x^2 + 1$ and the line $x + y = 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

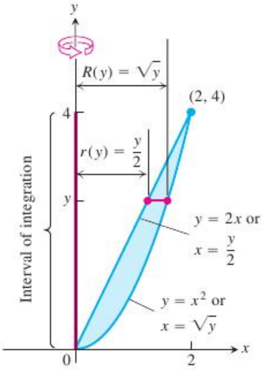


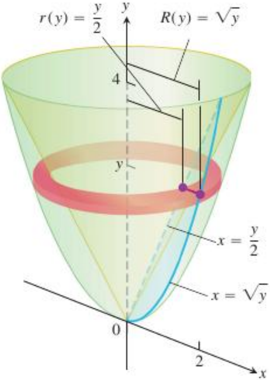
The outer radius is $R(x) = -x + 3$.
The inner radius is $r(x) = x^2 + 1$.
The limits of integration are obtained by finding the points of intersection of the given curves:

$$x^2 + 1 = -x + 3 \Rightarrow x = -2,\ 1.$$

Example 7

Find the volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = 2x$, about the y-axis.





The given curves intersect at $y = 0$ and $y = 4$. The required volume is

$$V = \int_0^4 \pi[(R(y))^2 - (r(y))^2]\,dy = \int_0^4 \pi[(\sqrt{y})^2 - (y/2)^2]\,dy = \frac{8\pi}{3}.$$

Example 8

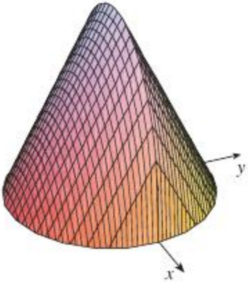
Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2+4)$ and the lines $x = 0,\ x = 2,\ y = 0.$

The volume is $V = \displaystyle\int_0^2 \pi \frac{16}{(x^2+4)^2}\,dx.$

Substitute $x = 2\tan t$. $dx = 2\sec^2 t\,dt$, $(x^2+4)^2 = 16\sec^4 t$ for $0 \le t \le \pi/4$. So,

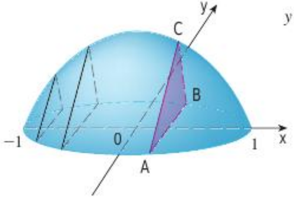
$$V = \int_0^{\pi/4} 16\pi \frac{2\sec^2 t}{16\sec^4 t}\,dt = \int_0^{\pi/4} 2\pi\cos^2 t\,dt = \pi\left(\frac{\pi}{4} + \frac{1}{2}\right).$$

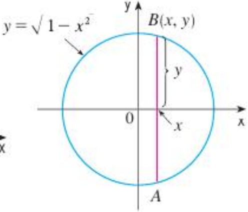
Example 9

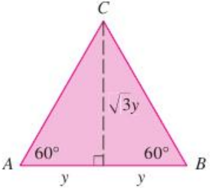


In the figure is shown a solid with a circular base of radius 1. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

Take the base of the solid as the disk $x^2 + y^2 \le 1$. The solid, its base, and a typical triangle at a distance x from the origin are:







Example 9 Contd.

The point B lies on the circle $y = \sqrt{1 - x^2}$.

The length of AB is $2\sqrt{1 - x^2}$.

The triangle is equilateral with height $\sqrt{3}\sqrt{1 - x^2}$.

The cross sectional area is

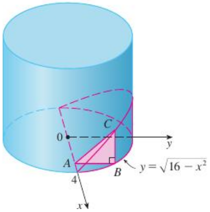
$$A(x) = \frac{1}{2}\, 2\sqrt{1 - x^2}\,\sqrt{3}\,\sqrt{1 - x^2} = \sqrt{3}\,(1 - x^2).$$

Thus, the volume of the solid is

$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\, dx = \frac{4}{\sqrt{3}}.$$

Example 10

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.



Place the x-axis along the diameter where the planes meet.
The base of the solid is the semicircle:
$y = \sqrt{16 - x^2}, \ -4 \le x \le 4.$
A cross-section perpendicular to x-axis at a distance x from the origin is the triangle ABC. The base and height of ABC are:

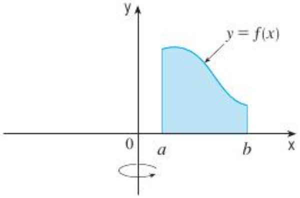
Base: $y = \sqrt{16 - x^2}$; Height: $|BC| = y \tan 30^\circ = \sqrt{16 - x^2}/\sqrt{3}$.

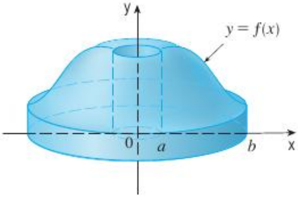
Cross sectional area $A(x) = \frac{1}{2}\sqrt{16 - x^2}\,\frac{\sqrt{16-x^2}}{\sqrt{3}} = \frac{16-x^2}{2\sqrt{3}}$.

Volume of the wedge $V = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16-x^2}{2\sqrt{3}}\,dx = \frac{128}{3\sqrt{3}}$.

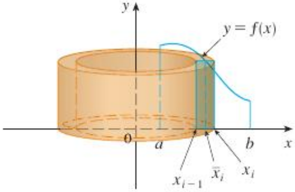
Cylindrical Shell Method

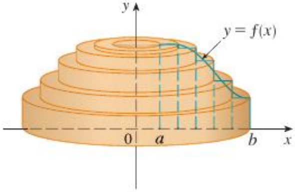
Let S be a solid obtained by revolving about the y-axis the region bounded by $y = f(x)$ and the lines $y = 0,\ x = a,\ x = b$, where $f(x) > 0, 0 < a < b$.



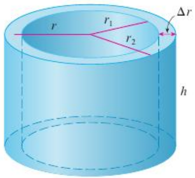


Approximate the volume of the solid by slicing into cylindrical shells. When the width of the cylindrical shells approach zero, as in the Riemann sums, we would obtain the volume as a limit.





Using the Idea



The volume of each cylindrical shell is

$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = 2\pi r h \Delta r$$

where $r = (r_1 + r_2)/2$ and $\Delta r = r_2 - r_1$.

So, for the volume of the solid, divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and take $\bar{x}_i$ as the mid-point of the subinterval.
If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is revolved about the y-axis, the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$ and thickness Δx.
Thus the volume of the shell is $V_i = (2\pi\bar{x}_i) f(\bar{x}_i)\,\Delta x$. Therefore, the approximation to the volume V is given by

$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} (2\pi\bar{x}_i) f(\bar{x}_i)\,\Delta x.$$

Axis of revolution

By taking n approach ∞, we get the required volume as

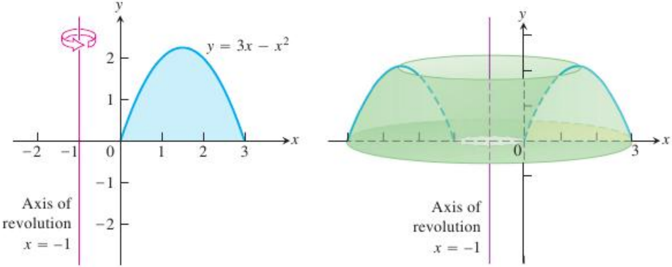
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} V_i = \int_a^b 2\pi x f(x)\, dx.$$

Instead of taking the axis of revolution as the y-axis, we may take the vertical line $x = \ell$. In that case, the shell radius will be $x - \ell$ instead of $x = x - 0$. Therefore, we have the following:

> The volume V of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x)$ with $f(x) \geq 0$ and $\ell \leq a \leq x \leq b$, about a vertical line $x = \ell$ is given by
>
> $$V = \int_a^b 2\pi(x-\ell)f(x)\, dx = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx.$$

Example 11

Find the volume of the solid generated by revolving the region bounded by the parabola $y = 3x - x^2$ and the x-axis, about the line $x = -1$.



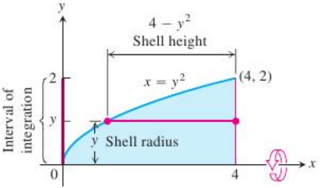
The parabola intersects the x-axis at $x = 0$ and $x = 3$.

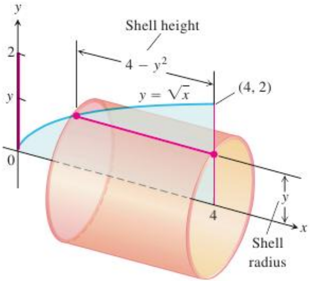
The required volume is

$$V = \int_0^3 2\pi(x+1)(3x - x^2)\,dx = 2\pi \int_0^3 (2x^2 + 3x - x^3)\,dx = \frac{45\pi}{2}.$$

Example 12

The region bounded by the x-axis, the line $x = 4$, and the curve $y = \sqrt{x}$ is revolved about the x-axis. Find the volume of the solid of revolution.



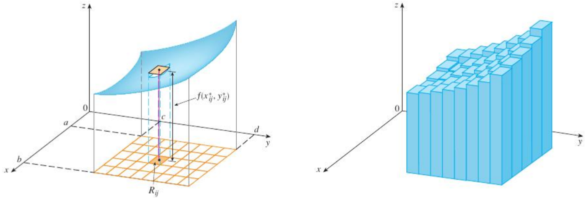


Here, the shell thickness variable is y.
The limits of integration are $y = 0$ and $y = 2$.
The shell radius is y and the shell height is $4 - y^2$.
Thus the volume of the solid of revolution is

$$V = \int_0^2 2\pi\, y(4 - y^2)\, dy = 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi.$$

Volume of general solids

Let $f(x, y)$ be defined on the rectangle $R: \quad a \le x \le b,\ c \le y \le d.$
For simplicity, take $f(x, y) \ge 0$. The graph of f is the surface
$z = f(x, y)$. We approximate the volume of the solid
$S: \quad \{(x, y, z) : (x, y) \in R,\ 0 \le z \le f(x, y)\}$ by partitioning R and then
adding up the volumes of the solid rods:



So, consider a partition of R as $P: \quad R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for
$1 \le i \le m$, $1 \le j \le n$, $a = x_0$, $b = x_m$, $c = y_0$, $d = y_n$.
Denote by $\|P\| = \max\{\text{area of } R_{ij}\}$, the norm of P.
Denote by $A(R_{ij})$ the area of such a rectangle R_{ij}.
Choose sample points $(x_i^*, y_j^*) \in R_{ij}$. An approximation to the volume
of S is the Riemann sum $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) A(R_{ij})$.

Continuous Functions on a Rectangle

If limit of S_{mn} exists as $\|P\| \to 0$, then this limit is called the double integral of $f(x, y)$. It is denoted by $\iint_R f(x, y)dA$. Whenever the integral exists, it is also enough to consider uniform partitions, that is, $x_i - x_{i-1} = (b - a)/m = \Delta x$ and $y_j - y_{j-1} = (d - c)/n = \Delta y$. In this case, we write $A(R_{ij}) = \Delta A = \Delta x \Delta y$. Then

$$\iint_R f(x, y)dA = \lim_{\|P\| \to 0} S_{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x \Delta y.$$

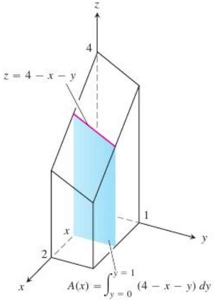
Since $f(x, y) \geq 0$, the value of this integral is the volume of the solid S bounded by the rectangle R and the surface $z = f(x, y)$.
When the integral of $f(x, y)$ exists, we say that f is Riemann integrable or just integrable.

Riemann sum is well defined even if f is not a positive function.

Theorem 1: Each continuous function defined on a closed bounded rectangle is integrable.

Iterated Integrals

Find the volume V of the solid raised over the rectangle $R: \quad [0,2] \times [0,1]$ and bounded above by the plane $z = 4 - x - y$.



Suppose $A(x)$ is the cross sectional are at x. Then $V = \int_0^2 A(x)dx$.
Now, $A(x) = \int_0^1 (4-x-y)dy$. Thus, $V = \int_0^2 \int_0^1 (4-x-y)dydx$.

Therefore, $\displaystyle\iint_R (4-x-y)dA = \int_0^2 \int_0^1 (4-x-y)dydx.$

The expression on the left is a double integral and on the right is an iterated integral.

Fubuni's

Theorem 2: Let R be the rectangle $[a, b] \times [c, d]$. Let $f : R \to \mathbb{R}$ be a continuous function. Then

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

Example 13: Evaluate $\iint_R (1 - 6x^2y)dA$, where $R = [0, 2] \times [-1, 1]$.

$$\iint_R (1 - 6x^2y)dA = \int_{-1}^1 \int_0^2 (1 - 6x^2y)dxdy = \int_{-1}^1 (2 - 16y)dy = 4.$$

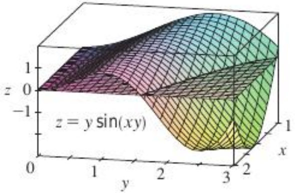
Also, reversing the order of integration, we have

$$\iint_R (1 - 6x^2y)dA = \int_0^2 \int_{-1}^1 (1 - 6x^2y)dydx = \int_0^2 2dx = 4.$$

Example 14

Evaluate $\iint_R y\sin(xy)dA$, where $R = [1,2] \times [0,\pi]$.

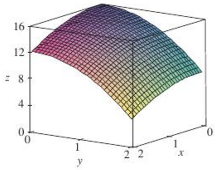
$$\iint_R y\sin(xy)dA = \int_0^\pi \int_1^2 y\sin(xy)\,dxdy$$
$$= \int_0^\pi (-\cos 2y + \cos y)dy = 0.$$



Notice that the volume of the solid above R and below the surface $z = y\sin(xy)$ is the same as the volume below R and above the surface. Therefore, the net volume is zero.

Example 15

Find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, planes $x = 2$ and $y = 2$, and the three co-ordinate planes.

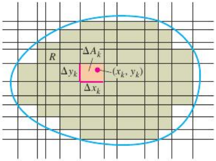


Let R be the rectangle $[0, 2] \times [0, 2]$. The solid is above R and below the surface defined by $z = f(x, y) = 16 - x^2 - 2y^2$, where f is defined on R.

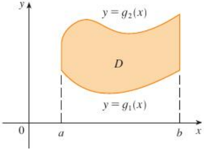
$$V = \iint_R (16 - x^2 - 2y^2)dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)dxdy = 48.$$

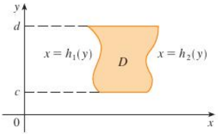
Non-rectangular Region

Given a function defined over a bounded non-rectangular region, we follow the same path: partition the region into smaller rectangles, form the Riemann sum, take its limit as the norm of the partition goes to zero.



The double integral of f over such a bounded region R can also be evaluated using iterated integrals. Look at R bounded by two continuous functions $g_1(x)$ and $g_2(x)$.
Or, as one bounded by two continuous functions $h_1(y)$ and $h_2(y)$.





Fubuni Again

Theorem 3: Let $f(x, y)$ be a continuous on a region R.

1. If R is given by $a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)$, where $g_1, g_2 : [a, b] \to \mathbb{R}$ are continuous, then

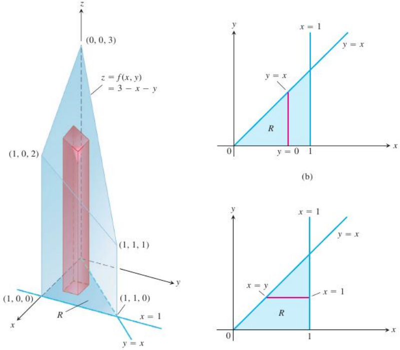
$$\iint_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx.$$

2. If R is given by $c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)$, where $h_1, h_2 : [c, d] \to \mathbb{R}$ are continuous, then

$$\iint_R f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy.$$

Example 16

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the lines $y = 0$, $x = 1$ and $y = x$, and whose top lies in the plane $z = 3 - x - y$.



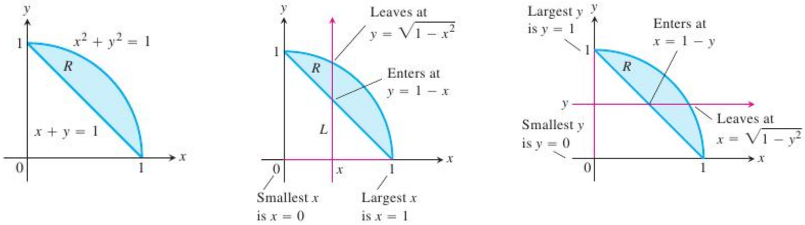
$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 (3x - 3x^2/2)\,dx = 1.$$

Also, $V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 (5/2 - 4y + 3y^2/2)\,dy = 1.$

Get the Limits Right

Suppose R is the region bounded by the line $x + y = 1$ and the portion of the circle $x^2 + y^2 = 1$ in the first quadrant.

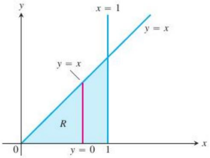
Sketch it, find the limits, and then write the appropriate integrals:

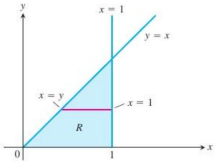


$$\iint_R f(x, y)\,dA = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x, y)\,dy\,dx = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\,dx\,dy.$$

First x or First y?

Example 17: Evaluate $\iint_R \frac{\sin x}{x} dA$, where R is the triangle in the xy-plane bounded by the lines $y = 0, x = 1, y = x$.





$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_y^1 \frac{\sin x}{x} dx dy.$$

We are stuck. No way to proceed further. On the other hand,

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_0^x \frac{\sin x}{x} dy dx = \int_0^1 \sin x dx = -\cos(1) + 1.$$

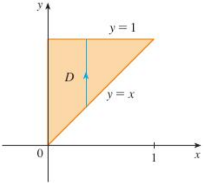
For evaluating a double integral as an iterated integral, choose some order: first x, next y. If it does not work, choose the reverse order.

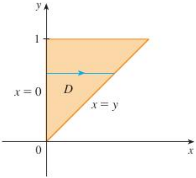
Example 18

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)dydx$.

$$\int_0^1 \int_x^1 \sin(y^2)dydx = \iint_D \sin(y^2)dA,$$

where $D: \quad 0 \le x \le 1, \; x \le y \le 1.$





Then

$\int_0^1 \int_x^1 \sin(y^2)dydx = \iint_D \sin(y^2)dA = \int_0^1 \int_0^y \sin(y^2)dxdy$

$= \int_0^1 y\sin(y^2)dy = \frac{1}{2}(1 - \cos(1)).$

Properties of Double Integrals

Let D and R be non-overlapping regions in the plane. Let $f(x, y)$ and $g(x, y)$ be continuous on D, R. Let c be a constant.

1. Constant Multiple: $\iint_D cf(x, y)dA = c \iint_D f(x, y)dA$.

2. Sum-Difference:

$\iint_D [f(x, y) \pm g(x, y)]dA = \iint_D f(x, y)dA \pm \iint_D g(x, y)dA$.

3. Additivity: $\iint_{D \cup R} f(x, y)dA = \iint_D f(x, y)dA + \iint_R f(x, y)dA$.

4. Domination:

If $f(x, y) \leq g(x, y)$ in D, then $\iint_D f(x, y)dA \leq \iint_D g(x, y)dA$.

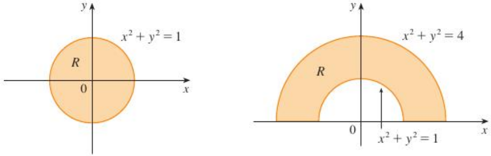
5. Area: $\iint_R 1\, dA = \Delta(R) =$ Area of R.

6. Boundedness:

If $m \leq f(x, y) \leq M$ in R, then $m\Delta(R) \leq \iint_R f(x, y)dA \leq M\Delta(R)$.

Polar Rectangles

Suppose R is one of the following regions in the plane:

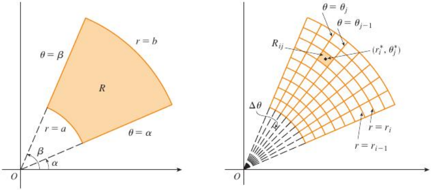


(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

It is easy to describe as polar rectangles. A polar rectangle is

$$R = \{(r, \theta) : a \leq r \leq b,\ \alpha \leq \theta \leq \beta, \beta - \alpha \leq 2\pi\}.$$

for some $a, b, \alpha, \beta \in \mathbb{R}$. We can divide a polar rectangle into polar subrectangles as in the following:



$$R_{ij} = \{(r, \theta) : r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}.$$

Riemann Sum in Polar

Suppose f is a real valued function defined on a polar rectangle R. Let P be a partition of R into smaller polar rectangles R_{ij}. The area of R_{ij} is $\Delta(R_{ij}) = \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j - \theta_{j-1})$.

Take a uniform grid dividing r into m equal parts and θ into n equal parts. Write $r_i - r_{i-1} = \Delta r$ and $\theta_j - \theta_{j-1} = \Delta\theta$. Also write the mid-point of r_{i-1} and r_i as $r_i^* = \frac{1}{2}(r_i + r_{i-1})$, similarly, $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$. Then the Riemann sum for $f(r, \theta)$ can be written as

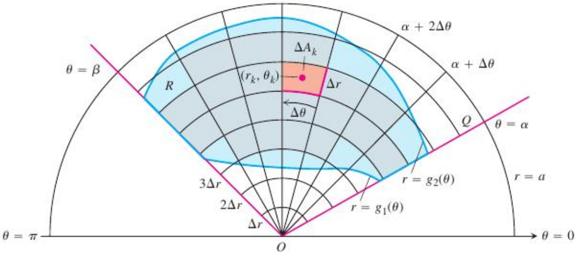
$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^*, \theta_j^*)\Delta(R_{ij}) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^*, \theta_j^*) r_i^* \Delta r \Delta\theta.$$

Therefore, if $f(r, \theta)$ is continuous on the polar rectangle R, then

$$\iint_R f(r, \theta)dA = \iint_R f(r, \theta) r dr d\theta, \quad \iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Fubini

Let f be a continuous function defined over a region bounded by the rays $\theta = \alpha,\ \theta = \beta$ and the continuous curves $r = g_1(\theta),\ r = g_2(\theta)$.



Then

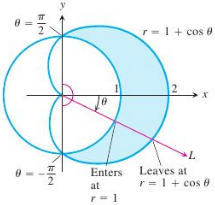
$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r\, dr\, d\theta.$$

Caution: Do not forget the r on the right hand side.

Finding Limits of Integration

Example 19: Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $x^2 + y^2 = 1$.

Better write the circle as $r = 1$. Now, R is the region:

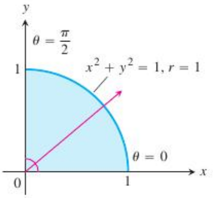


$$\iint_R f(r,\theta)dA = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} f(r,\theta)r\,dr\,d\theta.$$

Example 20

Evaluate $I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$.

The limits of integration say that the region is the quarter of the unit disk in the first quadrant:



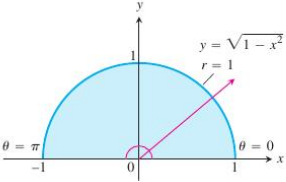
The region in polar co-ordinates is $R: \quad 0 \le r \le 1, \ 0 \le \theta \le \pi/2$. Changing to polar co-ordinates, we have $x = r\cos\theta, y = r\sin\theta$ and then

$$I = \int_0^{\pi/2} \int_0^1 r^2 \, r \, d\theta \, dr = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{\pi}{8}.$$

Example 21

Evaluate $I = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\, dydx.$

The region is the upper semi-unit-disk:



Its polar description is

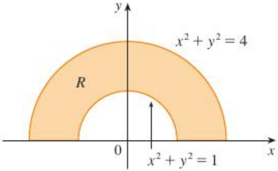
$$R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi\}.$$

Then $I = \iint_R e^{x^2+y^2} dA$. Using integration in polar form,

$$I = \int_0^{\pi} \int_0^{1} e^{r^2} r\, dr\, d\theta = \int_0^{\pi} \left[\frac{1}{2} e^{r^2}\right]_0^1 d\theta = \int_0^{\pi} (e-1)/2 d\theta = \frac{\pi}{2}(e-1).$$

Example 22

Evaluate $\iint_R (3x + 4y^2)dA$, where R is the region in the upper half plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

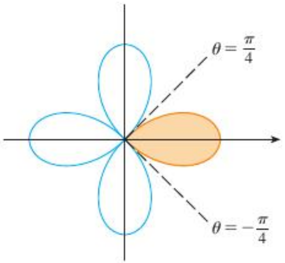


$R = \{(r, \theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$. Therefore,

$$\iint_R (3x + 4y^2)dA = \int_0^{\pi} \int_1^{2} (3r\cos\theta + 4r^2 \sin^2\theta) r\, dr\, d\theta = \frac{15\pi}{2}.$$

Example 23

Find the area enclosed by one of the four leaves of the curve $r = |\cos(2\theta)|$.

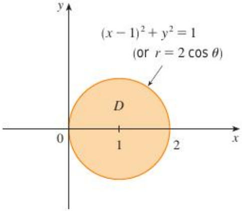


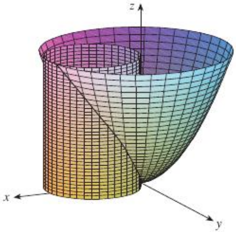
The region is $R = \{(r, \theta) : -\pi/4 \le \theta \le \pi/4,\ 0 \le r \le \cos(2\theta)\}$. Then the required area is

$$\iint_R dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} r\, dr\, d\theta = \pi/8.$$

Example 24

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.





The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$, or $(x-1)^2 + y^2 = 1$. In polar co-ordinates, the boundary of D is $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$.

The disk $D = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}$.
Then the required volume V is given by

$$V = \iint_D (x^2 + y^2)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \frac{3\pi}{2}.$$

Triple Integral

Let $f(x, y, z)$ be a real valued function defined on a bounded region D is 3d space. As earlier we divide the region into smaller cubes enclosed by planes parallel to the co-ordinate planes. The set of these smaller cubes is called a partition P. The norm of the partition is the maximum volume enclosed by any smaller cube.
Then form the Riemann sum S and take its limit as the cubes become smaller and smaller. If the limit exists, we say that the limit is the triple integral of the function over the region D.

$$\iiint_D f(x, y, z)dV = \lim_{\|P\|\to 0} \sum f(x_i^*, y_j^*, z_k^*)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}),$$

where (x_i^*, y_j^*, z_k^*) is a point in the (i, j, k)-th cube in the partition.
As earlier, Fubuni's theorem says that for continuous functions, if the region D can be written as

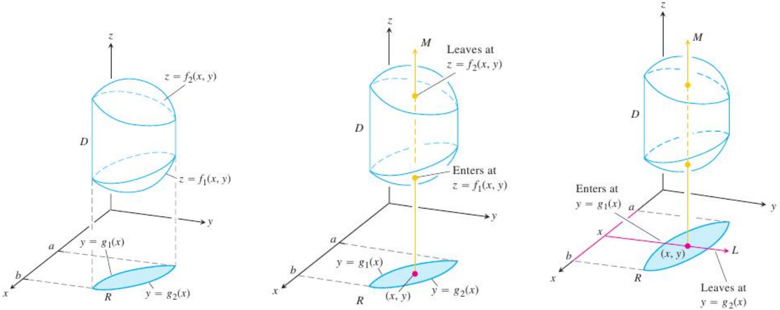
$$D = \{(x, y, z) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ h_1(x, y) \leq z \leq h_2(x, y)\},$$

then the triple integral can be written as an iterated integral:

$$\iiint_D f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)dz\,dy\,dx.$$

Finding Limits of Integration

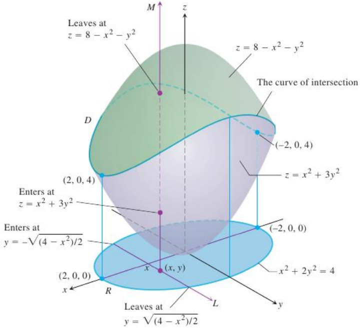
Sketch D along with its shadow on the xy-plane.
Find the z-limits, then y-limits and then x-limits.



Observe: Volume of D is $\iiint_D 1\, dV$.
All properties for double integrals hold analogously for triple integrals.

Example 25

Find the volume of the solid enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

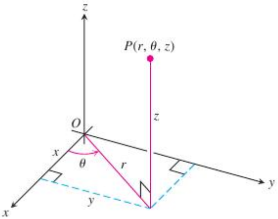


Eliminating z from the two equations, we get the projection of the curve of intersection on the xy-plane, which is $x^2 + 2y^2 = 4$. This gives the limits of integration for y as $\pm\sqrt{(4 - x^2)/2}$. Thus

$$V = \iiint_D dV = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx = \frac{8\pi}{\sqrt{2}}.$$

Cylindrical Co-ordinates

Cylindrical co-ordinates express a point P in space as a triple (r, θ, z), where (r, θ) is the polar representation of the vertical projection of P on the xy-plane.



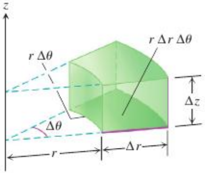
If P has Cartesian representation (x, y, z) and cylindrical representation (r, θ, z), then

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ r^2 = x^2 + y^2,\ \tan\theta = y/x.$$

In cylindrical co-ordinates,
$r = a$ describes a cylinder with axis as z-axis.
$\theta = \alpha$ describes a plane containing the z-axis.
$z = b$ describes a plane perpendicular to z-axis.

Triple Integral in Cylindrical Co-ordinates

The Riemann sum of $f(r, \theta, z)$ uses a partition of D into cylindrical wedges:



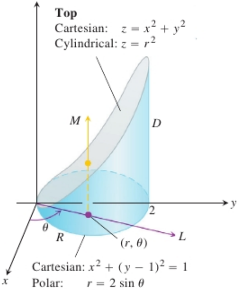
The volume element $dV = r\,dr\,d\theta dz$. Thus the triple integral is

$$\iiint_D f(r, \theta, z) dV = \iiint_D f(r, \theta, z) r\,dr\,d\theta\,dz.$$

Its conversion to iterated integrals uses a similar technique of determining the limits of integration.

Example 26

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

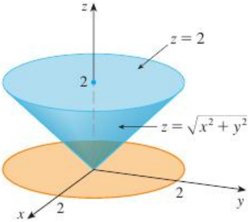


The projection of D onto the xy-plane gives the disk R enclosed by the circle $x^2 + (y-1)^2 = 1$, whose polar form is $r = 2\sin\theta$.
A line through a point $(r, \theta) \in R$ enters D at $z = 0$ and leaves D at $z = x^2 + y^2 = r^2$.
A line in the (r, θ)-plane through the origin enters R at $r = 0$ and leaves R at $r = 2\sin\theta$.
As this line sweeps through R it starts at $\theta = 0$ and ends at $\theta = \pi$.

Hence $$\iiint f(r, \theta, z)\,dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, r\, dz\, dr\, d\theta.$$

Example 27

Evaluate $I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.$



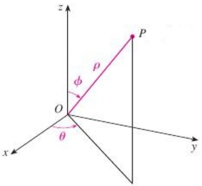
Its projection onto xy-plane is a disk; so cylindrical co-ordinates will be easier.

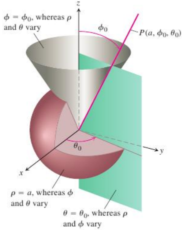
The projected disk gives the limits as $0 \le \theta \le 2\pi,\ 0 \le r \le 2$ whereas $\sqrt{x^2 + y^2} = r \le z \le 2$. Thus

$$I = \iiint_D (x^2 + y^2) dV = \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta = \frac{16}{5}\pi.$$

Spherical Co-ordinates

Spherical co-ordinates express a point P in space as a triple (ρ, ϕ, θ), where ρ is the distance of P from the origin O, ϕ is the angle between z-axis and the line OP, and θ is the angle between the projected line of OP on the xy-plane and the x-axis. This θ is the same as the 'cylindrical' θ. Moreover, $\rho \geq 0,\ 0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$.





If $P(x, y, z)$ has spherical representation (ρ, ϕ, θ), then

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho\cos\phi,\ r = \rho\sin\phi, \rho = \sqrt{x^2 + y^2 + z^2}.$$

In spherical co-ordinates,

$\rho = a$ describes a sphere centered at origin.

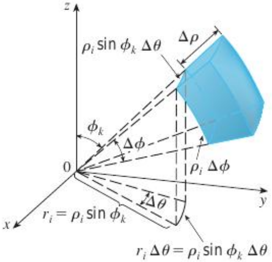
$\phi = \phi_0$ describes a cone with axis as z-axis.

$\theta = \theta_0$ describes the plane containing z-axis and OP.

Triple Integral in Spherecal Co-ordinates

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by changes by $\Delta\rho_k, \Delta\phi_k, \Delta\theta_k$ in ρ, ϕ, θ.

Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k \Delta\theta_k$ and thickness $\Delta\rho_k$. The volume of this spherical wedge is (approximately) $\Delta V_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$.

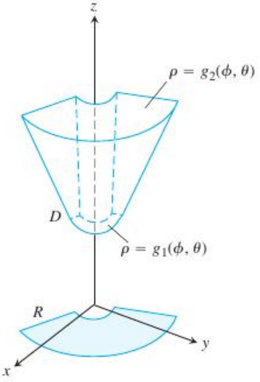


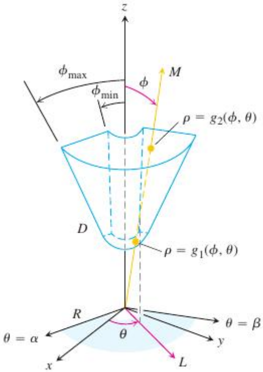
The corresponding Riemann sum is $S = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$. Accordingly,

$$\iiint_D f(\rho, \phi, \theta)dV = \iiint_D f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Iterated Integrals

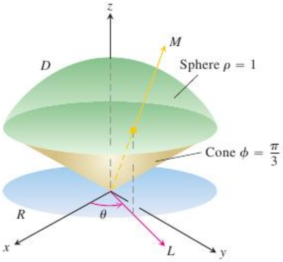
Sketch the region D and its projection on xy-plane. Then find the ρ limit, ϕ limit and θ limit.





$$\iiint_D f(\rho, \phi, \theta) dV = \int_\alpha^\beta \int_{\phi-min}^{\phi-max} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho, \phi, \theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example 28



Find the volume of the solid D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.
Draw a ray M through D from the origin making an angle ϕ with z-axis. Draw also its projection L on xy-plane along with it making angle θ with x-axis. Let R be the projected region of D in xy-plane.

M enters D at $\rho = 0$ and leaves D at $\rho = 1$.
Angle ϕ runs through 0 to $\pi/3$, since D is bounded by the cone $\phi = \pi/3$.
L sweeps through R as θ varies from 0 to 2π. Thus

$$V = \iiint_D \rho^2 \sin\phi \, dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{3}.$$

Example 29

Evaluate $I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} dz\, dy\, dx.$

$I = \iiint_D e^{(x^2+y^2+z^2)^{3/2}} dV$, where D is the solid unit sphere.

Writing in spherical co-ordinates, $I = \iiint_D e^{\rho^3} dV$.

Then converting to iterated integral,

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} e^{\rho^3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

Since the integrand is a product of separate functions of ρ, of ϕ, of θ,

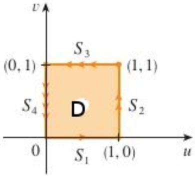
$$I = \int_0^1 e^{\rho^3} \rho^2\, d\rho \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta = \frac{4\pi}{3}(e-1).$$

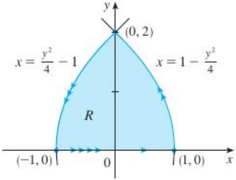
Change of Variables

Suppose f maps a region D in $\mathbb{R}^2$ to a region R in $\mathbb{R}^2$ in a one-one manner. For convenience, we say that D is a region in uv-plane and R is a region in xy-plane; and f maps (u, v) to (x, y). Then f can be thought of as a pair of maps: (f_1, f_2). That is, $x = f_1(u, v)$ and $y = f_2(u, v)$. We often show this dependence implicitly by writing

$$x = x(u, v), \quad y = y(u, v).$$

What is the image of $D = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$ under the map given by $x = u^2 - v^2,\ y = 2uv$?
Hint: Consider the boundary lines.





Area of a Rectangle

If $(u, v) \mapsto (x, y)$, then how does area of a small rectangle change?

A typical small rectangle with sides Δu and Δv has corners

$$A_1 = (a, b),\ A_2 = (a + \Delta u, b),\ A_3 = (a, b + \Delta v),\ A_4 = (a + \Delta u, b + \Delta v).$$

Let the images of A_k under $(u, v) \mapsto (x, y)$ be $B_k = (a_k, b_k)$ for $k = 1, 2, 3, 4$. Then

$$\begin{aligned}
a_1 &= x(a, b) \\
a_2 &= x(a + \Delta u, b) \approx x(a, b) + x_u \Delta u \\
a_3 &= x(a, b + \Delta v) \approx x(a, b) + x_v \Delta v \\
a_4 &= x(a + \Delta u, b + \Delta v) \approx x(a, b) + x_u \Delta u + x_v \Delta v
\end{aligned}$$

Here, $x_u = x_u(a, b)$ and $x_v = x_v(a, b)$.

Similar approximations hold for b_1, b_2, b_3, b_4.

Area of a Rectangle

Now, Area of the image of the rectangle $A_1A_2A_3A_4$ is approximately equal to the area of the parallelogram $B_1B_2B_3B_4$ in xy-plane, which is twice the area of the triangle $B_1B_2B_4$ and is

$$|(a_4 - a_1)(b_4 - b_2) - (a_4 - a_2)(b_4 - b_1)| = \left|\det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\right| (a, b)\Delta u \Delta v.$$

This determinant is called the Jacobian of the map $(u, v) \mapsto (x, y)$; and is denoted by $J(x(u, v), y(u, v))$ and also as $\dfrac{\partial(x, y)}{\partial(u, v)}$.

We write this as Area of image of a rectangle centered at (a, b) of sides Δu and Δv is approximately $\left|\dfrac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v$, where the Jacobian $J(\cdot, \cdot)$ is evaluated at (a, b).

In deriving this approximation, we have assumed that x_u, x_v, y_u, y_v are continuous.

Change of Variables

Assume that $x = x(u, v)$ and $y = y(u, v)$ have continuous partial derivatives with respect to u and v. Assume also that a region D in the uv-plane is in one-one correspondence with a region R in the xy-plane by the map $(u, v) \mapsto (x, y)$. Let $f(x, y)$ be a real valued continuous function on a region R. Then we have the map $\tilde{f}(u, v) = f(x(u, v), y(u, v))$.

How are the integrals of f over R and integral of $\tilde{f}$ over D related?

Divide D in the uv-plane into smaller rectangles. Now, the images of the smaller rectangles are related by

$$\text{Area of } R = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \text{Area of } D.$$

By forming the Riemann sum and taking the limit, we obtain:

$$\iint_R f(x, y)dA = \iint_D \tilde{f}(u, v)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA.$$

Change of variables Contd.

In the case of polar co-ordinates, we have

$$x = x(r, \theta) = r\cos\theta, \quad y = y(r, \theta) = r\sin\theta.$$

The absolute value of the Jacobian is

$$\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = |x_r y_\theta - x_\theta y_r| = |\cos\theta(r\cos\theta) - (-r\sin\theta)\sin\theta| = r.$$

Therefore, the double integral in polar co-ordinates for a function $f(x, y)$ takes the form

$$\iint_R f(x, y)dA = \iint_D f(r\cos\theta, r\sin\theta) r\, dA.$$

Suppose $x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w)$. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det\begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}.$$

If R is the region in space on which f has been defined and D is the region in the uvw-space so that the functions x, y, z map D onto R, then

$$\iiint_R f(x, y, z)dV = \iiint_D f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw.$$

Cylindrical-Spherical

For the cylindrical co-ordinates, $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. The absolute values of the Jacobian is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \left|\det\begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix}\right| = r.$$

$$\iiint_R f(x,y,z)dV = \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

For the spherical co-ordinates, we see that

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$$

The triple integral looks like

$$\iiint_R f(x,y,z)dV = \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

Example 30

Evaluate $\iint_R (y-x)dA$, where R is the region bounded by the lines $y-x=1,\ y-x=-3,\ 3y+x=7,\ 3y+x=15.$

Take $u=y-x,\ v=3y+x.$

That is, $x=\frac{1}{4}(v-3u), y=\frac{1}{4}(u+v)$. Then

$$D=\{(u,v):-3\le u\le 1,\ 7\le v\le 15\}.$$

The Jacobian is

$$J=x_u y_v - x_v y_u = (-3/4)(1/4)-(1/4)(1/4) = -1/4.$$

Therefore,

$$\iint_R (y-x)dA = \iint_D u\,\frac{1}{4}\,dA = \int_{-3}^{1}\int_{7}^{15}\frac{1}{4}u\,dv\,du = -8.$$

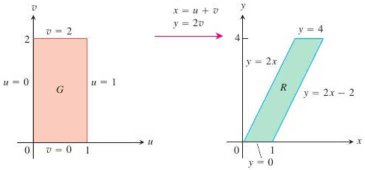
Example 31

Evaluate $\int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2} dxdy$ by using the transformation $u = x - y/2,\ v = y/2.$

Here, $x = u + v,\ y = 2v,\ f(x, y) = x - y/2 = u.$ The regions are

$$R = \{(x, y) : 0 \le y \le 4, y/2 \le x \le 1 + y/2\},$$

$$G = \{(u, v) : 0 \le u \le 1, 0 \le v \le 2\}.$$



And $|J(x(u, v), y(u, v)| = |x_u y_v - x_v y_u| = |(1)(2) - (0)(1)| = 2$. So,

$$\int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2} dxdy = \iint_R \frac{2x-y}{2} dA = \iint_G 2u dA = \int_0^2 \int_0^1 2u\, du\, dv = \int_0^2 1^2\, dv = 2.$$

Caution

The change of variables formula turns an xy-integral into a uv-integral. But the map that changes the variables goes from uv-region onto xy-region. This map must be one-one on the interior of the uv-region.

Sometimes it is easier to get such a map from xy-region to uv-region. Then we will be tackling with the inverse of such an easy map. Here, the following fact helps us:

The Jacobian of the inverse map is the inverse of the Jacobian of the original map.

This may be expressed as $\dfrac{\partial(x,y)}{\partial(u,v)} = \left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$.

Similarly, triple integrals undergo change of variables by using the inverse of the Jacobian.

Example 32

Integrate $f(x, y) = xy(x^2 + y^2)$ over the region

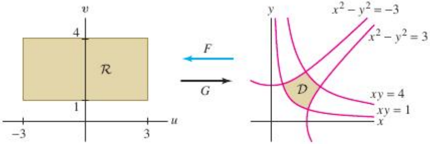
$$D: \quad -3 \le x^2 - y^2 \le 3,\ 1 \le xy \le 4.$$

There is a simple map that goes in the *wrong direction*:

$$u = x^2 - y^2,\ v = xy.$$

In the uv-plane R becomes the rectangle

$$R: \quad -3 \le u \le 3,\ 1 \le v \le 4,$$



Then we have $F : D \to R$ defined by $F(x, y) = (u, v) = (x^2 - y^2, xy)$.
And $G = F^{-1}$ is the map $G : R \to D$.

Example 32 Contd.

Instead of computing the map G, we go for the Jacobian.

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2(x^2+y^2).$$

Therefore, $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2(x^2+y^2)}$. Then

$$\iint_R xy(x^2+y^2)\,dA = \iint_D \left[xy(x^2+y^2) \left| \frac{1}{2(x^2+y^2)} \right| \right] dA.$$

The integral on the right side is in the uv-plane and the bracketed term inside $[\,\cdot\,]$ is a function of (u,v). Since the bracketed term simplifies to $xy/2$ which is equal to $v/2$, we have the integral as

$$\iint_D \frac{v}{2}\,dA = \frac{1}{2}\int_{-3}^{3}\int_{1}^{4} v\,dv\,du = \frac{45}{2}.$$

Review Problems

Problem 1: Find the area of the region bounded by the curves $y = x$ and $y = 2 - x^2$.

The points of intersection of the curves are $(-2, -2)$ and $(1, 1)$. Hence the area is

$$\int_{-2}^{1} \int_{x}^{2-x^2} dydx = \int_{-2}^{1} (2 - x^2 - x)dx = \frac{9}{2}.$$

Problem 2: Evaluate $I = \iint_D (4 - x^2 - y^2)\, dA$ if D is the region bounded by the straight lines $x = 0,\ x = 1,\ y = 0$ and $y = 3/2$.

$$I = \int_0^{3/2} \int_0^1 (4 - x^2 - y^2)dxdy = \int_0^{3/2} \left[4x - y^2x - x^3/3\right]_0^1 dy = \frac{35}{8}.$$

Problem 3: Evaluate the double integral I of $f(x, y) = 1 + x + y$ over a region bounded by the lines $y = -x,\ x = \sqrt{y}$, and $y = 2$.

$$I = \int_0^2 \int_{-y}^{\sqrt{y}} (1+x+y)dx\, dy = \int_0^2 \left(\sqrt{y} + \frac{3y}{2} + y\sqrt{y} + \frac{y^2}{2}\right)dy = \frac{44}{15}\sqrt{2} + \frac{13}{3}.$$

Review Problems Contd.

Problem 4: Change the order of integration in $\int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dydx$.

The region D of integration is bounded by the straight line $y = x$ and the parabola $y = \sqrt{x}$.
Every straight line parallel to x-axis cuts the boundary of D in no more than two points, and it remains in between y^2 to y. Also, y lies between 0 and 1. Hence

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dydx = \int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy.$$

Problem 5: Evaluate $I = \iint_D e^{y/x}\,dA$, where D is a triangle bounded by the straight lines $y = x$, $y = 0$, and $x = 1$.

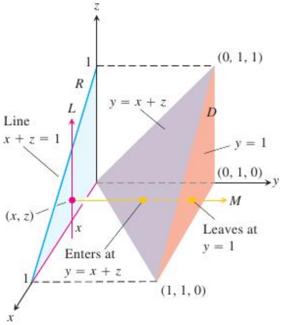
In D, the variable x remains in between 0 and 1, and y lies between 0 and x. Hence

$$I = \int_0^1 \int_0^x e^{y/x} dy\,dx = \int_0^1 x(e-1)\,dx = (e-1)/2.$$

Problem 6

Write the integral of $f(x, y, z)$ over a tetrahedron with vertices at $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$ as iterated integrals in the order $dydzdx$ and also in the order $dzdydx$.

Sketch the region D to see the limits geometrically.



The right hand side bounding surface of D lies in the plane $y = 1$.
The left hand side bounding surface lies in the plane $y = z + x$.
The projection of D on the zx-plane is R.
The upper boundary of R is the line $z = 1 - x$.
So, $R = \{(x, z) : 0 \leq x \leq 1,\ 0 \leq z \leq 1 - x\}$.

Then, $D = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq z \leq 1 - x,\ x + z \leq y \leq 1\}$.

Problem 6 Contd.

$D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le z \le 1 - x,\ x + z \le y \le 1\}$.
Thus the triple integral of a function $f(x, y, z)$ over D is given by

$$\iiint_D f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_{x+z}^1 f(x, y, z)\, dy\, dz\, dx.$$

To express in the order $dzdydx$, project D on the xy-plane.
A line parallel to z-axis through (x, y) in the xy-plane enters D at $z = 0$ and leaves D through the upper plane $z = y - x$.
For the y-limits, on the xy-plane, where $z = 0$, the sloped side of D crosses the plane along the line $y = x$.
A line through (x, y) parallel to y-axis enters the xy-plane at $y = x$ and leaves at $y = 1$.
The x-limits are as earlier.
Therefore $D = \{(x, y, z) : 0 \le x \le 1,\ x \le y \le 1,\ 0 \le z \le y - x\}$. Then

$$\iiint_D f(x, y, z)\, dV = \int_0^1 \int_x^1 \int_0^{y-x} f(x, y, z)\, dz\, dy\, dx.$$

Problem 7

Evaluate $\int_0^1 \int_0^z \int_0^y e^{(1-x)^3}\, dx\, dy\, dz$ by changing the order of integration.
The region is $D = \{(x, y, z) : 0 \le z \le 1,\ 0 \le y \le z,\ 0 \le x \le y\}$.
Sketch the region.

We plan to change the order of integration from $dxdydz$ to $dzdydx$.

Its projection on the xy-plane is the triangle bounded by the lines $x = 0, y = 1$ and $y = x$.

It is expressed as $\{(x, y) : 0 \le x \le 1,\ x \le y \le 1\}$.

Then $D = \{(x, y, z) : 0 \le x \le 1,\ x \le y \le 1,\ y \le z \le 1\}$.

Therefore, $\displaystyle\int_0^1 \int_0^z \int_0^y e^{(1-x)^3}\, dx\, dy\, dz = \int_0^1 \int_x^1 \int_y^1 e^{(1-x)^3}\, dz\, dy\, dx$

$$= \int_0^1 \int_x^1 (1-y)e^{(1-x)^3}\, dy\, dx = \int_0^1 \frac{(1-x)^2}{2} e^{(1-x)^3}\, dx$$

$$= -\int_{(1-0)^3}^0 \frac{e^t}{6}\, dt = \frac{e-1}{6}. \qquad \text{with } t = (1-x)^3$$

Problem 8

Find $I = \iint_D e^{x+y}\, dA$, where D is the annular region bounded by two squares of sides 2 and 4, each has center $(0,0)$ and sides parallel to the axes.

Draw the picture.
Let D_1 be the inner square and D_2 be the outer square. Then

$$I = \iint_{D_2} e^{x+y}\, dA - \iint_{D_1} e^{x+y}\, dA.$$

Converting each integral to an iterated integral, we have

$$\begin{aligned} I &= \int_{-2}^{2}\int_{-2}^{2} e^{x+y} dydx - \int_{-1}^{1}\int_{-1}^{1} e^{x+y} dydx \\ &= e^4 - 2 - e^{-4} - (e^2 - 2 - e^{-2}) \\ &= 2\cosh(4) - 2\cosh(2) \\ &= 4\sinh(3)\,\sinh(1). \end{aligned}$$

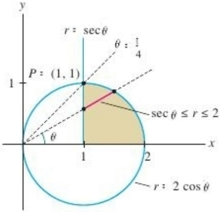
Review Problems Contd.

Problem 9: Calculate the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

The volume $V = \iint_D (1 - x - y)dA$, where D is the base of the solid on the xy-plane. D is the triangular region bounded by the straight lines $x = 0$, $y = 0$, $x + y = 1$. Thus,

$$V = \int_0^1 \int_0^{1-x} (1 - x - y)dydx = \int_0^1 \frac{1}{2}(1 - x)^2 dx = \frac{1}{6}.$$

Problem 10: Evaluate $\iint_D (x^2 + y^2)^{-2}\, dA$, where D is the shaded region



The integrand in polar co-ordinates is $f(r, \theta) = r^{-4}$. The region D is given by $0 \le \theta \le \pi/4$, $\sec\theta \le r \le 2\cos\theta$.

Therefore

$$\iint_D (x^2+y^2)^{-2} dA = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-4} r dr d\theta = \frac{1}{8} \int_0^{\pi/4} (4\cos^2\theta - \sec^2\theta) d\theta = \frac{\pi}{16}.$$

Problem 11

Compute the volume V of the solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$, the cylinder $x^2 + y^2 = 2ay$, where $a > 0$, and which is inside the cylinder.

The region of integration is the circle $x^2 + y^2 - 2ay = 0$, which is $x^2 + (y - a)^2 = a^2$. We calculate $V/4$, the volume of the portion of the solid in the first octant. Then the region of integration D is the semicircular disk whose boundaries in polar co-ordinates are given by

$$r = g_1(\theta) = 0, \ r = g_2(\theta) = 2a\sin\theta, \ 0 \le \theta \le \pi/2.$$

The integrand is $z = \sqrt{4a^2 - x^2 - y^2}$. In polar co-ordinates, $f(r,\theta) = \sqrt{4a^2 - r^2}$. For the limits of integration, use $x^2 + y^2 = r^2$, $y = r\sin\theta$ to get:

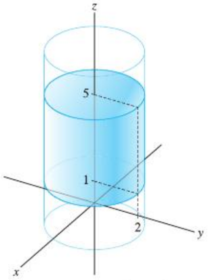
$$x^2 + y^2 - 2ay = 0 \Rightarrow r^2 - 2ar\sin\theta = 0 \Rightarrow r = 2a\sin\theta.$$

Therefore,

$$V = 4\int_0^{\pi/2}\int_0^{2a\sin\theta} \sqrt{4a^2 - r^2}\, r\, dr\, d\theta$$

Problem 12

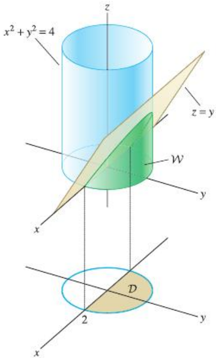
Integrate $f(x, y, z) = z\sqrt{x^2 + y^2}$ over the solid cylinder $x^2 + y^2 \leq 4$ for $1 \leq z \leq 5$.



The region of integration D in cylindrical co-ordinates is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, $1 \leq z \leq 5$. The integrand is zr. Thus

$$\iiint_D z\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^2 \int_1^5 (zr)\, r\, dz\, dr\, d\theta = 64\pi.$$

Problem 13



Integrate $f(x, y, z) = z$ over the part of the solid cylinder $x^2 + y^2 \leq 4$ for $0 \leq z \leq y$.

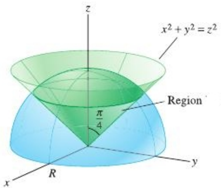
The region W has the projection D on the xy-plane as the semicircle depicted in the figure. The z-co-ordinate varies from 0 to y and $y = r\sin\theta$. Thus W is given by $0 \leq \theta \leq \pi$, $0 \leq r \leq 2$, $0 \leq z \leq r\sin\theta$.

In cylindrical co-ordinates,

$$\iiint_W z\,dV = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z\,r\,d\theta\,dr\,dz = \int_0^{\pi}\int_0^{2}\frac{1}{2}(r\sin\theta)^2\,r\,d\theta\,dr = \pi.$$

Problem 14

Compute $\iiint_D z\,dV$, where D is the solid lying above the cone $x^2 + y^2 = z^2$ and below the unit sphere.



The upper branch of the cone, which is relevant to D, has the equation $\phi = \pi/4$ in spherical co-ordinates.
The sphere has the equation $\rho = 1$.
Thus D is given by $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le 1$.
Since $z = \rho\cos\phi$, the required integral is

$$\iiint_D z\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\int_0^{\pi/4}\int_0^1 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi = \frac{\pi}{2}\int_0^{\pi/4}\cos\phi\sin\phi\,d\phi = \frac{\pi}{8}.$$

Problem 15

Evaluate $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

$$I^2 = \lim_{a\to\infty}\Big(\int_{-a}^{a} e^{-x^2}\,dx\Big)^2 = \lim_{a\to\infty}\Big[\Big(\int_{-a}^{a} e^{-x^2}\,dx\Big)\Big(\int_{-a}^{a} e^{-y^2}\,dy\Big)\Big]$$

$$= \lim_{a\to\infty}\Big[\int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy\Big] = \lim_{a\to\infty}\iint_R e^{-x^2-y^2}\,dA,$$

where R is the rectangle $[-a,a]\times[-a,a]$ for $a>0$.

Let $D = B(0,a)$ and $S = B(0,\sqrt{2}\,a)$. Then $D\subseteq R\subseteq S$. Since $e^{-x^2-y^2} > 0$ for all $(x,y)\in\mathbb{R}^2$, we have $\iint_D e^{-x^2-y^2}\,dA \le \iint_R e^{-x^2-y^2}\,dA \le \iint_S e^{-x^2-y^2}\,dA$.

Also,

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta = \pi(1-e^{-a^2}).$$

Similarly, $\iint_S e^{-x^2-y^2}\,dA = \pi(1-e^{-2a^2})$. Hence

$$\lim_{a\to\infty}\iint_D e^{-x^2-y^2}\,dA = \pi, \qquad \lim_{a\to\infty}\iint_S e^{-x^2-y^2}\,dA = \pi.$$

Therefore, $I^2 = \lim_{a\to\infty}\iint_R e^{-x^2-y^2}\,dA = \pi \Rightarrow I = \sqrt{\pi}$.

Problem 16

Compute the volume V of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

Projection of this solid on the xy-plane is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Therefore, $V = \displaystyle\int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$

$$= 2c \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx.$$

Substitute $y = b(1 - x^2/a^2)^{1/2} \sin t$. Then $dy = b(1 - x^2/a^2) \cos t\, dt$ and $-\pi/2 \le t \le \pi/2$. Therefore,

$$V = 2c \int_{-a}^{a} \int_{-\pi/2}^{\pi/2} \Big[\big(1 - \frac{x^2}{a^2}\big) - \big(1 - \frac{x^2}{a^2}\big) \sin^2 t\Big]^{1/2} b\big(1 - \frac{x^2}{a^2}\big) \cos t\, dt\, dx$$

$$= \frac{bc\pi}{a^2} \int_{-a}^{a} (a^2 - x^2)\, dx = \frac{4\pi abc}{3}.$$

Problem 17

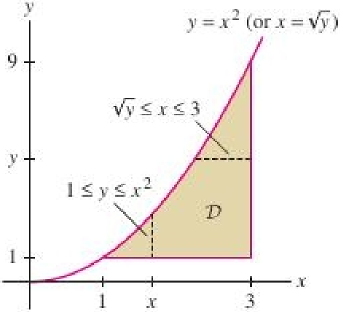
Evaluate $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx$ for $a > 0, b > 0$.

$$\begin{aligned}
\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx &= \int_0^\infty \int_a^b e^{-yx} dy\, dx \\
&= \int_a^b \int_0^\infty e^{-yx} dx\, dy \\
&= \int_a^b \frac{1}{y}\, dy \\
&= \ln \frac{b}{a}.
\end{aligned}$$

Notice the change in order of integration above.

Problem 16

Evaluate $\int_1^9 \int_{\sqrt{y}}^3 xe^y \, dx \, dy$.



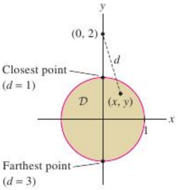
The region of integration is given by $1 \le y \le 9, \ \sqrt{y} \le x \le 3$.
The same is expressed as $1 \le x \le 3, \ 1 \le y \le x^2$.
Changing the order of integration, we have

$$\int_1^9 \int_{\sqrt{y}}^3 xe^y \, dx \, dy = \int_1^3 \int_1^{x^2} xe^y \, dx \, dy = \int_1^3 (xe^{x^2} - e\,x) dx = \frac{1}{2}(e^9 - 9e).$$

Problem 18

Let D be the unit disc. Show that

$$\frac{\pi}{3} \leq \iint_D \frac{dA}{\sqrt{x^2+(y-2)^2}} \leq \pi$$



The quantity $f(x,y) = \sqrt{x^2+(y-2)^2}$ is the distance of any point (x,y) from $(0,2)$. For $(x,y) \in D$, maximum of $f(x,y)$ is thus 3 and minimum is 1. Therefore, $\frac{1}{3} \leq \frac{1}{\sqrt{x^2+(y-2)^2}} \leq 1$.

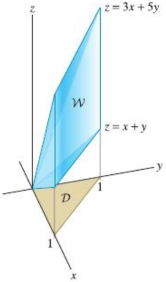
Integrating over D, we have

$$\iint_D \frac{1}{3}\, dA \leq \iint_D \frac{1}{\sqrt{x^2+(y-2)^2}}\, dA \leq \iint_D 1\, dA.$$

Since $\iint_D dA$ = area of D, we obtain

$$\frac{\pi}{3} \leq \iint_D \frac{dA}{\sqrt{x^2+(y-2)^2}} \leq \pi.$$

Problem 19



Evaluate $\iiint_W z\,dV$, where W is the solid bounded by the planes $x = 0$, $y = 0$, $x + y = 1$, $z = x + y$, and $z = 3x + 5y$ in the first octant.

W lies over the triangle D in the xy-plane defined by $0 \le x \le 1,\ 0 \le y \le 1 - x$. Therefore,

$$\iiint_D z\,dV = \int_0^1 \int_0^{1-x} \int_{x+y}^{3x+5y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (4x^2 + 14xy + 12y^2)\,dy\,dx$$

$$= \int_0^1 (4 - 5x + 2x^2 - x^3)\,dx = \frac{23}{12}.$$

Fun Problem

The n-dimensional cube with side a has volume a^n.

What is the volume of an n-dimensional ball?

Denote by $V_n(r)$ the volume of the n-dimensional ball with radius r.
Also, write $A_n = V_n(1)$.
$A_1 = 2,\ V_1(r) = 2r$. $A_2 = \pi,\ V_2 = \pi r^2$.
$A_3 = 4\pi/3$ and $V_3(r) = 4\pi r^3/3$.

Exercise 1: By induction, show that volume of an n-dimensional ball of radius r is $A_n r^n$.

Suppose $V_{n-1}(r) = A_{n-1}r^{n-1}$. The slice of the n-dimensional ball

$$x_1^2 + \cdots x_{n-1}^2 + x_n^2 = r^n$$

at the height $x_n = c$ has the equation

$$x_1^2 + \cdots x_{n-1}^2 + c^2 = r^2.$$

Fun Problem Contd.

This slice has the radius $\sqrt{r^2 - c^2}$. Thus

$$V_n(r) = \int_{-r}^{r} V_{n-1}\sqrt{r^2 - x_n^2}\, dx_n = A_{n-1}\int_{-r}^{r} (\sqrt{r^2 - x_n^2})^{n-1}\, dx_n.$$

Substitute $x_n = r\sin\theta$. So, $dx_n = r\cos\theta$ and $-\pi/2 \leq \theta \leq \pi/2$. Then

$$V_n(r) = A_{n-1} r^n \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta = A_{n-1} C_n r^n,$$

where $C_n = \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta$. This says that $A_n = A_{n-1} C_n$.

Exercise 2: Prove that $C_3 = 4/5, C_4 = 3\pi/8$ and $C_n = \frac{n-1}{n} C_{n-2}$.

Exercise 2: Prove that $A_{2m} = \dfrac{\pi^m}{m!}$ and $A_{2m+1} = \dfrac{2^{m+1}\pi^m}{1 \cdot 3 \cdots (2m+1)}$.

This sequence of numbers have a curious property:

A_n increases up to $n = 5$ and then it decreases to 0 as $n \longrightarrow \infty$.