Limit

Let $f : D \to \mathbb{R}$ be a function, where D is a region. Let $(a, b) \in \overline{D}$.

The limit of $f(x, y)$ as (x, y) approaches (a, b) is L iff

corresponding to each $\epsilon > 0$, there exists $\delta > 0$ such that

for all $(x, y) \in D$ with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$,

we have $|f(x, y) - L| < \epsilon$.

In this case, we write $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

We also say that L is the limit of f at (a, b).

If for no real number L, the above happens, then limit of f at (a, b) does not exist.

It is often difficult to show that limit of a function does not exist at a point. We will come back to this question soon.

Limit-see

The limit of $f(x,y)$ as (x,y) approaches (a,b) is L

$$f(x,y) \to L \text{ as } (x,y) \to (a,b).$$

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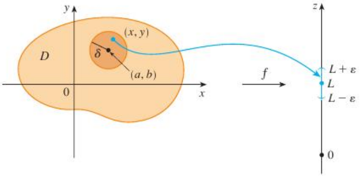

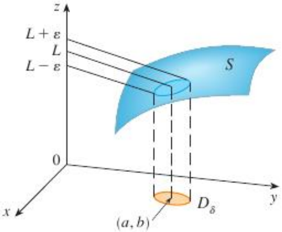

The distance between $f(x,y)$ and L can be made arbitrarily small by making the distance between (x,y) and (a,b) sufficiently small but not zero.

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Observe: Domain D of f is $\mathbb{R}^2 \setminus \{(0,0)\}$.
$f(0,y) = 0$ for $y \neq 0$. Also, $f(x,0) = 0$ for $x \neq 0$. We guess that if the limit exists, it would be 0.

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$$\text{If} \quad 0 < \sqrt{x^2+y^2} < \delta, \quad \text{then} \quad \left|\tfrac{4xy^2}{x^2+y^2}\right| < \epsilon.$$

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So, we choose $\delta = \epsilon/4$. Assume that $0 < \sqrt{x^2+y^2} < \delta$. Then

$$\left|\frac{4xy^2}{x^2+y^2}\right| \leq 4\sqrt{x^2+y^2} < 4\delta = \epsilon.$$

$$\text{Hence} \quad \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

Example 2

Consider $f(x, y) = \sqrt{1 - x^2 - y^2}$ when $D = \{(x, y) : x^2 + y^2 \leq 1\}$.
We guess that limit $f(x, y)$ is 1 as $(x, y) \to (0, 0)$.

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To see this, Let $\epsilon > 0$.

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Observe: Look at the requirement of the limit. If we have found a δ corresponding to a smaller ϵ, then the same δ works for a larger ϵ.

So, assume that $0 < \epsilon < 1$. Choose $\delta = \sqrt{1 - (1 - \epsilon)^2}$.
Let $|(x, y) - (0, 0)| < \delta$. Then

$$x^2 + y^2 < 1 - (1 - \epsilon)^2 \Rightarrow 1 - x^2 - y^2 > (1 - \epsilon)^2 \Rightarrow f(x, y) > 1 - \epsilon.$$

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$$x^2 + y^2 < 1 - (1 - \epsilon)^2 \Rightarrow 1 - x^2 - y^2 > (1 - \epsilon)^2 \Rightarrow f(x, y) > 1 - \epsilon.$$

That is, $|f(x, y) - 1| = 1 - f(x, y) < \epsilon$.

Therefore, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$.

Uniqueness of limit

Theorem 1: Let $f(x, y)$ be a real valued function defined on a region $D \subseteq \mathbb{R}^2$. Let $(a, b) \in \overline{D}$. If limit of $f(x, y)$ as (x, y) approaches (a, b) exists, then it is unique.

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Proof: Suppose $f(x,y) \longrightarrow \ell$ and also $f(x,y) \longrightarrow m$ as $(x,y) \longrightarrow (a,b)$. Let $\epsilon > 0$. For $\epsilon/2$, we have $\delta_1 > 0,\ \delta_2 > 0$ such that

$$0 < (x-a)^2 + (y-b)^2 < \delta_1^2 \Rightarrow |f(x,y) - \ell| < \epsilon/2,$$
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Choose a point (α, β) so that both of the following hold:

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Now, $|\ell - m| \leq |\ell - f(\alpha, \beta)| + |f(\alpha, \beta) - m| < \epsilon/2 + \epsilon/2 = \epsilon$.
That is, for any $\epsilon > 0$, we have $|\ell - m| < \epsilon$. Hence $\ell = m$. $\square$

Non-existence of Limit

Proposition: If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, and $L_1 \neq L_2$, then limit of $f(x, y)$ as $(x, y) \to (a, b)$ does not exist.

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When $x = 0$, limit of $f(x,y)$ as $y \to 0$ is $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$.

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That is, $f(x,y) \to -1$ as $(x,y) \to (0,0)$ along the y-axis.

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

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Does it say that limit of $f(x,y)$ as $(x,y) \longrightarrow (0,0)$ is 0?

When $y = x$, limit of $f(x,y)$ as $(x,y) \longrightarrow 0$ is $\lim_{x\to 0} \frac{x^2}{x^2+x^2} = 1/2$.

That is, $f(x,y) \longrightarrow 1/2$ as $(x,y) \longrightarrow (0,0)$ along the line $y = x$.

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Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

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If $y = mx$, for some $m \in \mathbb{R}$, then $f(x, y) = \frac{m^2 x}{1+m^4 x^2}$.
So, $\lim_{(x,y)\to(0,0)} f(x, y)$, along all straight lines is 0.

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If $x = y^2$, then $f(x,y) = \frac{y^4}{y^4+y^4} = 1/2$ for $y \neq 0$.
As $(x,y) \to (0,0)$ along the parabola $x = y^2$, $\lim f(x,y)$ is $1/2$.

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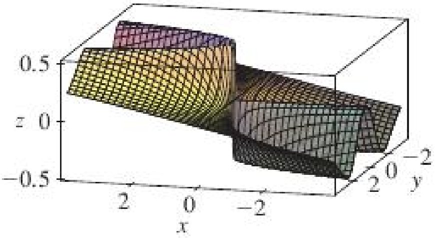
So, $\lim_{(x,y)\to(0,0)} f(x,y)$, along all straight lines is 0.

If $x = y^2$, then $f(x,y) = \frac{y^4}{y^4+y^4} = 1/2$ for $y \neq 0$.
As $(x,y) \to (0,0)$ along the parabola $x = y^2$, $\lim f(x,y)$ is $1/2$.

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



Iterated Limit

Are the following same:

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Example 6: $f(x,y) = \frac{(y-x)(1+x)}{(y+x)(1+y)}$ for $x+y \neq 0, -1 < x,\ y < 1$. Then

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For different values of m, we get the last limit value different. So, limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist.

Iterated Limit Contd.

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Hence existence of the limit of $f(x, y)$ as $(x, y) \to (a, b)$ and the two iterated limits have no connection.

Limit Properties

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5. Power: If $p, q \in \mathbb{Z}, q \neq 0, L^{p/q} \in \mathbb{R}$ and $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim_{(x,y)\to(a,b)} (f(x,y))^{p/q} = L^{p/q}$.

Continuity

Let $f(x, y)$ be a real valued function on a subset D of $\mathbb{R}^2$.
$f(x, y)$ is continuous at a point (a, b) iff for each $\epsilon > 0$, there exists $\delta > 0$ such that for all points $(x, y) \in D$ with $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ we have $|f(x, y) - f(a, b)| < \epsilon$.

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1. $f(a, b)$ is well defined, that is, $(a, b) \in D$;
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$f(x,y)$ is continuous on a subset of D iff $f(x,y)$ is continuous at all points of the subset.

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Therefore, the function $g(x, y)$ defined on $\mathbb{R}^2$ by

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

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The function y/x is continuous everywhere except when $x = 0$.
The function $\tan^{-1}$ is continuous everywhere on $\mathbb{R}$.
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As earlier, composition of continuous functions is continuous:

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For example, e^{x-y} is continuous at all points in the plane.

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The function $(x^2 + y^2 + z^2 - 1)^{-1}$ is continuous everywhere except on the sphere $x^2 + y^2 + z^2 = 1$, where it is not defined.

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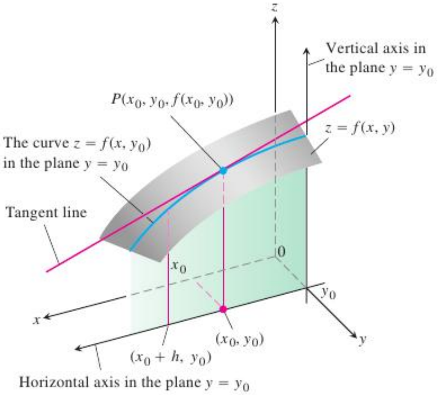

Let C be the curve of intersection of the surface $z = f(x, y)$ with the plane $y = y_0$.

The slope of the tangent line to C at $(x_0, y_0, f(x_0, y_0))$ is the partial derivative of $f(x, y)$ with respect to x at (x_0, y_0).

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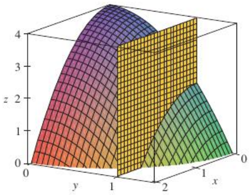

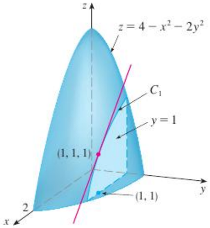

The vertical plane $y = 1$ crosses the paraboloid in the curve C_1 : $z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1,1,1)$ (which corresponds to $(x,y) = (1,1)$) is $f_x(1,1) = -2$.

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$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d(1 + y^2)}{dy}\right|_{y=2} = (2y)|_{y=2} = 4.$$

Higher Order Partial derivatives

For a function $f(x, y)$, partial derivatives of second order are:

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Observe: $f_x(a,b)$ is not the same as $\lim_{(x,y)\to(a,b)} f_x(x,y)$. Why?

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Continuity of both of f_{xy} and f_{yx} implies their equality.

Clairaut's Theorem

Theorem 2: Let $D \subseteq \mathbb{R}^2$ be a region. Let $f : D \to \mathbb{R}$. Suppose that f_x, f_y, f_{xy} and f_{yx} are continuous on D. Then $f_{xy} = f_{yx}$.

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Hence, $f_{xy}(a, b) = f_{yx}(a, b)$. $\square$

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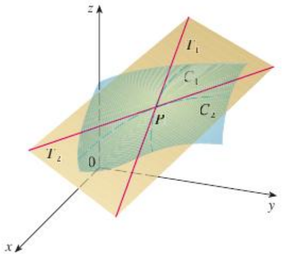
Similarly, $f_{yx}(0, 0)$ does not exist.

Tangent Planes

Let S be the surface $z = f(x, y)$, where f_x, f_y are continuous on the region D, the domain of f. Let $(a, b) \in D$. Let C_1 and C_2 be the curves of intersection of the planes $x = a$ and of $y = b$ with S.



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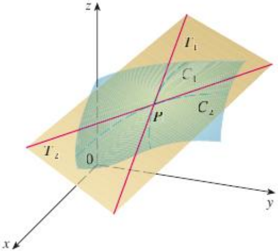

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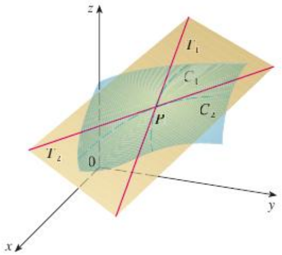

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The tangent plane to S at P consists of all possible tangent lines at P to the curves C that lie on S and pass through P. This plane approximates S at P most closely.

Equation of the tangent plane

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To see this: $z_x = 4x, z_y = 2y$. So, $z_x(1, 1) = 4, z_y(1, 1) = 2$. Then Eqn is $z - 3 = 4(x - 1) + 2(y - 1)$.

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$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

This formula holds true for all points $(x, y, f(x, y))$ on the tangent plane at $(a, b, f(a, b))$. For approximating $f(x, y)$ for (x, y) close to (a, b), we may take

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The RHS is called the standard linear approximation of $f(x,y)$. Writing in the increment form,

$$f(a+h,b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k.$$

This gives rise to the total increment $f(a+h,b+k) - f(a,b)$.

Increment Theorem

Theorem 3: Let $f : D \longrightarrow \mathbb{R}$, D be a region in $\mathbb{R}^2$, f_x and f_y be continuous on D. Then $f(x, y)$ is continuous on D and the total increment $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$ at $(a, b) \in D$ can be written as

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By MVT, there exist $c \in [a, a + h]$ and $d \in [b, b + k]$ such that

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Increment Theorem

Theorem 3: Let $f : D \to \mathbb{R}$, D be a region in $\mathbb{R}^2$, f_x and f_y be continuous on D. Then $f(x, y)$ is continuous on D and the total increment $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$ at $(a, b) \in D$ can be written as

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Then $f(x,y)$ is a continuous function. □

Differntiability

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However, we continue using the Increment Theorem directly. Remember: We can always replace the continuity of f_x, f_y with differentiablity of f in all our results.

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Given that $|\Delta x|, |\Delta y|, |\Delta z| \leq 0.2$cm, the largest error in cubic cm is

$$|\Delta V| \approx |dV| = 60 \times 40 \times 0.2 + 40 \times 75 \times 0.2 + 75 \times 60 \times 0.2 = 1980.$$

Chain Rule 1

Theorem 6: Let $x(t)$ and $y(t)$ be differentiable. Let $f(x, y)$ be such that f_x and f_y are continuous. Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

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Proof: Use the increments theorem at a point P.

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As $\Delta t \longrightarrow 0$, we have $\Delta x \longrightarrow 0$, $\Delta y \longrightarrow 0$, $\epsilon_1 \longrightarrow 0$, $\epsilon_2 \longrightarrow 0$ and $\frac{\Delta x}{\Delta t} \longrightarrow \frac{dx}{dt}$, $\frac{\Delta y}{\Delta t} \longrightarrow \frac{dy}{dt}$. □

For example, if $z = xy$ and $x = \sin t$, $y = \cos t$, then

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Check: $z(t) = \sin t \cos t$. So, $z'(t) = \cos^2 t - \sin^2 t$.

Chain Rule 2

Using a similar argument, we obtain the following result.

Theorem 7: Let $f(x, y)$ be a function, where f_x and f_y are continuous. Suppose $x = x(s, t)$ and $y = y(s, t)$ are functions such that x_s, x_t, y_s and y_t are also continuous. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

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Example 18: Let $z = e^x \sin y,\ x = st^2,\ y = s^2 t$. Then

$$\frac{\partial z}{\partial s} = (e^x \sin y)t^2 + (e^x \cos y)2st = te^{st^2}\left(t \sin(s^2 t) + 2s \cos(s^2 t)\right).$$

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Similar formulas hold for functions of more than two variables.

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Given that $z = f(x, y)$ has continuous second order partial derivatives and that $x = r^2 + s^2$ and $y = 2rs$. Find z_{rr}.

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Functions can be differentiated implicitly.
If F is defined within a sphere S containing a point (a, b, c), where $F(a, b, c) = 0,\ F_z(a, b, c) \neq 0$, and F_x, F_y, F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines a function $z = f(x, y)$ in a sphere containing (a, b, c) and contained in the sphere S. Moreover, with $z = f(x, y)$, we have $z_x = -F_x/F_z,\ z_y = -F_y/F_z$, which are continuous.

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We differentiate 'the equation' with respect to x and y as follows:

$$3x^2 + 3z^2 z_x + 6y(z + xz_x) = 0$$

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Evaluate $\frac{\partial w}{\partial x}(2, -1, 1)$ given that $w = x^2 + y^2 + z^2$ and $z(x, y)$ satisfies $z^3 - xy + yz + y^3 = 1$.

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Evaluating it at $(2, -1, 1)$ gives $\frac{\partial w}{\partial x}(2, -1, 1) = 3$.

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A function $f(x, y)$ is homogeneous of degree n in a region $D \subseteq \mathbb{R}^2$ if for all $(x, y) \in D$, and for each positive λ, $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

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Here, f_x means partial differentiation w.r.t. first variable, similarly, f_y means partial differentiation w.r.t. second variable.
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Since (a,b) is any arbitrary point in D, we have

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

□

Directional Derivative

Recall: If $f(x, y)$ is a function, then $f_x(x_0, y_0)$ is the rate of change in f w.r.t. change in x at (x_0, y_0), that is, in the direction $\hat{\imath}$.

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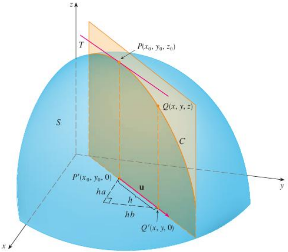

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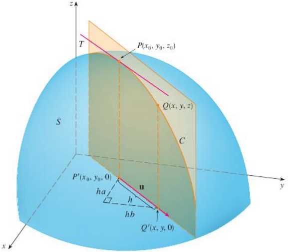

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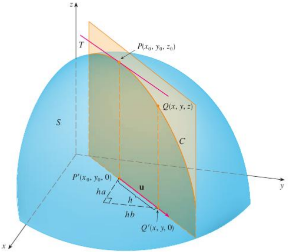

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The slope of the tangent line T to C at P is the rate of change of z in the direction of $\hat{u}$.

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$$D_u z(1, 2) = \lim_{h \to 0} \frac{f(1 + h/\sqrt{2}, 2 + h/\sqrt{2}) - f(1, 2)}{h} = 6/\sqrt{2}.$$

Directional Derivative

Let $f(x, y)$ be a function defined in a region D. Let $(x_0, y_0) \in D$. The directional derivative of $f(x, y)$ in the direction of a unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ at (x_0, y_0) is given by

$$(D_u f)(x_0, y_0) = \left(\frac{df}{ds}\right)_u \Bigg|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

Example 21: Find the derivative of $z = x^2 + y^2$ at $(1, 2)$ in the direction of $\hat{u} = (1/\sqrt{2})\hat{\imath} + (1/\sqrt{2})\hat{\jmath}$.

$$D_u z(1, 2) = \lim_{h \to 0} \frac{f(1 + h/\sqrt{2}, 2 + h/\sqrt{2}) - f(1, 2)}{h} = 6/\sqrt{2}.$$

Notice that
$f_x(1, 2)(1/\sqrt{2}) + f_y(1, 2)(1/\sqrt{2}) = (2 + 2(2))(1/\sqrt{2}) = 6/\sqrt{2}.$

A formula

Theorem 9: If $f(x, y)$ is a function of x and y having continuous partial derivatives f_x and f_y, then f has a directional derivative at (x, y) in any direction $\hat{u} = a\hat{\imath} + b\hat{\jmath}$; and it is given by

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

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Proof: $g(h) = f(x_0 + ah, y_0 + bh)$ is a continuously differentiable function of h. Now,

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$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = D_u f(x_0, y_0)$. Hence

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Example 22: Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ in the direction of the line that makes an angle of $\pi/6$ with the x-axis.

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$$D_u f(x, y) = f_x \cos(\pi/6) + f_y \sin(\pi/6) = \frac{1}{2}[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y].$$

Gradient

The formula for the directional derivative in the direction of the unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ can be written as

$$D_u f = f_x a + f_y b = (f_x \hat{\imath} + f_y \hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath}).$$

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The vector operator $\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$ is called the gradient. The gradient of $f(x, y)$ is given by

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For example, for the function $f(x, y) = xe^y + \cos(xy)$,
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For example, for the function $f(x, y) = xe^y + \cos(xy)$,
$\text{grad}\, f|_{(2,0)} = \hat{\imath} + 2\hat{\jmath}$. Thus, the directional derivative of f in the direction of $3\hat{\imath} - 4\hat{\jmath}$ is $\text{grad}\, f|_{(1,2)} \cdot ((3/5)\hat{\imath} - (4/5)\hat{\jmath}) = -1$.

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How much the value of $y \sin x + 2yz$ change if the point (x, y, z) moves 0.1 units from $(0, 1, 0)$ toward $(2, 2, -2)$?

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$\vec{v} = \overrightarrow{PQ} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$. The unit vector in the direction of $\vec{v}$ is $\hat{u} = \frac{1}{3}\vec{v}$.
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$$\operatorname{grad} f = (y \cos x)\hat{\imath} + (\sin x + 2z)\hat{\jmath} + 2y\,\hat{k}.$$

Then

$$D_u(P) = \operatorname{grad} f|_{(0,1,0)} \cdot \vec{u} = (\hat{\imath} + 2\hat{k}) \cdot \hat{u} = -\frac{2}{3}.$$

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The change df in the direction of $\vec{u}$ in moving $ds = 0.1$ units is approximately

$$df \approx D_u(P)\, ds = -\frac{2}{3}\,(0.1) = -0.067 \text{ units.}$$

Gradient Again

Theorem 10: Let $f(x, y)$ have continuous first order partial derivatives. The maximum value of the directional derivative $D_u f(x, y)$ is $|\text{grad } f|$ and it is achieved when the unit vector $\hat{u}$ has the same direction as that of grad f.

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Since maximum of $\cos\theta$ is 1, maximum of $D_u f$ is $|\text{grad}\, f|$.
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This also says that
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$f(x, y)$ increases most rapidly in the direction of its gradient.
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$f(x, y)$ remains constant in any direction orthogonal to its gradient.

Example 24

Find the directions in which the function $f(x, y) = (x^2 + y^2)/2$ changes most, least, and not at all, at $(1, 1)$.

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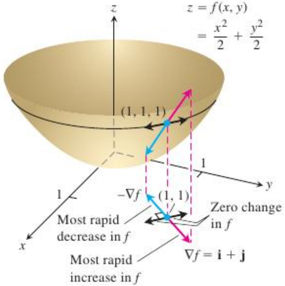
Thus the function $f(x, y)$ increases most at $(1, 1)$ in the direction $(\hat{\imath} + \hat{\jmath})/\sqrt{2}$. It decreases most at $(1, 1)$ in the direction $-(\hat{\imath} + \hat{\jmath})/\sqrt{2}$. And it does not change at $(1, 1)$ in the directions $\pm(\hat{\imath} - \hat{\jmath})/\sqrt{2}$.



Normal to the Level Curve

Let $z = f(x, y)$ be a given surface. Assume that f_x and f_y are continuous. Let c be a number in the range of f. Suppose $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ is a parametrization of the corresponding level curve. Then $f(x(t), y(t)) = c$.

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Differentiating w.r.t. t, we have $\frac{d}{dt}f(x(t), y(t)) = 0$. Or,

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Since $d\vec{r}/dt$ is the tangent to the curve, grad f is the normal to the level curve. That is,

Let $f(x, y)$ have continuous first order partial derivatives. At any point (x_0, y_0) in the domain of $f(x, y)$, its gradient grad f is the normal to the level curve that passes through (x_0, y_0), provided grad f is nonzero at (x_0, y_0).

Gradient Rules

In higher dimensions, if $f(x_1, \ldots, x_n)$ is a function of n independent variables defined on $D \subseteq \mathbb{R}^n$, then its gradient at any point is

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The algebraic rules for the gradient are:

1. Constant multiple: $\operatorname{grad}(kf) = k(\operatorname{grad} f)$ for $k \in \mathbb{R}$.

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$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

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The algebraic rules for the gradient are:

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Tangent Planes

Suppose $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is a smooth curve on the level surface $f(x, y, z) = c$. Then $f(x(t), y(t), z(t)) = c$ for all t. Differentiating this we get

$$\operatorname{grad} f \cdot \vec{r}\,'(t) = 0.$$

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Look at all such smooth curves that pass through a point P on the level surface. The velocity vectors $\vec{r}\,'(t)$ to all these smooth curves are orthogonal to the gradient at the point P.

Let $f(x, y, z)$ have continuous first order partial derivatives. The tangent plane at $P(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of the function $f(x, y, z)$ is the plane through P which is orthogonal to $\operatorname{grad} f$ at P.

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Example 25: Find the tangent plane and the normal line to the surface $x^2 + y^2 + z - 9 = 0$ at the point $(1, 2, 4)$.

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$f_x(1, 2, 4) = 2,\ f_y(1, 2, 4) = 4$ and $f_z(1, 2, 4) = 1$. The tangent plane is

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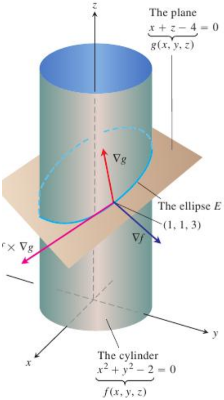
Example 26: Find the tangent plane to the surface $z = x \cos y - ye^x$ at the origin.
$f_x(0, 0) = 1,\ f_y(0, 0) = -1$. The tangent plane is

$$x - y - z = 0.$$

Example 27

Find the tangent line to the curve of intersection of the surfaces $f(x, y, z) := x^2 + y^2 - 2 = 0$ and $g(x, y, z) := x + z - 4 = 0$ at the point $(1, 1, 3)$.



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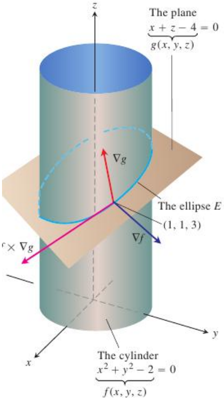

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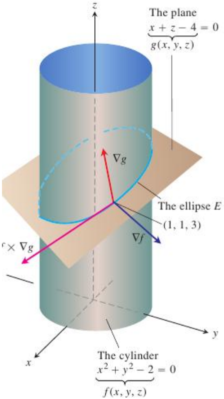

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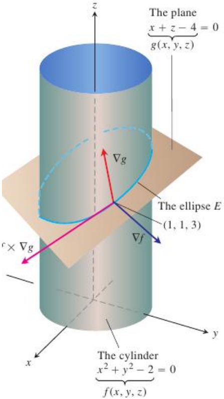

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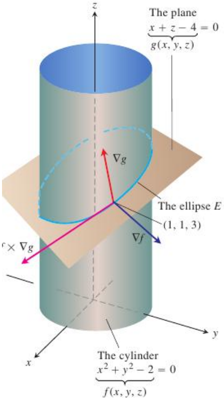

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The tangent line is $x = 1 + 2t,\ y = 1 - 2t,\ z = 3 - 2t$.

Taylor's Theorem

Recall: Let $f : (a, b) \to \mathbb{R}$ be a function having continuous derivatives up to order $n + 1$. Let $x_0 \in (a, b)$. Then for each $x \in (a, b)$, there exists c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{m=1}^{n} \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

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Theorem 11: Let D be a region in $\mathbb{R}^2$. Let (a, b) be an interior point of D. Let $f : D \longrightarrow \mathbb{R}$ have continuous partial derivatives of order up to $n+1$ in some open disk D_0 centered at (a, b) and contained in D. Then for any $(a+h, b+k) \in D_0$, we have

$$\begin{aligned} f(a+h, b+k) &= f(a,b) + \sum_{m=1}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a,b) \\ &+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k) \end{aligned}$$

for some θ with $0 \leq \theta \leq 1$.

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Substitute the expressions for $\phi(1),\ \phi(0),\ \phi_{(m)}(0)$ and $\phi^{(n+1)}(\theta)$. □

Upper bound for the Error

Recall: The standard linearization (linear approximation) of $f(x, y)$ at (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

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The error in the standard linearization at (a, b) is (Taylor's)

$$E(x, y) = f(x, y) - L(x, y) = \frac{1}{2!}\left((x-a)^2 f_{xx} + 2(x-a)(y-b) f_{xy} + (y-b)^2 f_{yy}\right)\Big|_{(c,d)},$$

where $c = a + \theta(x - a),\ d = b + \theta(y - b)$ for some $\theta \in [0, 1]$.

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Theorem 12: Let $f(x, y)$ defined on an open set $D \subseteq \mathbb{R}^2$ have continuous first and second partial derivatives. Let R be a rectangle centered at (x_0, y_0) and contained in D. Suppose there exists an $M \in \mathbb{R}$ such that $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ for all points in R. Then

$$|E(x, y)| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2.$$

Example 28

Find the standard linearization of $f(x, y) = x^2 - xy + y^2/2 - 3$ at $(3, 2)$. Also find an upper bound of the error in the linearization in the rectangle $|x - 3| \leq 0.1,\ |y - 2| \leq 0.1$.

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Then $|E(x,y)| \leq (|x-3| + |y-2|)^2 \leq (0.1+0.1)^2 = 0.04.$

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$|x - 2| \leq 0.01, |y - 1| \leq 0.02, |z| \leq 0.01.$

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All double derivatives are bounded above by 2. So,

$$E(x,y,z)|_P \le \frac{1}{2}(2)(|x-2| + |y-1| + |z|)^2 \le 0.0016.$$

Extreme Values

Let D be a region in $\mathbb{R}^2$, (a, b) an interior point of D, and $f : D \longrightarrow \mathbb{R}$.

We say that $f(x, y)$ has a **local maximum** at (a, b) iff $f(x, y) \leq f(a, b)$ for all $(x, y) \in D$ near (a, b). That is, for all (x, y) in some open disk centered at (a, b) and contained in D, $f(x, y) \leq f(a, b)$.

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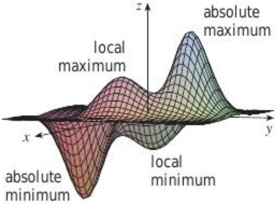

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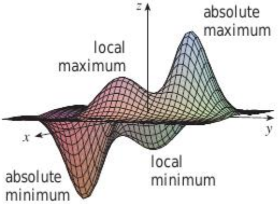

Replace all $\le$ by $\ge$ in the above definition;
and call all those minimum instead of maximum.

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Theorem 13: Let D be a region in $\mathbb{R}^2$; $f : D \to \mathbb{R}$. Let (a, b) be an interior point of D. If f has a local extremum at (a, b), then (a, b) is a critical point of f.

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Geometrically, it says that if at an interior point (a, b), there exists a tangent plane to the surface $z = f(x, y)$, then there exists a horizontal tangent plane to the surface at (a, b).

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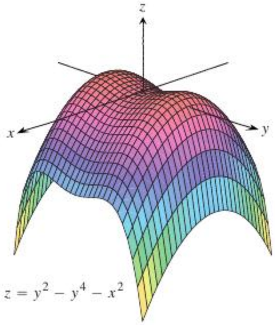

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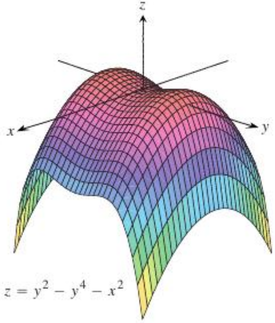

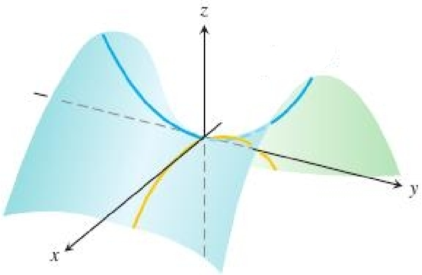

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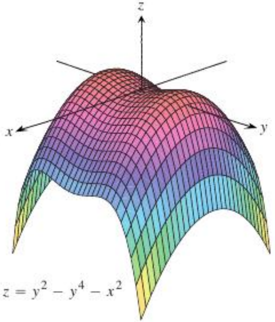

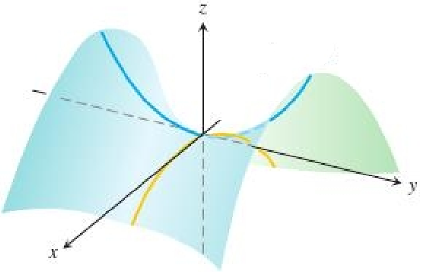

At a saddle point, the function has neither a local maximum nor a local minimum; the surface crosses its tangent plane.

Second Derivative Test

Theorem 14: Let $f : D \to \mathbb{R}$ have continuous first and second partial derivatives in an open disk centered at $(a, b) \in D$. Define the Hessian of f as

$$H(f) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

If $H(f)(a, b) > 0$, then the surface $z = f(x, y)$ curves the same way in all directions near (a, b).

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3. If $H(f)(a, b) < 0$ then f has a saddle point at (a, b).
4. If $H(f)(a, b) = 0$, then nothing can be said, in general.

See the classnotes for proof of (1)-(4).

Proof

Let $(a+h, b+k)$ be in an open disk centered at (a, b) and contained in D.

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By continuity of functions involved, $f_{xx}(a+\theta h, b+\theta k) < 0$. The RHS is positive. Therefore, $f(a+h, b+k) - f(a, b) < 0$. That is, (a, b) is a local maximum point.

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(3A) $f_{xx}(a,b) \neq 0$. (3B) $f_{yy}(a,b) \neq 0$, (3C) $f_{xx}(a,b) = f_{yy}(a,b) = 0$.

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Since $H(f)(a,b) < 0$, these two limits have opposite signs. Thus, for small values of t, $f(a+h,b+k) - f(a,b)$ will have opposite signs.

(3B) Let $H(f)(a,b) < 0$ and $f_{yy}(a,b) \neq 0$. This is similar to (3A).

(3C) Let $H(f)(a,b) < 0$ and $f_{xx}(a,b) = f_{yy}(a,b) = 0$.

First, set $h = k = t$. Then

$$\lim_{t\to 0} \frac{f(a+h,b+k) - f(a,b)}{t^2} = f_{xy}(a,b).$$

Next, set $h = t,\ k = -t$. Then

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As in (3A), we conclude that for small values of t, $f(a+h,b+k) - f(a,b)$ will have opposite signs. □

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Notice that the case $H(f)(a,b) > 0$ and $f_{xx}(a,b) = 0$ is not possible.

Example 31

Find the extreme values of $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.

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$f_{xx}(-2, -2) = -2$, $f_{xy}(-2, -2) = 1$, $f_{yy}(-2, -2) = -2$.

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$f_{xx}(-2, -2) = -2$, $f_{xy}(-2, -2) = 1$, $f_{yy}(-2, -2) = -2$.

Then $H(f)(-2, -2) = 3 > 0$, $f_{xx} < 0$.

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Here also f has absolute maximum and the maximum value is $f(-2, -2) = 8$.

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Investigate $f(x, y) = x^4 + y^4 - 4xy + 1$ for extreme values.

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The critical points are at (x, y) where $f_x = 4x^3 - 4y = 0 = f_y = 4y^3 - 4x$.

That is, when $x^3 = y$ and $y^3 = x$. Giving $x^9 = x$ which has solutions $x = 0, 1, -1$ in $\mathbb{R}$. The corresponding y values are $0, 1, -1$.

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Now, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Thus $H(f) = 144x^2y^2 - 16$.

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At $x = 0, y = 0$, $H(f) = -16$. Thus f has a saddle point at $(0, 0)$.

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At $x = 1, y = 1$, $H(f) > 0$, $f_{xx} > 0$. Thus f has a local minimum at $(1, 1)$.

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At $x = -1, y = -1$, $H(f) > 0$, $f_{xx} > 0$. Thus f has a local minimum at $(-1, -1)$.

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Investigate $f(x, y) = x^4 + y^4 - 4xy + 1$ for extreme values.

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At $x = -1, y = -1$, $H(f) > 0$, $f_{xx} > 0$. Thus f has a local minimum at $(-1, -1)$.

The local minimum values are $f(1, 1) = -1$ and $f(-1, -1) = -1$. Both are absolute minima.

Example 33

Find absolute extrema of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ defined on the triangular region bounded by the straight lines $x = 0$, $y = 0$, and $x + y = 9$.

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This accounts for the interior points of the region.

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2. Draw the picture. The vertices of the triangle are $A(0, 0)$, $B(0, 9)$, $C(9, 0)$. These are possible extremum points.
This accounts for the vertices which are on the boundary.

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3(a). On the line segment AB, f is given by $(x = 0)$:
$g(y) = f(0, y) = 2 + 2y - y^2$ for $0 \leq y \leq 9$.

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Taking $g'(y) = 0$, we see that $y = 1$.
Thus, a possible extremum point is $(0, 1)$.

Example 33 Contd.

3(b). Similarly, on the line segment AC, f is given by $(y = 0)$:
$g(x) = f(x, 0) = 2 + 2x - x^2$ for $0 \leq x \leq 9$.

Example 33 Contd.

3(b). Similarly, on the line segment AC, f is given by ($y = 0$):
$g(x) = f(x, 0) = 2 + 2x - x^2$ for $0 \leq x \leq 9$. Now, $g'(x) = 0 \Rightarrow x = 1$.
Thus $(1, 0)$ is another possible extremum point.

Example 33 Contd.

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3(c). On the line segment BC, f is given by $(x + y = 9)$:
$g(x) = f(x, 9 - x) = -61 + 18x - 2x^2$ for $0 \leq x \leq 9$.

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$g(x) = f(x, 9 - x) = -61 + 18x - 2x^2$ for $0 \leq x \leq 9$.
$g'(x) = 0$ implies that $x = 9/2$, $y = 9 - x = 9/2$.
Thus $(9/2, 9/2)$ is a possible extremum point.

Example 33 Contd.

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Thus $(9/2, 9/2)$ is a possible extremum point.

The values at these possible extrema are

$f(1, 1) = 4, f(0, 0) = 2, f(0, 9) = -61, f(9, 0) = -61, f(1, 0) = 3,$
$f(0, 1) = 3, f(9/2, 9/2) = -41/2.$

Example 33 Contd.

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$g(x) = f(x, 0) = 2 + 2x - x^2$ for $0 \le x \le 9$. Now, $g'(x) = 0 \Rightarrow x = 1$.
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Therefore, f has absolute minimum at $(0, 9)$ and $(9, 0)$ and its minimum value is -61.

Example 33 Contd.

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Therefore, f has absolute minimum at $(0, 9)$ and $(9, 0)$ and its minimum value is -61.

It has absolute maximum at $(1, 1)$ and its maximum value is 4.

Example 34

Maximize the volume of a box of length x, width y and height z subject to the condition that $x + 2y + 2z = 108$.

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Maximize the volume of a box of length x, width y and height z subject to the condition that $x + 2y + 2z = 108$.

$f := V = xyz = (108 - 2y - 2z)yz$. Then

$$f_y = (108 - 4y - 2z)z, \quad f_z = (108 - 2y - 4z)y.$$

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$f := V = xyz = (108 - 2y - 2z)yz$. Then

$$f_y = (108 - 4y - 2z)z, \quad f_z = (108 - 2y - 4z)y.$$

Thus the critical points (where $f_y = 0 = f_z$) are $(0, 0)$, $(0, 54)$, $(54, 0)$ and $(18, 18)$. The volume is 0 at the first three points. The only possibility is $(18, 18)$. To see that this a point where f is maximum, consider

$$f_{yy} = -4z, \; f_{yz} = 108 - 4y - 4z, \; f_{zz} = -4y.$$

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$$f_{yy} = -4z, \; f_{yz} = 108 - 4y - 4z, \; f_{zz} = -4y.$$

At $(18, 18)$, that is, when $y = z = 18$, $f_{yy} < 0$, and

$$H(f) = f_{yy} - f_{zz} - f_{yz}^2 = 16 \times 18 \times 18 - 16(-9)^2 > 0.$$

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$$f_{yy} = -4z, \; f_{yz} = 108 - 4y - 4z, \; f_{zz} = -4y.$$

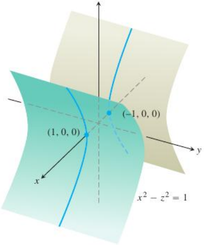
At $(18,18)$, that is, when $y = z = 18$, $f_{yy} < 0$, and

$$H(f) = f_{yy} - f_{zz} - f_{yz}^2 = 16 \times 18 \times 18 - 16(-9)^2 > 0.$$

Hence the volume of the box is maximum when its length is $108 - 36 - 36 = 36$, width is 18 and height is 18 units.
The maximum volume is 11664 cubic units.

Example 35

Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 = 1$.



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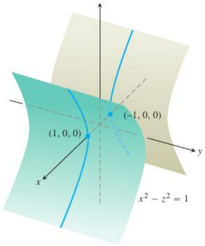

We seek a point (x, y, z) that minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - z^2 = 1$.

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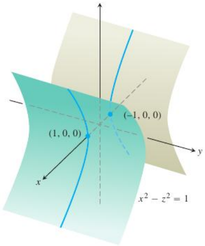
Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 = 1$.



We seek a point (x, y, z) that minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - z^2 = 1$.

As earlier, taking $z^2 = x^2 - 1$, we seek (x, y) such that $g(x, y) = x^2 + y^2 + x^2 - 1 = 2x^2 + y^2 - 1$ is minimum.

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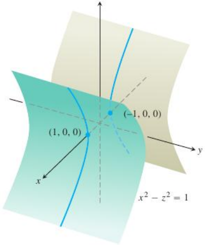

We seek a point (x, y, z) that minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - z^2 = 1$.
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$g_x = 4x,\ g_y = 2y$. But $x = 0$ does not correspond to any point on the surface $x^2 - z^2 = 1$.
So, the method fails!

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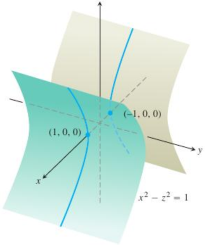

We seek a point (x, y, z) that minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - z^2 = 1$.
As earlier, taking $z^2 = x^2 - 1$, we seek (x, y) such that $g(x, y) = x^2 + y^2 + x^2 - 1 = 2x^2 + y^2 - 1$ is minimum.
$g_x = 4x,\ g_y = 2y$. But $x = 0$ does not correspond to any point on the surface $x^2 - z^2 = 1$.
So, the method fails!
Had you eliminated x, your $g(y, z)$ would have been $y^2 + 2z^2 + 1$. And $g_y = 0 = g_z$ would have given a point on the surface.

An Observation

Let S be a surface given by $g(x, y, z) = 0$.
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Let C be a curve given by $\vec{r}(t) = x(y)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ that lies on S and passes through P. Let $P = \vec{r}(t_0)$.

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A result

Our discussion may be summarized as follows:

Theorem: Let $D \subseteq \mathbb{R}^2$ be a region. Let $f, g : D \to \mathbb{R}^2$ have continuous first order partial derivatives. If $g_x^2 + g_y^2 > 0$ for all $(x, y) \in D$, then each point (a, b) on the curve $g(x, y) = 0$, where $f(x, y)$ has maxima or minima corresponds to a solution (a, b, λ) of the system of equations

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The points thus obtained by solving the above equations give possible points where the extrema may be achieved. Other verifications are required to determine whether they are actually maxima or minima.

Example 35 Contd.

$$f(x, y, z) = x^2 + y^2 + z^2, \ g(x, y, z) = x^2 - z^2 - 1.$$

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$f_x + \lambda g_x = 2x + \lambda 2x = 0, \; f_y + \lambda g_y = 2y = 0,$

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Thus the points closest to the origin on the cylinder are $(\pm 1, 0, 0)$.

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Notice that if we set $F(x, y, z, \lambda) := f(x, y, z) + \lambda g(x, y, z)$, then

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We get all our required equations as earlier.

Lagrange Multipliers

Requirement: Find extrema of the function $f(x_1, \ldots, x_n)$ subject to the conditions $g_1(x_1, \ldots, x_n) = 0, \; \cdots, \; g_m(x_1, \ldots, x_n) = 0.$

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Remember that the method succeeds under the condition that such extreme values exist where grad $g \neq 0$.

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Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $g(x, y, z) := x - y + z - 1 = 0$ and the cylinder $h(x, y, z) := x^2 + y^2 - 1 = 0$.

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The corresponding values of $f(x_0, y_0, z_0)$ show that the maximum value of f is $3 + \sqrt{29}$.

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$$L = \lim_{y=-x, x\longrightarrow 0} f(x,y) = \lim_{x\longrightarrow 0} \frac{-2x^2}{2x^2} = -1$$

It is a contradiction.

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It is known that in computing the co-ordinates of a point (x, y, z, t) certain (small) errors such as $\Delta x, \Delta y, \Delta z, \Delta t$ might have been committed. Find the maximum absolute error so committed when we evaluate a function $f(x, y, z, t)$ at that point.

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Determine the directions in which the function

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Determine the directions in which the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

has directional derivatives at $(0,0)$.

Consider a unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$.

At $(0,0)$, the directional derivative of $f(x,y)$ is

$$\lim_{h\to 0} \frac{f(ah,bh) - f(0,0)}{h} = \lim_{h\to 0} \frac{ab^2}{a^2 + bh^2} = \begin{cases} b^2/a & \text{for } a \neq 0 \\ 0 & \text{for } a = 0. \end{cases}$$

Hence directional derivatives of $f(x,y)$ at $(0,0)$ exist in all directions.

However, grad f at $(0,0)$ is $0\hat{\imath} + 0\hat{\jmath}$. If you use the formula blindly, then the directional derivative of $f(x,y)$ at $(0,0)$ in any direction would turn out to be 0. Why is it wrong?

Problem 5

The hypotenuse c and the side a of a right angled triangle ABC determined with maximum absolute errors $|\Delta c| = 0.2, \ |\Delta a| = 0.1$ are, respectively, $c = 75, \ a = 32$. Determine the angle A from the formula $A = \sin(a/c)$ and determine the maximum absolute error ΔA in the calculation of the angle A.

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Therefore

$$\sin^{-1} \frac{32}{75} - 0.00273 \leq A \leq \sin^{-1} \frac{32}{75} + 0.00273.$$

Problem 6

Let $f(x, y, z) = x^2 + y^2 + z^2$. Find $\left(\frac{\partial f}{\partial s}\right)_{\vec{v}}(1, 1, 1)$, where $\vec{v} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

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$$\operatorname{grad} f(1,1,1) = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k})(1,1,1) = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}.$$

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Then

$$\left(\frac{\partial f}{\partial s}\right)_{\vec{v}}(1, 1, 1) = (\operatorname{grad} f \cdot \hat{u})(1, 1, 1) = \frac{12}{\sqrt{14}}.$$

Problem 7

Find a point in the plane where the function $f(x, y) = \frac{1}{2} - \sin(x^2 + y^2)$ has a local maximum.

We see that at $(0, 0)$, the function has a maximum value of $\frac{1}{2}$.

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Therefore, $(0, 0)$ is a local maximum point of $f(x, y)$.

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Comparing $f(P_1),\ f(P_2),\ f(P_3), f(P_4)$, we get the required decomposition of a as $a = \frac{a}{3} + \frac{a}{3} + \frac{a}{3}$.

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Hence P_2 is a saddle point of the surface.

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Since $x > 0, y > 0, z > 0$, these equations imply $x = y = z$. Then $xy + zx + yz = a$ gives $x = y = z = \sqrt{a/3}$.
The corresponding value of w cannot be minimum, since by reducing x, y close to 0, and taking z close to a so that $xy + zx + yz = a$ is satisfied, w can be made as small as possible.

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The auxiliary function is $F(x, y, z, \lambda) = xyz + \lambda(xy + zx + yz - a) = 0$. Equating its partial derivatives to zero, we have

$$yz + \lambda(y + z) = 0, \; xz + \lambda(x + z) = 0, \; xy + \lambda(x + y) = 0.$$

Multiply the first by x, the second by y, and the third by z and add to obtain: $3xyz + 2\lambda(xy + zx + yz) = 0$.
Since $xy + zx + yz = a$, we have $\lambda = -\frac{3xyz}{2a}$. Substitute this value of λ in the equations above to get

$$yz(2a - 3x(y + z)) = 0, \; xz(2a - 3y(x + z)) = 0, \; xy(2a - 3z(x + y)) = 0.$$

Since $x > 0, y > 0, z > 0$, these equations imply $x = y = z$. Then $xy + zx + yz = a$ gives $x = y = z = \sqrt{a/3}$.
The corresponding value of w cannot be minimum, since by reducing x, y close to 0, and taking z close to a so that $xy + zx + yz = a$ is satisfied, w can be made as small as possible. Hence w has a maximum at $(\sqrt{a/3}, \sqrt{a/3}, \sqrt{a/3})$. Then $w_{max} = (a/3)^{3/2}$.

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This gives an alternative proof that the geometric mean of n positive numbers is no more than the arithmetic mean of those numbers.