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MA2020 Classnotes
Differential Equations

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1

First Order ODE

1.1 Introduction

A **differential equation** is an equation that involves derivatives of a variable that depends on other independent variables. For instance,

$$\frac{dy}{dx} = 3x^2 \sin(x+y), \qquad \frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)^4 - y = 0$$

are differential equations, where the variable y is supposed to be the variable that depends on the independent variable x. When the equation involves only one independent variable, the equation is said to be an *ordinary differential equation*, an **ODE**.

The **order** of an ODE is the order of the highest derivative of the dependent variable. In the above equations, the first one is of first order and the second one is of third order.

A **solution** of an ODE is a function which when replaces the dependent variable, it is seen that the equation is satisfied. If the dependent variable is y and the independent variable is x in an ODE of order k, then a solution of such an equation is $y = y(x)$ which is k times differentiable and which satisfies the given equation. For example, $y(x) = 2\sin x - \frac{1}{3}\cos(2x)$ is a solution of the ODE

$$\frac{d^2y}{dx^2} + y = \cos(2x).$$

(We also write y' for dy/dx, $y^{(n)}$ for d^ny/dx^n etc.) This claim is verified as follows:

$$\begin{aligned} y = 2\sin x - \frac{1}{3}\cos(2x) \;&\Rightarrow\; y' = 2\cos x + \frac{2}{3}\sin(2x) \\ &\Rightarrow\; y'' = -2\sin x + \frac{4}{3}\cos(2x) = -y + \cos(2x). \end{aligned}$$

Often an ODE comes with the restriction that the independent variable varies in a particular subset of $\mathbb{R}$. In that case, the domain of the dependent variable is *assumed* to be that subset. For example, in the ODE

$$xy' + y = 0, \quad x \neq 0$$

it is assumed that the domain of $y = y(x)$ is $\mathbb{R} \setminus \{0\}$. In this case, the function $y(x) = 1/x$ is a solution. Reason:

$$y = 1/x \implies xy' + y = x\frac{d(1/x)}{dx} + \frac{1}{x} = x\Big(-\frac{1}{x^2}\Big) + \frac{1}{x} = 0 \quad \text{for } x \neq 0.$$

It is easy to see that $y(x) = c/x$ for any $c \in \mathbb{R}$ is also a solution. In such a case, we say that c is an arbitrary constant.

Are there other *types* of solutions to this equation? Well, suppose, $y(x)$ is a solution of $xy' + y = 0$. Write $z(x) = xy$. Then

$$\frac{dz}{dx} = xy' + 1 \cdot y = 0 \implies z(x) = c \implies xy = c \implies y = c/x.$$

That is, any solution of $xy' + y = 0$ is in the form $y = c/x$ for $x \neq 0$.

Observe that a solution of an ODE need not be unique. However, if we have another condition on the function $y(x)$ such as $y(1) = 1$, then substituting $x = 1$ in our solution $y = c/x$, we have

$$1 = y(1) = c/1 = c.$$

We thus obtain the unique solution $y = 1/x$. The condition $y(1) = 1$ is called an **initial condition** for the ODE. In fact, when a condition on the dependent variable is given by prescribing its value at a single point, it is called an initial condition. An ODE with a given initial condition is called an **initial value problem**, an **IVP**.

It follows that $y = c/x = 1/x$ is the only solution to the initial value problem

$$xy' + y = 0, \quad y(1) = 1, \quad x \neq 0.$$

A general first order ODE may be given by an equation using x, y, y', which would then look like

$$h(x, y, y') = 0$$

for some specific expression $h(\cdot, \cdot, \cdot)$. For simplicity, we may only consider equations which can be solved for y'; that is, an ODE in the form:

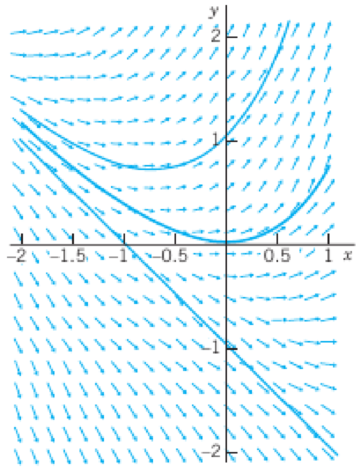
$$y' = g(x, y)$$

with a given domain, a subset of $\mathbb{R}$, where x varies. Geometrically, consider the xy-plane. At a particular point (with an admissible x-value), say (x_0, y_0), the ODE gives the value of y'. That is, $y'(x_0) = g(x_0, y_0)$; it is a number which represents the slope of the tangent to $y = y(x)$ at $x = x_0$. By varying x throughout its domain and with all possible values of y, the ODE prescribes slopes at each admissible point. The set of all these slopes is called the *direction field* for the ODE.

By joining these slopes geometrically we may get many solution curves $y = y(x)$ to the ODE. In general, we *accept continuous curves in the xy-plane* as solutions to

ODEs rather than functions $y = y(x)$. Once an initial value is prescribed, whenever there exists a unique solution, we would obtain only one solution curve that passes through the point (x_0, y_0).

The direction field for the ODE $y' = x + y$ is plotted in the following figure. Also plotted are three approximate solution curves passing through the points $(0, 1)$, $(0, 0)$ and $(0, -1)$, respectively.



It is a fact that even all initial value problems do not have unique solutions. Anything can happen. There are IVPs having no solutions, having more than one solutions, and there are IVPs having a unique solution. We will use the following result without proof.

(1.1) ***Theorem*** (Existence-Uniqueness)

Let $g(x, y)$ *and* $\dfrac{\partial g}{\partial x}$ *be continuous in the rectangle* $R: \ x_0 \le x \le x_0 + a, \ |y - y_0| \le b$. *Compute* $M = \max\{|g(x, y)| : (x, y) \in R\}$ *and* $\alpha = \min\{a, b/M\}$. *Then, the IVP* $y' = g(x, y), \ y(x_0) = y_0$ *has a unique solution in the interval* $x_0 \le x \le x_0 + \alpha$.

(1.2) ***Example***

Consider the IVP: $y' = \sin(2x) y^{1/3}, \ y(0) = 0$.
It has a solution as $y(x) = 0$, the zero function.
Verify that $y(x) = \pm\sqrt{8/27}\sin^3 x$ are solutions of the same IVP.
Notice that $f(x, y) = \sin(2x) y^{1/3}$ has no partial derivative at $y = 0$. ▯

1.2 Variables Separable

Sadly, all ODEs of the form $y' = g(x, y)$ cannot be solved since it would ask us to integrate $g(x, y)$ with respect to x, where y is an unknown function of x. A simpler case, which we may think of solving is when $g(x, y)$ is a function of x alone. So, we consider an ODE in the form

$$y' = f(x).$$

Of course, we cannot even solve all equations in this form. For instance, we do not know how to solve

$$y' = e^{x^2}$$

since our data base for integrating algebraic expressions does not include such a function. Knowing this fact very well, we will attempt solving first order ODEs; in fact, whichever we can. In general, if we know how to integrate the function $f(x)$, we can get a solution of the ODE. In fact,

$$y' = f(x) \;\Rightarrow\; y(x) = \int f(x)\, dx.$$

For example, the ODE $y' = x^r$ for $r > -1$ may be solved by taking

$$y = \int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad \text{for an arbitrary constant } C.$$

We can slightly generalize this method to solve most ODEs in the form

$$g(y)y' = f(x) \tag{1.2.1}$$

by using the differentials. Recall that $dy = y'\, dx$. Using this, we obtain

$$g(y)y' = f(x) \;\Rightarrow\; \int g(y)\, dy = \int g(y)y'\, dx = \int f(x)\, dx.$$

This amounts to the following formal manipulation:

$$g(y)\frac{dy}{dx} = f(x) \;\Rightarrow\; g(y)\, dy = f(x)\, dx \;\Rightarrow\; \int g(y)\, dy = \int f(x)\, dx.$$

Of course, we also add an arbitrary constant to the result of any one integral. This is the reason, the ODE in (1.2.1) is called a *variables separable ODE*, and this method is called the *method of variables separable*. The solution so obtained this way is called the *general solution* of the ODE (1.2.1) since any solution can be put in this form.

(1.3) *Example*

Find the general solutions to the following ODEs:
(a) $y' = x^2/y^2$ (b) $y' = 1 + y^2$ (c) $y' = (x+1)e^{-x}y^2$.

(a) Separating the variables, we have $y^2 y' = x^2$. Integrating,

$$\int y^2\,dy = \int x^2\,dx \;\Rightarrow\; \frac{y^3}{3} = \frac{x^3}{3} + C_1 \;\Rightarrow\; y^3 = x^3 + C.$$

(b) $y' = 1 + y^2 \;\Rightarrow\; (1+y^2)^{-1}y' = 1 \;\Rightarrow\; \displaystyle\int \frac{dy}{1+y^2} = \int 1\,dx \;\Rightarrow\; \tan^{-1} y = x + C$
$\;\Rightarrow\; y = \tan(x + C)$.

(c) $y' = (x+1)e^{-x}y^2 \;\Rightarrow\; y^{-2}y' = (x+1)e^{-x} \;\Rightarrow\; \displaystyle\int y^{-2}\,dy = \int (x+1)e^{-x}\,dx$
$\;\Rightarrow\; -y^{-1} = -(x+2)e^{-x} + C \;\Rightarrow\; y = [(x+2)e^{-x} - C]^{-1}$. ∎

(1.4) ***Example***

Find solutions to the following IVPs:
(a) $y' = -2xy,\ y(0) = 1.8$ (b) $e^y y' = x + x^3,\ y(1) = 1$.

(a) $y' = -2xy \;\Rightarrow\; y^{-1}y' = -2x \;\Rightarrow\; \displaystyle\int y^{-1}\,dy = \int (-2x)\,dx \;\Rightarrow\; \log|y| = -x^2 + C_1$
$\;\Rightarrow\; |y| = e^{-x^2+C_1} \;\Rightarrow\; |y| = Ce^{-x^2} \;\Rightarrow\; y = \pm Ce^{-x^2}$.

As C is an arbitrary constant, which may be any real number, $y = Ce^{-x^2}$.
Then $y(0) = 1.8 \;\Rightarrow\; Ce^0 = 1.8 \;\Rightarrow\; C = 1.8$. Hence, $y(x) = 1.8\,e^{-x^2}$.

(b) $e^y y' = x + x^3 \;\Rightarrow\; \displaystyle\int e^y\,dy = \int (x + x^3)\,dx \;\Rightarrow\; e^y = \frac{x^2}{2} + \frac{x^4}{4} + C$
$\;\Rightarrow\; y = \log\left(\dfrac{x^2}{2} + \dfrac{x^4}{4} + C\right)$ for $C \ge 0$.

$y(1) = 1 \;\Rightarrow\; e^1 = \frac{1}{2} + \frac{1}{4} + C \;\Rightarrow\; C = e - \frac{3}{4}$. Hence, the solution is
$y(x) = \log\left(\dfrac{x^2}{2} + \dfrac{x^4}{4} + e - \dfrac{3}{4}\right)$. ∎

Most often, the differential equation does not signal in any way that its solutions are not defined at certain points. This can even happen for IVPs.

(1.5) ***Example***

Solve the IVPs: (a) $y' = 1 + y^2,\ y(0) = 0$ (b) $y' = 1 + y^2,\ y(0) = 1$.

(a) $y' = 1 + y^2 \;\Rightarrow\; \displaystyle\int (1+y^2)^{-1}\,dy = \int dx \;\Rightarrow\; \tan^{-1} y = x + C \;\Rightarrow\; y = \tan(x + C)$.

$y(0) = 0 \;\Rightarrow\; 0 = \tan(C) \;\Rightarrow\; C = 0$. Hence, the solution is $y = \tan x$.

This solution is not defined at $x = \pm\pi/2$. Yet, the ODE does not signal anything about this! The solution exists in $(-\pi/2, \pi/2)$.

(b) As in (a), $y = \tan(x + C)$. $y(0) = 1 \;\Rightarrow\; 1 = \tan C \;\Rightarrow\; C = \pi/4$. So, the solution is $y = \tan(x + \pi/4)$. Again, this solution exists in $(-3\pi/4, \pi/4)$. ∎

(1.6) ***Example***

Find the solution of the IVP $y' = (1+y)x,\ y(0) = -1$.

$$y' = (1+y)x \;\Rightarrow\; \int \frac{dy}{1+y} = \int x\,dx \;\Rightarrow\; \log|1+y| = \frac{x^2}{2} + C.$$

This solution is defined for $y \neq -1$. But the initial condition says otherwise. We observe that $y(x) = -1$ is a solution. Due to our existence-uniqueness theorem, $y(x) = -1$ is the only solution of the IVP. ∎

(1.7) ***Example***

Find the solution to the IVP $yy' + (1+y^2)\sin x = 0,\ y(0) = 1$.

$$yy' + (1+y^2)\sin x = 0 \;\Rightarrow\; y(1+y^2)^{-1}y' = -\sin x$$
$$\Rightarrow\; \int \frac{2y\,dy}{1+y^2} = -\int 2\sin x \;\Rightarrow\; \log(1+y^2) = 2\cos x + C.$$

$y(0) = 1 \;\Rightarrow\; \log 2 = 2 + C \;\Rightarrow\; C = \log 2 - 2$.

So, $\log(1+y^2) = 2\cos x + \log 2 - 2$. Or,

$$y^2 = e^{2\cos x - 2 + \log 2} - 1 = 2e^{2(\cos x - 1)} - 1 = 2e^{-4\sin^2(x/2)} - 1.$$

Since $y(0) > 0$, we take the positive sign in the square root. That is,

$$y = \sqrt{2e^{-4\sin^2(x/2)} - 1}.$$

This solution is defined for $2e^{-4\sin^2(x/2)}$. However,

$$2e^{-4\sin^2(x/2)} \geq 1 \Leftrightarrow e^{-4\sin^2(x/2)} \geq 1/2 \Leftrightarrow e^{4\sin^2(x/2)} \leq 2$$
$$\Leftrightarrow 4\sin^2(x/2) \leq \log 2 \Leftrightarrow |x/2| \leq \sin^{-1}\frac{\sqrt{\log 2}}{2}.$$

That is, the solution exists in the interval $(-a, a)$, where $a = \sin^{-1}\frac{\sqrt{\log 2}}{2}$. ∎

(1.8) ***Example***

Find all solutions of $y' = -x/y$.

$$y' = -x/y \;\Rightarrow\; \int y\,dy = -\int x\,dx \;\Rightarrow\; \frac{y^2}{2} = -\frac{x^2}{2} + C_1 \;\Rightarrow\; x^2 + y^2 = C.$$

In this case, we cannot find y as a function of x. However, the solutions are solution curves in the xy-plane. ∎

(1.9) ***Example***

Solve the IVP $(1+e^y)y' = \cos x,\ y(\pi/2) = 3$.

$$(1+e^y)y' = \cos x \;\Rightarrow\; \int (1+e^y)\,dy = \int \cos x\,dx \;\Rightarrow\; y + e^y = \sin x + C.$$

$y(\pi/2) = 3 \implies 3 + e^3 = 1 + C \implies C = 2 + e^3$. So, the solution is given by

$$y + e^y = \sin x + 2 + e^3.$$

Here, we cannot express y in terms of x explicitly. In general, we accept solutions given implicitly. □

1.3 Reducible to variables separable

Sometimes we use a suitable substitution so that a given ODE will become amenable to the variables separable method. A specific case is when the ODE looks like

$$y' = f(y/x),$$

where the right hand side is a an expression depending directly on y/x. In this case, we substitute $u = y/x$. Then, $u = y/x \implies y = ux \implies y' = u'x + u$. The ODE becomes

$$u'x + u = f(u) \implies \frac{du}{dx} = f(u) - u \implies \int \frac{du}{f(u) - u} = \int \frac{dx}{x}.$$

In fact, we do not remember the last formula. It only shows that the substitution $y = ux$ reduces the ODE to a case of variables separable.

(1.10) ***Example***

Consider the ODE $2xyy' = y^2 - x^2$.

Here, $y' = \dfrac{y^2 - x^2}{2xy} = \dfrac{y}{2x} - \dfrac{x}{2y}$. Take $y = ux$ to get

$$u'x + u = \frac{u}{2} - \frac{1}{2u} \implies u'x = -\frac{1 + u^2}{2u} \implies \frac{2u}{1 + u^2}\frac{du}{dx} = -\frac{1}{x}.$$

Integrating, we obtain

$$\int \frac{2u}{1 + u^2}\, du = -\int \frac{dx}{x} \implies \log(1 + u^2) = -\log|x| + C \implies 1 + u^2 = \frac{C}{x}.$$

Since C is arbitrary, we write C/x instead of $C/|x|$.
Substituting back $u = y/x$, we get $1 + (y/x)^2 = C/x$ or, $x^2 + y^2 = Cx$ or,

$$\left(x - \frac{C}{2}\right)^2 + y^2 = \frac{C^2}{4}.$$

The solutions comprise a family of circles passing through the origin with center on the x-axis. ∎

Another type of ODEs can be reduced to variables separable form. They are equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}.$$

Here, a, b, c, a', b', c' are some real numbers. We consider two cases.

Case 1: Suppose the coefficients of x and y are in ratio. That is, $\frac{a}{a'} = \frac{b}{b'}$.

In this case, the ODE is in the form

$$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c'}.$$

We substitute $u = ax + by$ so that $u' = a + by'$ and the ODE is reduced to

$$u' = a + by' = a + b\,\frac{u + c}{mu + c'}.$$

Here, the variables are separated.

Case 2: Suppose the the coefficients of x and y are not in ratio. That is, $\frac{a}{a'} \neq \frac{b}{b'}$.

In this case, we shift both the independent and dependent variables; that is, we take $x = X + h$ and $y = Y + k$ for some constants h, k to be determined suitably.

With this change of variables, we have

$$\frac{dY}{dX} = \frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} = \frac{aX + bY + ah + bk + c}{a'X + b'Y + a'h + b'k + c'}.$$

The trick is to take h, k in such a way that the last expression is simplified. So, we take

$$ah + bk + c = 0, \quad a'h + b'k + c' = 0. \tag{1.3.1}$$

Then, the ODE is simplified to

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y} = \frac{a + b(Y/X)}{a' + b'(Y/X)} = f(Y/X).$$

Now, the earlier method of substituting $Y = uX$ will separate the variables. Then, solution curves will be obtained in the form $g(X, Y) = 0$, or, $g(x - h, y - k) = 0$.

(1.11) *Example*

Solve the ODE $(x + y - 1)y' = 2x + 2y + 1$.

Here, the coefficients of x, y are in ratio. We substitute $u = x + y$ so that $u' = 1 + y'$, and the ODE is reduced to

$$u' - 1 = \frac{2u+1}{u-1} \;\Rightarrow\; \frac{du}{dx} = u' = \frac{2u+1}{u-1} + 1 = \frac{3u}{u-1}.$$

Integrating, we get

$$\int \frac{u-1}{3u}\,du = \int dx \;\Rightarrow\; \frac{u}{3} - \frac{1}{3}\log|u| = x + C_1.$$

Substituting $u = x + y$ and simplifying we obtain

$$y - 2x - C_2 = \log|x+y| \;\Rightarrow\; Ce^{y-2x} = x + y$$

for an arbitrary constant C. This gives the solution curves of the ODE. ∎

(1.12) ***Example***

Solve the ODE $(3y - 7x + 7) + (7y - 3x + 3)y' = 0$.

The ODE is $y' = \dfrac{7x - 3y - 7}{-3x + 7y + 3}$. The coefficients of x, y in both linear expressions are not in ratio. So, we substitute $x = X + h,\ y = Y + k$. Equation 1.3.1 gives $7h - 3k - 7 = 0 = -3h + 7k + 3$. Solving these, we get $h = 1,\ k = 0$. That is, we take $x = X + 1,\ y = Y$ so that the ODE is reduced to

$$\big[3Y - 7(X+1) + 7\big] + \big[7Y - 3(X+1) + 3\big]\frac{dY}{dX} = 0 \;\Rightarrow\; \frac{dY}{dX} = \frac{7X - 3Y}{-3X + 7Y} = \frac{7 - 3(Y/X)}{-3 + 7(Y/X)}.$$

Substitute $Y = uX$ so that $\dfrac{dY}{dX} = \dfrac{du}{dX}X + u = \dfrac{7 - 3u}{-3 + 7u}$. This gives

$$\frac{du}{dX}X = \frac{7-3u}{-3+7u} - u = \frac{7 - 3u + 3u - 7u^2}{-3+7u} = \frac{7 - 7u^2}{-3+7u}.$$

Separating the variables, we obtain $\displaystyle\int \frac{7u-3}{7-7u^2}\,du = \int \frac{dX}{X}$.

Now, $\dfrac{7u-3}{7-7u^2} = -\dfrac{1}{2}\dfrac{2u}{u^2-1} + \dfrac{3}{14}\dfrac{1}{u-1} - \dfrac{3}{14}\dfrac{1}{u+1}$. Then the above gives

$$-\frac{1}{2}\log|u^2 - 1| + \frac{3}{14}\log|u-1| - \frac{3}{14}\log|u+1| = \log|X| + C_1.$$

Taking exponential of both sides and simplifying we get

$$C_2|X| = |u-1|^{-2/7}|u+1|^{-5/7}.$$

Substituting $u = Y/X,\ X = x - 1,\ Y = y$ and simplifying we obtain

$$(y + x - 1)^5 (y - x + 1)^2 = C$$

for some arbitrary constant C. This gives the solution curves. ∎

1.4 Exact Equations

Sometimes observing simple identities about the differentials help in solving ODEs. For instance, consider the ODE

$$xy' + y - 2x = 0.$$

Notice that $\dfrac{d(xy)}{dx} = xy' + y$. Then the ODE can be solved as follows:

$$\frac{d(xy)}{dx} = 2x \;\Rightarrow\; \int d(xy) = \int 2x\,dx \;\Rightarrow\; xy = x^2 + C.$$

Using differentials, the ODE can be written as

$$x\,dy + y\,dx - 2x\,dx = 0.$$

This can be solved as

$$d(xy) - d(x^2) = 0 \;\Rightarrow\; \int d(xy) - \int d(x^2) = C \;\Rightarrow\; xy - x^2 = C.$$

In fact, we will write an ODE of the first order such as $xy' + y - 2x = 0$ as

$$x\,dy + (y - 2x)\,dx = 0$$

using the differentials. In general, we consider first order ODEs in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

This also covers the variables separable case since an ODE in the form $g(y)y' = f(x)$ can be rewritten as

$$-f(x)\,dx + g(y)\,dy = 0.$$

We can solve the general ODE above provided we find that the expression on the left is a differential $d\big(u(x, y)\big)$. In this case, we may integrate to obtain the general solution as $u(x, y) = C$. So, the question is when can we get a function $u(x, y)$ so that

$$d\big(u(x, y)\big) = M\,dx + N\,dy$$

is true. First, we look for some necessary conditions. Suppose that there exists a function $u(x, y)$ such that

$$du = M\,dx + N\,dy$$

From calculus we know that

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial x}\,dy.$$

Hence, the necessary condition is that

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y}.$$

If we assume that the second derivatives of $u(x, y)$ are continuous, then $u_{xy} = u_{yx}$. The above condition would imply that

$$\frac{\partial M}{\partial y} = u_{xy} = \frac{\partial N}{\partial x}.$$

In fact, this condition is also sufficient as the following result shows.

(1.13) *Theorem*

Let $M(x, y)$ and $N(x, y)$ be real valued functions having continuous partial derivatives on the rectangle $R : a < x < b,\ c < y < d$. Then, the following are equivalent:

(1) *There exists a function $u(x, y)$ defined on R such that $du = M\,dx + N\,dy$.*

(2) *$M_y = N_x$ holds in R.*

(3) *There exists a function $u(x, y)$ satisfying $M = u_x$ and $N = u_y$.*

Proof. (1) ⇒ (2): Suppose $du = M\,dx + N\,dy$ is true in R. Then, u_{xy} and u_{yx} exist and are continuous. By the Chain rule, $du = u_x dx + u_y dy$. Comparing these two equations, we get $M = u_x$ and $N = u_y$. Thus, $M_y = u_{xy}$ and $N_x = u_{yx}$. Since u_{xy} and u_{yx} are continuous, they are equal. Hence, $M_y = N_x$ holds in R.

(2) ⇒ (3): Suppose that $M_y = N_x$. Integrate with respect to x to get

$$N(x, y) = \int M_y\,dx + g(y).$$

Here, $g(y)$ is an arbitrary function of y alone. Define

$$u(x, y) = \int M(x, y)\,dx + \int g(y)\,dy.$$

Then,

$$u_x = M(x, y) + \frac{\partial}{\partial x}\int g(y)\,dy = M(x, y) + 0 = M(x, y).$$

$$u_y = \int M_y\,dx + g(y) = N(x, y).$$

(3) ⇒ (1): Suppose that there exists $u(x, y)$ such that $M = u_x$ and $N = u_y$. Then $M\,dx + N\,dy = u_x\,dx + u_y\,dy = du$. ∎

In view of this result, we say that

the ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is an **exact equation** iff $M_y = N_x$.

The proof of (1.13) shows how to compute a function $u(x,y)$ if the condition $M_y = N_x$ is satisfied. It is:

$$\begin{aligned} u(x,y) &= \int M(x,y)\,dx + \int g(y)\,dy \\ &= \int M(x,y)\,dx + \int N(x,y)\,dy - \int\int M_y\,dx\,dy. \end{aligned}$$

Since this formula holds under the assumption $M_y = N_x$, we also have

$$u(x,y) = \int M(x,y)\,dx + \int N(x,y)\,dy - \int\int N_x\,dy\,dx.$$

We will not memorize these formulas. Instead, we understand the method and then use it in any particular problem. This understanding gives rise to three ways of solving an exact equation. So, let the given exact equation be

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

The exactness implies that $M_y = N_x$, which, due to (1.13) guarantees the existence of a function $u(x,y)$ such that $M = u_x$ and $N = u_y$.

First method: Since $M = u_x$, we have $u = \int M(x,y)\,dx + g(y)$. Differentiating with respect to y, we get

$$g'(y) = u_y - \int M_y\,dx = N(x,y) - \int \frac{\partial M}{\partial y}\,dx.$$

Then, we determine $g(y)$ from this and substitute back to get $u(x,y)$. Recall that the solution curves are given by $u(x,y) = C$.

Second method: As $N = u_y$, we have $u = \int N(x,y)\,dy + h(x)$. Differentiating with respect to x, we obtain

$$h'(x) = u_x - \int N_x\,dy = M(x,y) - \int \frac{\partial N}{\partial x}\,dy.$$

We determine $h(x)$ from this and substitute back to obtain $u(x,y)$.

Third method: Using both $M = u_x$ and $N = u_y$, we get

$$u(x,y) = \int M(x,y)\,dx + g(y), \quad u(x,y) = \int N(x,y)\,dy + h(x).$$

Inspecting these two expression, we determine $g(y)$, $h(x)$; and then $u(x,y)$.

(1.14) *Example*

Find the general solution of the ODE $3y + e^x + (3x + \cos y)y' = 0$.

The ODE is $M\,dx + N\,dy = 0$ with $M = 3y + e^x$ and $N = 3x + \cos y$.
We find that $M_y = 3$ and $N_x = 3$. So, it is an exact equation. Hence, there exists a function $u(x, y)$ such that

$$\text{(a)}\ \ M = 3y + e^x = u_x(x, y), \quad \text{(b)}\ \ N = 3x + \cos y = u_y(x, y).$$

We illustrate the three methods to determine $u(x, y)$.

First method: Integrating (a) with respect to x, we get

$$u = \int (3y + e^x)\,dx = 3xy + e^x + g(y).$$

Differentiating with respect to y and using (b), we have

$$u_y = 3x + g'(y) \ \Rightarrow\ 3x + \cos y = 3x + g'(y) \ \Rightarrow\ g'(y) = \cos y \ \Rightarrow\ g(y) = \sin y.$$

Here, we need not consider the constant of integration, since in the solution this constant will re-appear as $u(x, y) = C$. Also, we need just one such $g(y)$.

Then, $u = 3xy + e^x + g(y) = 3xy + e^x + \sin y$. The solution curves are given by $u(x, y) = C$ or, $3xy + e^x + \sin y = C$.

Second method: Integrate (b) with respect to y to get

$$u = \int (3x + \cos y)\,dy = 3xy + \sin y + h(x).$$

Differentiate with respect to x and use (a) to get

$$u_x = 3y + h'(x) \ \Rightarrow\ 3y + e^x = 3y + h'(x) \ \Rightarrow\ h'(x) = e^x \ \Rightarrow\ h(x) = e^x.$$

Again, we neglect the constant of integration. It says that

$$u(x, y) = 3xy + \sin y + h(x) = 3xy + \sin y + e^x.$$

And, the solution curves are given by $u(x, y) = C$ or, $3xy + \sin y + e^x = C$.

Third method: We integrate (a) with respect to x and also integrate (b) with respect to y to obtain

$$u = \int (3y + e^x)dx = 3xy + e^x + g(y),\ u = \int (3x + \cos y)dy = 3xy + \sin y + h(x).$$

Matching them we find that $g(y) = \sin y$ and $h(x) = e^x$. Then $u = 3xy + e^x + \sin y$ gives the solution curves as $3xy + e^x + \sin y = C$. ∎

Out of the three, the third method is the easiest provided one is able to guess correctly. One should also get familiarized with other methods.

(1.15) *Example*

Solve $\cos(x+y)\,dx + \big(3y^2 + 2y + \cos(x+y)\big)\,dy = 0.$

Here, $M = \cos(x+y)$ and $N = 3y^2 + 2y + \cos(x+y)$. Then $M_y = -\sin(x+y)$ and $N_x = -\sin(x+y) = M_y$. Hence, it is an exact equation. Thus, there exists a function $u(x,y)$ such that

$$\text{(a)}\ \ u_x = M = \cos(x+y), \quad \text{(b)}\ \ u_y = N = 3y^2 + 2y + \cos(x+y).$$

We determine $u(x,y)$ by inspection (Third method) as follows.
Integrate (a) with respect to x and integrate (b) with respect to y to get

$$u = \sin(x+y) + g(y), \quad u = y^3 + y^2 + \sin(x+y) + h(x).$$

Matching these, we find that $g(y) = y^3 + y^2$ and $h(x) = 0$. Then the solution curves are given by $u(x,y) = C$ or, $\sin(x+y) + y^3 + y^2 = C$. ∎

(1.16) *Example*

Solve the IVP $(\cos y \sinh x + 1)\,dx - \sin y \cosh x\,dy = 0,\ y(1) = 2.$

Here, $M = \cos y \sinh x + 1$ and $N = -\sin y \cosh x$. It gives $M_y = -\sin y \sinh x$ and $N_x = -\sin y \sinh x$. As $M_y = N_x$, the ODE is exact. Then, there exists a function $u(x,y)$ such that $u_x = M$ and $u_y = N$. To determine u, we integrate $u_x = M$ with respect to x to obtain

$$u = \cos y \cosh x + x + g(y).$$

Differentiating with respect to y and using $u_y = N$, we have

$$u_y = -\sin y \cosh x + g'(y) \ \Rightarrow\ g'(y) = -\sin y \cosh x + \sin y \cosh x = 0 \ \Rightarrow\ g(y) = K.$$

Since we need only one such u, we take $K = 0$ so that

$$u(x,y) = \cos y \cosh x + x + g(y) = \cos y \cosh x + x.$$

A general solution is $\cos y \cosh x + x = C$. Using $y(1) = 2$, we have $\cos 2 \cosh 1 + 1 = C$. Then the solution to the IVP is given by

$$\cos y \cosh x + x = \cos 2 \cosh 1 + 1.$$

∎

(1.17) *Example*

Solve the IVP $3x^2y + 8xy^2 + (x^3 + 8x^2y + 12y^2)y' = 0,\ y(2) = 1.$

Here, $M = 3x^2y + 8xy^2$ and $N = x^3 + 8x^2y + 12y^2$. Then $M_y = 3x^2 + 16xy$ and $N_x = 3x^2 + 16xy + 0 = M_y$. So, the ODE is exact. Then, there exists a function $u(x,y)$ such that

$$\text{(a)}\ \ u_x = 3x^2y + 8xy^2, \quad \text{(b)}\ \ u_y = x^3 + 8x^2y + 12y^2.$$

Integrating (a) and (b) with respect to x and y, respectively, we get

$$u = x^3y + 4x^2y^2 + g(y), \quad u = x^3y + 4x^2y^2 + 4y^3 + h(x).$$

Matching these we have $g(y) = 4y^3$ and $h(x) = 0$. Then the solution curves are given by $u(x, y) = C$ or, $x^3y+4x^2y^2+4y^3 = C$. Using the initial condition: $y(2) = 1$, we see that

$$2^3 \cdot 1 + 4 \cdot 2^2 \cdot 1^2 + 4 \cdot 1^3 = C \Rightarrow C = 28.$$

Hence, the solution to the IVP is $x^3y + 4x^2y^2 + 4y^3 = 28$. ∎

1.5 Integrating factors

We must be cautious while using the three methods discussed in the last section. Remember that the methods work for exact equations. If the ODE is not exact, then the methods need not give solutions to the ODE, or even, they may fail towards obtaining a solution. See the following example.

(1.18) ***Example***

Solve the ODE $-y\,dx + x\,dy = 0$.

here, $M = -y$ and $N = x$. We find $u(x, y)$ so that $u_x = M = -y$ and $u_y = N = x$. Integrating first with respect to x, we get $u = -xy + g(y)$. Differentiating with respect to y, we have $u_y = -x + g'(y)$. Since $u_y = x$, we have $g'(y) = -2x$. There is something wrong, since our method assumes that $g(y)$ is a function of y alone.

We find that the ODE is not exact, because $M_y = -1$ whereas $N_x = 1$. Thus, none of the three methods above are applicable.

However, we can separate the variables and solve it as follows:

$$x\,dy = y\,dx \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \log|y| = \log|x| + C_1 \Rightarrow y = Cx.$$ ∎

So, before applying any one of the three methods, one must check that the ODE is exact.

Though the ODE in (1.18) is not exact, it can be made exact. Look at the solution curves. They are $y/x = C$. So, our function of two variables is $u(x, y) = y/x$. Now, its differential is

$$du = u_x\,dx + u_y\,dy = -\frac{y}{x^2}\,dx + \frac{dy}{x}.$$

The ODE is given as $-y\,dx + x\,dy = 0$. Comparing these, we find that if we multiply $1/x^2$ to the given ODE, we would obtain an exact equation. We explore this possibility further.

For the ODE $M\,dx + N\,dy = 0$, a function $\mu(x, y)$ is called an **integrating factor** iff $\mu(x, y)M\,dx + \mu(x, y)N\,dy = 0$ is an exact equation; that is, multiplying $\mu(x, y)$ the new ODE becomes exact.

Notice that $\mu(x, y)M\,dx + \mu(x, y)N\,dy = 0$ is exact when

$$\frac{\partial\big[\mu(x, y)M\big]}{\partial y} = \frac{\partial\big[\mu(x, y)N\big]}{\partial x}.$$

Using the Chain rule, it means that $\mu(x, y)$ is an integrating factor of the ODE $M\,dx + N\,dy = 0$ iff

$$M\frac{\partial\mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial\mu}{\partial x} + \mu\frac{\partial N}{\partial x}.$$

However, solving such an equation for determining $\mu(x, y)$ may be more difficult than solving the original ODE. So, we look for some special cases.

We ask whether it is possible for the function $\mu(x, y)$ to depend on x alone? What could be the conditions that yield this situation? When $\mu = \mu(x)$, its partial derivative with respect to y becomes 0 so that the above equation simplifies to

$$\mu\frac{\partial M}{\partial y} = N\frac{d\mu}{dx} + \mu\frac{\partial N}{\partial x} \Leftrightarrow \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\,\mu.$$

Notice that this expression is meaningless unless $\dfrac{M_y - N_x}{N}$ is a function of x alone. So, suppose

$$\frac{M_y - N_x}{N} = f(x).$$

Then μ is obtained by solving $\mu'(x) = f(x)\mu$. We need just one such μ; so we ignore the constants of integration. By separating the variables, we have

$$\int\frac{d\mu}{\mu} = \int f(x)\,dx \;\Rightarrow\; \log\mu = \int f(x)\,dx \;\Rightarrow\; \mu(x) = \exp\Big(\int f(x)\,dx\Big).$$

Here, we do not bother about taking $|\mu|$ since our requirement is one such μ. Our method boils down to the following:

Integrating factor 1: If the ODE $M\,dx + N\,dy = 0$ is not exact and

$$\frac{M_y - N_x}{N} = f(x),$$

a function of x alone, then $\mu(x) = \exp\left(\int f(x)\,dx\right)$ is an integrating factor of the ODE.

Similarly, we have the following method when an analogous expression is a function of y alone.

Integrating factor 2: If the ODE $M\,dx + N\,dy = 0$ is not exact and

$$\frac{M_y - N_x}{M} = g(y),$$

a function of y alone, then $\mu(y) = \exp\big(-\int g(y)\,dy\big)$ is an integrating factor of the ODE.

We illustrate these methods in the following examples.

(1.19) ***Example***

Solve the ODE $\dfrac{y^2}{2} + 2ye^x + (y + e^x)y' = 0$.

Here, $M = y^2/2 + 2ye^x,\ N = y + e^x \implies M_y = y + 2e^x,\ N_x = e^x \neq M_y$. Hence, it is not an exact equation. Now,

$$\frac{M_y - N_x}{N} = \frac{y + e^x}{y + e^x} = 1 = f(x).$$

It is a function of x alone. Hence, $\mu = \exp\big(\int f(x)\,dx\big) = \exp\big(\int dx\big) = e^x$ is an integrating factor. There exists a function $u(x, y)$ such that

$$\text{(a) } u_x = \mu M = \frac{e^x y^2}{2} + 2ye^{2x}, \quad \text{(b) } u_y = \mu N = ye^x + e^{2x}.$$

Integrating (a) with respect to x and (b) with respect to y, we have

$$u = \frac{e^x y^2}{2} + ye^{2x} + g(y), \quad u = \frac{e^x y^2}{2} + ye^{2x} + h(x).$$

Matching these, we take $g(y) = h(x) = 0$ to get the solution curve as $u(x, y) = C$ or, $\dfrac{e^x y^2}{2} + ye^{2x} = C$. ∎

(1.20) ***Example***

Solve the IVP $\big(e^{x+y} + ye^y\big)\,dx + \big(xe^y - 1\big)\,dy = 0,\ y(0) = -1$.

Here, $M = e^{x+y} + ye^y,\ N = xe^y - 1 \implies M_y - N_x = e^{x+y} + ye^y$. So, it is not an exact equation. We find that

$$\frac{M_y - N_x}{N} = \frac{e^{x+y} + ye^y}{xe^y - 1}$$

is not a function of x alone. And,

$$\frac{M_y - N_x}{M} = \frac{e^{x+y} + ye^y}{e^{x+y} + ye^y} = 1 = g(y)$$

is a function of y alone. So, we take the integrating factor as

$$\mu(y) = \exp\Big(-\int g(y)\,dy\Big) = \exp\Big(-\int dy\Big) = e^{-y}.$$

Multiplying it with the ODE, we have

$$\big(e^x + y\big)\, dx + \big(x - e^{-y}\big)\, dy = 0.$$

Since it is an exact equation, there exists a function $u(x, y)$ such that

$$\text{(a) } u_x = e^x + y, \quad \text{(b) } u_y = x - e^{-y}.$$

Integrating (a) with respect to x and (b) with respect to y we get

$$u(x, y) = e^x + xy + g_1(y), \quad u(x, y) = xy + e^{-y} + h_1(x).$$

Matching these we see that $g_1(y) = e^{-y}$ and $h_1(x) = e^x$. Then the solution curves are given by $u(x, y) = C$ or,

$$e^x + xy + e^{-y} = C.$$

Since $y(0) = -1$, we get $e^0 + 0(-1) + e^1 = C \Rightarrow C = 1 + e$. Then the solution of the IVP is given by $e^x + xy + e^{-y} = 1 + e$. ∎

1.6 Linear equations

A very special type of ODE that often comes up in applications is a linear equation. A **linear first order ODE** is an ODE in the form

$$y' + p(x)\, y = r(x).$$

When the right hand side is 0, that is, $r(x) = 0$, the equation is called a linear first order **homogeneous** ODE.

The linear ODE can be written in the differential form as

$$\big(p(x)y - r(x)\big)dx + dy = 0.$$

Here, $M = p(x)y - r(x),\ N = 1$ so that $M_y - N_x = p(x)$. Thus, the equation is exact when $p(x) = 0$. In that case, the equation is $y' = r(x)$ whose solution can be written as $y = \int r(x)\, dx + C$. In case $p(x) \neq 0$, we should seek an integrating factor. We observe that

$$\frac{M_y - N_x}{N} = \frac{p(x)}{1} = p(x)$$

is a function of x alone. Hence, an integrating factor is given by

$$\mu(x) = \exp\Big(\int p(x)\, dx\Big).$$

Multiplying the ODE with $\mu(x)$, we have $\mu(x)\big(p(x)y - r(x)\big)dx + \mu(x)\,dy = 0$, or,

$$\mu(x)p(x)y\,dx + \mu(x)\,dy = \mu(x)r(x)\,dx.$$

We see that

$$\mu'(x) = \frac{d}{dx}\exp\Big(\int p(x)\,dx\Big) = \exp\Big(\int p(x)\,dx\Big)\frac{d}{dx}\Big(\int p(x)\,dx\Big) = \mu(x)p(x).$$

Thus, the ODE reduces to

$$\mu'(x)y\,dx + \mu(x)\,dy = \mu(x)r(x)\,dx \;\Rightarrow\; d\big(\mu(x)y\big) = \mu(x)r(x)\,dx.$$

Integrating, we obtain $\mu(x)y = \int \mu(x)r(x)\,dx$. Along with the constant of integration, we obtain

$$y = [\mu(x)]^{-1}\Big[\int \mu(x)r(x)\,dx + C\Big], \quad \text{where } \mu(x) = \exp\Big(\int p(x)\,dx\Big).$$

This is the general solution of the linear first order ODE. We need not remember this formula, but use the method by multiplying the linear ODE with the integrating factor $\mu(x)$. It is enough remember that $\mu'(x) = \mu(x)p(x)$.

(1.21) ***Example***

Find the general solution of the ODE $y' - 2xy = x$.

It is a linear first order ODE with $p(x) = -2x$. Its integrating factor is

$$\mu(x) = \exp\Big(\int p(x)\,dx\Big) = \exp\Big(\int (-2x)\,dx\Big) = e^{-x^2}.$$

Multiplying it with the equation, we get

$$e^{-x^2}y' - 2xye^{-x^2} = xe^{-x^2} \;\Rightarrow\; \Big(e^{-x^2}y\Big)' = xe^{-x^2}.$$

Integrating, we obtain

$$e^{-x^2}y = \int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C \;\Rightarrow\; y = Ce^{x^2} - \frac{1}{2}.$$

□

(1.22) ***Example***

Solve the IVP $y' + 2xy = x,\ y(1) = 2$.

It is a linear first order ODE with $p(x) = 2x$ so that its integrating factor is

$$\mu(x) = \exp\Big(\int p(x)\,dx\Big) = \exp\Big(\int 2x\,dx\Big) = e^{x^2}.$$

Multiplying with the equation, we have

$$e^{x^2}y' + e^{x^2}2xy = e^{x^2}x \;\Rightarrow\; \big(e^{x^2}y\big) = xe^{x^2}.$$

Integrating, we obtain

$$e^{x^2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C.$$

As $y(1) = 2$, we get $e^1 \cdot 2 = \frac{1}{2}e^1 + C \;\Rightarrow\; C = \frac{3e}{2}$. Hence, the solution of the IVP is $y = Ce^{-x^2} + \frac{1}{2} = \frac{3}{2}e^{1-x^2} + \frac{1}{2}$. ∎

(1.23) ***Example***

Solve the IVP $y' + y\tan x = \sin(2x),\ y(0) = 1$.

It is a linear first order ODE with $p(x) = \tan x$. Its integrating factor is

$$\mu(x) = \exp\Big(\int \tan x\,dx\Big) = \exp(\log\sec x) = \sec x.$$

Multiplying it with the equation, we get

$$\sec x y' + \sec x\tan x y = 2\sin x \;\Rightarrow\; (\sec x y)' = 2\sin x.$$

Integrating, we obtain

$$\sec x y = -2\cos x + C \;\Rightarrow\; y = -2\cos^2 x + C\cos x.$$

Now, $y(0) = 1 \;\Rightarrow\; -2 + C = 1 \;\Rightarrow\; C = 3$. Then the solution to the IVP is $y = 3\cos x - 2\cos^2 x$. ∎

There are many ODEs that can be reduced to linear ODEs by suitable substitutions. One such is the **Bernoulli equation**:

$$y' + p(x)y = g(x)y^\alpha.$$

Notice that this first order ODE is linear for $\alpha = 0,\ 1$. So, suppose $\alpha \neq 0$ and $\alpha \neq 1$. Substitute $z(x) = [y(x)]^{1-\alpha}$. Then

$$\begin{aligned} z'(x) &= (1-\alpha)y^{-\alpha}y' = (1-\alpha)y^{-\alpha}\big[g(x)y^\alpha - p(x)y\big] \\ &= (1-\alpha)\big(g(x) - p(x)y^{1-\alpha}\big) = (1-\alpha)\big(g(x) - p(x)z(x)\big) \\ &= -(1-\alpha)p(x)z(x) + (1-\alpha)g(x). \end{aligned}$$

That is, we have the linear first order ODE

$$z'(x) + (1-\alpha)p(x)z(x) = (1-\alpha)g(x).$$

(1.24) ***Example***

Solve the *Logistic equation* $y' = Ay - By^2$.

Observe that it is a Bernoulli equation with $\alpha = 2$. We substitute $z(x) = y^{-1}$. Then

$$z' = -y^{-2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-1} = B - Az \implies z' + Az = B.$$

For this linear first order ODE, the integrating factor is $\mu = \exp(\int A\,dx) = e^{Ax}$. Multiplying, we have

$$e^{Ax}z' + Ae^{Ax}z = e^{Ax}B \implies \left(e^{Ax}z\right)' = e^{Ax}B.$$

It gives

$$e^{Ax}z = \int e^{Ax}B\,dx = \frac{B}{A}e^{Ax} + C \implies z = \frac{B}{A} + Ce^{-Ax}.$$

Substituting $z = y^{-1}$, we get $y = \left(B/A + Ce^{-Ax}\right)^{-1}$.
Notice that this general solution does not include the solution $y(x) = 0$. $\square$

2

Second Order ODE

2.1 Introduction

As we have seen all first order equations could not be solved. We could only solve exact equations and those which could be reduced to exact equations in two special cases. The second order equations put more difficult challenges. In general, a second order equation looks like

$$f(x, y, y', y'') = 0.$$

A special case is when we can solve such an equation for the second derivative. It then looks like

$$y'' = g(x, y, y').$$

Unfortunately, there is no method to solve even this special type. General methods are available to solve a still special class, and that to partially. The special class is the **second order linear ODEs**, which have the form

$$y'' + p(x)y' + q(x)y = r(x).$$

When $r(x) = 0$, such an ODE is called **homogeneous**, otherwise, **non-homogeneous**.

The **initial value problems** or **IVP**s involving second order equations come with two conditions given at a point such as

$$\text{Initial values : } y(x_0) = y_0,\ y'(x_0) = y'_0.$$

Thus, the initial, value problem with a homogeneous linear second order ODE looks like

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{2.1.1}$$

for some given real numbers x_0, y_0, and y'_0. We are interested in finding a solution of the IVP in an open interval that contains the point x_0. The existence and uniqueness of a solution to such an initial value problem is guaranteed under certain mild conditions.

(2.1) ***Theorem*** (Existence-Uniqueness)

Let the functions $p(x)$ and $q(x)$ be continuous in the open interval $a < x < b$ and let $a < x_0 < b$. Then, there exists a unique function $y = y(x)$ defined on the interval $a < x < b$ satisfying the IVP (2.1.1).

We will not prove this theorem. Observe that, in particular, if the initial conditions are zero conditions, that is, if $y_0 = 0 = y_0'$, then $y(x)$ is the zero function. This means, if $y(x)$ satisfies the homogeneous linear ODE and for some x_0 in the open interval, $y(x_0) = 0 = y'(x_0)$, then at all points x in the same open interval $y(x) = 0$.

2.2 The Wronskian

Before actually solving the homogeneous linear second order ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{2.2.1}$$

we will discuss some important properties of the solutions, or rather, properties of the set of all solutions. This will help us in solving the ODE. Due to the Existence-uniqueness theorem, we assume that $p(x)$ and $q(x)$ are continuous functions in a nontrivial open interval.

(2.2) ***Theorem***

Let $y_1(x)$ and $y_2(x)$ be two solutions of the ODE (2.2.1). Let c_1, c_2 be two constants. Then $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution of (2.2.1).

Proof. Since $y_1(x)$ and $y_2(x)$ are solutions of (2.2.1), we have

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 = y_2'' + p(x)y_2' + q(x)y_2.$$

Then multiplying the first equation with c_1 and the second with c_2, and adding, we obtain

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) = 0.$$ ∎

Of course, the above result does not hold for non-homogeneous ODEs.

(2.3) ***Example***

The ODE $y'' + y = 0$ has solutions $y_1(x) = \cos x$ and $y_2(x) = \sin x$. From (2.2) it follows that $y(x) = A\cos x + B\sin x$ is also a solution of the ODE. Indeed,

$$y'' = (A\cos x + B\sin x)'' = (-A\sin x + B\cos x)' = (-A\cos x - B\sin x) = -y.$$

It verifies what the theorem states. □

We will show that any solution of this ODE is in the form $y(x) = A\cos x + B\sin x$.

Instead of $\cos x$ and $\sin x$ suppose we take any two distinct functions, say, $y_1(x)$ and $y_2(x)$, with $y_1(x) \neq y_2(x)$ for some x. For instance, take $y_1 = \cos x$ and $y_2 = 2\cos x$. Then they are distinct functions but the solution $\sin x$ cannot be written as $c_1 \cos x + c_2(2\cos x)$. Thus, we need some condition on the functions y_1 and y_2 in order to write any solution as $c_1 y_1 + c_2 y_2$.

Let $y_1(x)$ and $y_2(x)$ be two continuously differentiable functions defined on a nontrivial open interval I. The **Wronskian** of $y_1(x)$ and $y_2(x)$, written $W[y_1, y_2](x)$, is defined by

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Notice that the Wronskian is a function of x. The reason for defining this is the following result.

(2.4) *Theorem*

Let $y_1(x)$ and $y_2(x)$ be two solutions of the ODE (2.2.1) on a nontrivial open interval I with $W[y_1, y_2](x) \neq 0$ for some $x \in I$. Then the general solution of the ODE (2.2.1) is $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1, c_2 are arbitrary constants.

Proof. Let $y(x)$ be a solution of (2.2.1). We need to find two constants c_1, c_2 such that $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for each $x \in I$. To this end, let $x_0 \in I$ be such that $W[y_1, y_2](x_0) \neq 0$. Let y_0 denote $y(x_0)$ and let y_0' denote $y'(x_0)$. If such constants c_1, c_2 exist, then evaluating at x_0, we must have

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0, \quad c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0'.$$

Since $W[y_1, y_2](x_0) \neq 0$, we have $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$. It follows that there exist unique constants c_1, c_2 satisfying the above two linear algebraic equations. Now, define

$$z(x) = c_1 y_1(x) + c_2 y_2(x) \quad \text{for } a < x < b.$$

Due to (2.2), $z(x)$ is a solution of (2.2.1). Further,

$$z(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0), \quad z'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0'.$$

That is, $z(x)$ is a solution to the IVP consisting of (2.2.1) and the initial conditions $z(x_0) = y_0$ and $z'(x_0) = y_0'$. But $y(x)$ is also a solution to the same IVP. Hence, by the Existence-uniqueness theorem, $y(x) = z(x)$. That is, $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for $a < x < b$. ∎

Look at the statement in (2.4). It looks that the same conclusion will hold irrespective of whether the Wronskian is nonzero at x_0 or at x_1 as long as $x_0, x_1 \in I$. In fact, if the Wronskian of two solutions of (2.2.1) is nonzero at some point in I, then it is nonzero at each point of I. We show this fact below.

(2.5) *Theorem*

Let $W(x)$ be the Wronskian of two solutions $y_1(x)$ and $y_2(x)$ of the ODE (2.2.1). Then $W'(x) + p(x)W(x) = 0$.

Proof. Since y_1 and y_2 are solutions of (2.2.1), we have

$$y_1'' = -p(x)y_1' - q(x)y_1, \quad y_2'' = -p(x)y_2' - q(x)y_2.$$

Using these, we obtain

$$\begin{aligned} W'(x) &= (y_1y_2' - y_1'y_2)' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\ &= y_1y_2'' - y_1''y_2 = y_1\big(-p(x)y_2' - q(x)y_2\big) + \big(p(x)y_1' + q(x)y_1\big)y_2 \\ &= -p(x)\big(y_1y_2' - y_1'y_2\big) = -p(x)W(x). \end{aligned}$$

∎

(2.6) *Theorem*

Let $p(x)$ and $q(x)$ be continuous on a nontrivial open interval I. Let $y_1(x)$ and $y_2(x)$ be two solutions of the ODE (2.2.1). Then, $W[y_1, y_2](x)$ is either identically zero, or is never zero for any $x \in I$.

Proof. Take any $x_0 \in I$. Write $W(t) = W[y_1, y_2](t)$. By (2.5), $W'(t) = -p(t)W(t)$ for $t \in I$. Separating the variables and integrating from x_0 to any $x \in I$, we have

$$\int_{x_0}^{x} \frac{W'(t)}{W(t)}\,dt = -\int_{x_0}^{x} p(t)\,dt \;\Rightarrow\; \log|W(x)| - \log|W(x_0)| = -\int_{x_0}^{x} p(t)\,dt$$

$$\Rightarrow\; |W(x)| = |W(x_0)| \exp\Big(-\int_{x_0}^{x} p(t)\,dt\Big).$$

The exponential term is never zero. Thus, $W(x) = 0$ iff $W(x_0) = 0$. That is, $W(x)$ is either identically zero or is never zero for any $x \in I$. ∎

The formula $|W(x)| = |W(x_0)| \exp\big(-\int_{x_0}^{x} p(t)\,dt\big)$ derived in the proof of (2.6) is called **Abel's formula**.

Caution: The Wronskian of any two arbitrary functions need not have the property proved in (2.6). It so happens only for solutions y_1 and y_2 of a homogeneous linear second order ODE. For instance, consider $y_1(x) = x$ and $y_2(x) = \sin x$. We find that

$$W(x) = W[y_1, y_2](x) = x(\sin x)' - x'\sin x = x\cos x - \sin x.$$

Then $W(0) = 0$ but $W(\pi/2) = -1$. That is, $W(x)$ is neither identically zero nor that it is never zero. It means that the functions $y_1(x) = x$ and $y_2(x) = \sin x$ cannot both be solutions of the same homogeneous linear second order ODE.

Suppose $y_(x)$ is a solution of (2.2.1). Then for any constant c, the function $y_2(x) = cy_1(x)$ is also a solution. We see that their Wronskian

$$W[y_1, y_2](x) = y_1(cy_1)' - y_1'(cy_1) = 0.$$

That is, when one solution is a constant multiple of the other, then their Wronskian is zero. We show that the converse is also true.

(2.7) ***Theorem***

Let $y_1(x)$ and $y_2(x)$ be solutions of the ODE (2.2.1) on a nontrivial open interval I. Suppose $W[y_1, y_2](x_0) = 0$ for some $x_0 \in I$. Then, one of these solutions is a constant multiple of the other.

Proof. Since $W[y_1, y_2](x_0) = (y_1 y_2' - y_1' y_2)(x_0) = 0$, the linear algebraic equations

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0, \quad c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

have a nontrivial solution. That is, there exist constants c_1, c_2 not both zero such that both the equations above are satisfied. With this choice of c_1, c_2, write $y(x) = c_1 y_1(x) + c_2 y_2(x)$. By (2.2), $y(x)$ is a solution of (2.2.1). The above two equations imply that $y(x_0) = 0$ and $y'(x_0) = 0$. Thus, by the Existence-uniqueness theorem, the IVP consisting of (2.2.1) and these two initial conditions has a unique solution. However, the zero function is a solution of this IVP. Hence, $y(x) = 0$, the zero function. That is,

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{for each } \; x \in I.$$

Now, if $c_1 \neq 0$, then $y_1(x) = -(c_2/c_1) y_2(x)$; and if $c_2 \neq 0$, then $y_2(x) = -(c_1/c_2) y_1(x)$. In either case, one is a multiple of the other. ∎

Again, we must remember that the above result is true only for solutions $y_1(x), y_2(x)$ of a homogeneous linear second order ODE. It need not be true for arbitrary functions $y_1(x)$ and $y_2(x)$.

Let $y_1(x)$ and $y_2(x)$ be two functions defined on an open interval $a < x < b$. We say that the functions y_1, y_2 are **linearly dependent** iff one of them is a constant multiple of the other. We say that y_1, y_2 are **linearly independent** iff they are not linearly dependent.

Further, two solutions $y_1(x)$ and $y_2(x)$ of (2.2.1) are said to form a **fundamental set of solutions** iff any solution of the ODE is expressible in the form $c_1 y_1 + c_2 y_2$ for suitable constants c_1, c_2.

Using these terminology, we can summarize our results as in the following.

(2.8) ***Theorem***

Let $y_1(x)$ and $y_2(x)$ be solutions of the ODE (2.2.1) in a nontrivial open interval I, where the functions $p(x)$ and $q(x)$ are continuous. Then the following are equivalent:

(1) *$y_1(x)$ and $y_2(x)$ are linearly independent.*

(2) *$W[y_1, y_2](x) \neq 0$ for some $x \in I$.*

(3) $W[y_1, y_2](x) \neq 0$ *for every* $x \in I$.

(4) $y_1(x)$ *and* $y_2(x)$ *form a fundamental set of solutions for (2.2.1).*

(2.9) ***Example***

The ODE $y'' + y = 0$ has solutions $y_1(x) = \cos x$ and $y_2(x) = \sin x$. So, $y(x) = A\cos x + B\sin x$ is also a solution of the ODE. We compute the Wronskian of $y_1(x)$ and $y_2(x)$ for any x in a nontrivial open interval I :

$$W[y_1, y_2](x) = y_1y_2' - y_1'y_2 = \cos x \cdot \cos x - (-\sin x)\sin x = 1 \neq 0.$$

Therefore, these two functions form a fundamental set; that is, any solution of the ODE $y'' + y = 0$ is in the form $c_1\cos x + c_2\sin x$ for some constants c_1 and c_2. □

2.3 Constant coefficients

Consider the simpler case of the ODE (2.2.1), where both $p(x)$ and $q(x)$ are constants. We may rewrite the simpler case as

$$ay'' + by' + cy = 0. \tag{2.3.1}$$

Since the ODE is of second order, we implicitly assume that $a \neq 0$. The theorems of the last section say that there are two linearly independent solutions which may be used to express all solutions. Unfortunately, the results do not tell us how to obtain a solution. We will have some sort of guess work. Observe that the functions $y(x)$, $y'(x)$ and $y''(x)$ should be such that they cancel among themselves and give us 0.

For example, if $y(x)$ is x^9, then $y'(x)$ is a constant times x^8 and y'' is a constant times y^7. They cannot cancel to give us 0. If $y(x)$ is $\cos x$, then $y'(x)$ will be a constant multiple of $\sin x$ and y'' a constant multiple of $\cos x$. Again, this is not a right candidate. If $y(x)$ is an exponential, say $e^{\lambda x}$, then $y'(x)$ and $y''(x)$ are also constant times $e^{\lambda x}$. It looks this is a possible choice. So, let us try $y(x) = e^{\lambda x}$. Then $y'(x) = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting these in (2.3.1), we get

$$a(e^{\lambda x})'' + b(e^{\lambda x})' + ce^{\lambda x} = 0 \implies (a\lambda^2 + b\lambda + c)e^{\lambda x} = 0.$$

Thus, $y(x) = e^{\lambda x}$ is a solution of (2.3.1) iff

$$a\lambda^2 + b\lambda + c = 0. \tag{2.3.2}$$

This equation is called the **characteristic equation** of (2.3.1). It has two roots λ_1, λ_2 given by

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Depending on the sign of $b^2 - 4ac$ we have three different cases.

Case 1 (Distinct Real Roots): First, suppose that $b^2 - 4ac > 0$.
Then $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$. We know that $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ are two distinct solutions of (2.3.1). Now,

$$W[y_1, y_2](x) = e^{\lambda_1 x}\big(e^{\lambda_2 x}\big)' - \big(e^{\lambda_1 x}\big)' e^{\lambda_2 x} = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)x}.$$

As $\lambda_2 \neq \lambda_1$, $W[y_1, y_2](x) \neq 0$ for any x. By (2.8), these two solutions form a fundamental set. That is, the general solution of the homogeneous linear second order ODE (2.3.1) is given by

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

Before discussing other cases, we solve some examples.

(2.10) *Example*

Find the general solution of $y'' + 5y' + 4y = 0$.

It is a homogeneous linear second order ODE with constant coefficients. Its characteristic equation is

$$\lambda^2 + 5\lambda + 4 = 0 \implies (\lambda + 1)(\lambda + 4) = 0.$$

So, the characteristic roots are $\lambda_1 = -1$ and $\lambda_2 = -4$; these are distinct and real. Thus, $y_1(x) = e^{-x}$ and $y_2(x) = e^{-4x}$ form a fundamental set of solutions. That is, the general solution is given by

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{-4x}$$

where c_1, c_2 are arbitrary constants. ∎

(2.11) *Example*

Solve the IVP: $y'' + y' - 2y = 0,\ y(0) = 4,\ y'(0) = -5$.

It is a homogeneous linear second order ODE with constant coefficients. Its characteristic equation is $\lambda^2 + \lambda - 2 = 0 \implies (\lambda - 1)(\lambda + 2) = 0$. The characteristic roots are $\lambda_1 = 1$ and $\lambda_2 = -2$. So, the general solution is

$$y(x) = c_1 e^{x} + c_2 e^{-2x}.$$

The initial conditions give

$$y(0) = c_1 + c_2 = 4, \quad y'(0) = c_1 - 2c_2 = -5.$$

Solving these equations, we have $c_1 = 1,\ c_2 = 3$. Thus, the solution to the IVP is $y(x) = e^x + 3e^{-2x}$. ∎

(2.12) *Example*

Find the general solution of the ODE $y'' + 4y' - 2y = 0$ and then solve the IVP $y'' + 4y' - 2y = 0$, $y(0) = 1$, $y'(0) = 2$.

It is a homogeneous linear second order ODE with constant coefficients. Its characteristic equation is

$$\lambda^2 + 4\lambda - 2 = 0.$$

Since $4^2 - 4(1)(-2) > 0$, there are two distinct real characteristic roots

$$\lambda_1 = \frac{-4 + \sqrt{16+8}}{2} = -2 + \sqrt{6}, \quad \lambda_2 = \frac{-4 - \sqrt{16+8}}{2} = -2 - \sqrt{6}.$$

The fundamental set of solutions comprise $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$. The general solution is

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$$

for arbitrary constants c_1, c_2. Using the initial conditions, we have

$$c_1 + c_2 = 1, \quad (-2+\sqrt{6})c_1 + (-2-\sqrt{6})c_2 = 2.$$

Solving these equations, we get $c_1 = 2/\sqrt{6}$ and $c_2 = 1/2 - 2/\sqrt{6}$. Then the solution of the IVP is

$$y(x) = \Big(\frac{1}{2} + \frac{2}{\sqrt{6}}\Big) e^{(-2+\sqrt{6})x} \Big(\frac{1}{2} - \frac{2}{\sqrt{6}}\Big) e^{-(2+\sqrt{6})x}.$$

□

Case 2 (Complex Conjugate Roots): Suppose that $b^2 - 4ac < 0$.
Then the characteristic roots λ_1, λ_2 are given by

$$\lambda_1 = \alpha + \beta i, \quad \lambda_2 = \alpha - \beta i, \quad \alpha = -\frac{b}{2a} \in \mathbb{R}, \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} \in \mathbb{R} \setminus \{0\}.$$

For the time being, pretend that we be satisfied with complex solutions. Then, as in Case 1, the two solutions will be

$$z_1 = e^{(\alpha + i\beta)x}, \quad z_2 = e^{(\alpha - i\beta)x}.$$

Their linear combinations, that is, any expression of the form $c_1 z_1 + c_2 z_2$ is also a solution. In particular,

$$y_1 = \frac{z_1 + z_2}{2} = e^{\alpha x}\cos(\beta x), \quad y_2 = \frac{z_1 - z_2}{2i} = e^{\alpha x}\sin(\beta x)$$

are also solutions. Notice that y_1, y_2 are real solutions. This suggests we try to show directly that y_1, y_2 are solutions of the ODE. For y_1 we proceed as follows, using the values of α, β as obtained earlier:

$$\begin{aligned}
y_1 &= e^{\alpha x}\cos(\beta x) \\
y_1' &= e^{\alpha x}\big(\alpha\cos(\beta x) - \beta\sin(\beta x)\big) \\
y_1'' &= \alpha e^{\alpha x}\big(\alpha\cos(\beta x) - \beta\sin(\beta x)\big) + e^{\alpha x}\big(-\alpha\beta\sin(\beta x) - \beta^2\cos(\beta x)\big) \\
&\Rightarrow a y_1'' + b y_1' + c y_1 \\
&= e^{\alpha x}\Big(\cos(\beta x)\big(a\alpha^2 - a\beta^2 + b\alpha + c\big) - (2a\alpha + b)\beta\sin(\beta x)\Big) \\
&= e^{\alpha x}\left(a\Big(\frac{b^2}{4a^2} - \frac{4ac - b^2}{4a^2}\Big) + b\,\frac{-b}{2a} + c\right)\cos(\beta x) = 0.
\end{aligned}$$

That is, $y_1(x)$ is a solution to the ODE. Similarly, it is easily verified that $y_2(x)$ is also a solution of the ODE. Clearly, these two solutions are linearly independent. Hence, the general solution is given by

$$y(x) = e^{\alpha x}\big(c_1 \cos(\beta x) + c_2 \sin(\beta x)\big).$$

Observe that $e^{(\alpha+i\beta)x} = y_1(x) + iy_2(x)$ and $e^{(\alpha-i\beta)x} = y_1(x) - iy_2(x)$. Thus, the two linearly independent solutions are the real and imaginary parts of $e^{\lambda x}$ where λ is a complex characteristic root.

(2.13) ***Example***

Find the general solution of $4y'' + 4y' + 5y = 0$.

It is a homogeneous linear second order ODE with constant coefficients. Its characteristic equation is $\lambda^2 + 4\lambda + 5 = 0$; its characteristic roots are

$$\lambda_1 = -\frac{1}{2} + i, \quad \lambda_2 = -\frac{1}{2} - i.$$

Hence, the two linearly independent solutions are

$$y_1(x) = e^{-x/2}\cos x, \quad y_2(x) = e^{-x/2}\sin x.$$

Thus, the general solution is $y(x) = e^{-x/2}(c_1 \cos x + c_2 \sin x)$. ▯

(2.14) ***Example***

Find the solution of the IVP: $y'' + 2y' + 4y = 0,\ y(0) = 1,\ y'(0) = 1$.

The characteristic equation is $\lambda^2 + 2\lambda + 4 = 0$. The characteristic roots are $-1 \pm \sqrt{3}\, i$. Hence, the two linearly independent solutions are

$$y_1(x) = e^{-x}\cos(\sqrt{3}\, x), \quad y_2(x) = e^{-x}\sin(\sqrt{3}\, x).$$

The general solution is $y(x) = e^{-x}\big(c_1\cos(\sqrt{3}\, x) + c_2 \sin(\sqrt{3}\, x)\big)$.

The constants $c_1,\ c_2$ are determined from the initial conditions

$$1 = y(0) = c_1, \quad 1 = y'(0) = -c_1 + \sqrt{3}\, c_2.$$

They give $c_1 = 1$ and $c_2 = 2/\sqrt{3}$. So, the solution to the IVP is

$$y(x) = e^{-x}\big[\cos(\sqrt{3}\, x) + (2/\sqrt{3})\sin(\sqrt{3}\, x)\big].$$ ▯

Case 3 (Equal Roots): Suppose $b^2 - 4ac = 0$.

Then the characteristic roots are real and equal; that is, $\lambda_1 = \lambda_2 = -b/(2a)$. We have at least one solution, namely, $y_1(x) = e^{\lambda_1 x} = e^{[-b/(2a)]x}$. The second solution, namely, $e^{\lambda_2 x}$ is same as $y_1(x)$; and we would not obtain a fundamental set. We use this known solution to obtain the second one in a clever way.

If $y_2(x)$ is another solution so that $y_1(x)$ and $y_2(x)$ are linearly independent, then $y_2(x)/y_1(x)$ is not a constant function. So, we start with $y_2(x) = y_1(x)u(x)$ and try to determine $u(x)$ from the ODE $ay'' + by' + cy = 0$. With this substitution, we have

$$y_2 = y_1 u, \quad y_2' = y_1' u + y_1 u', \quad y_2'' = y_1'' u + 2y_1' u' + y_1 u''.$$

Since y_2 satisfies the ODE, we get

$$\begin{aligned} 0 &= ay_2'' + by_2' + cy_2 \\ &= a(y_1''u + 2y_1'u' + y_1u'') + b(y_1'u + y_1u') + cy_1u \\ &= a(y_1'' + by_1' + cy_1)u + ay_1u'' + (2ay_1' + by_1)u'. \end{aligned}$$

As y_1 also satisfies the ODE, we have $ay_1'' + by_1' + cy_1 = 0$. Further,

$$2ay_1' + by_1 = 2a\big(e^{-\frac{b}{2a}x}\big)' + be^{-\frac{b}{2a}x} = 2a \cdot \frac{-b}{2a} e^{-\frac{b}{2a}x} + be^{-\frac{b}{2a}x} = 0.$$

Then the above equation reduces to $ay_1u'' = 0$. Also, $ay_1 \neq 0$. Hence $u'' = 0$ of which one solution is $u(x) = x$.

It follows that $y_2(x) = xy_1(x)$ is another solution of the same ODE. Clearly, $y_1(x)$ and $y_2(x)$ are linearly independent. Therefore, the general solution of (2.3.1) is given by

$$y(x) = (c_1 + c_2x)y_1(x) = (c_1 + c_2)e^{\lambda_1 x}, \quad \lambda_1 = -\frac{b}{2a}.$$

As a caution, we should remember that this $y(x)$ is not a solution of the ODE if λ_1 is not a double root of the characteristic equation.

Observe that we could have tried this solution in the beginning and got it immediately. However, it is good to familiarize with the method followed above. We will see the use of this method later in a more general setting.

(2.15) *Example*

Solve the IVP $y'' + 4y' + 4 = 0,\ y(0) = 1,\ y'(0) = 3.$

The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$. So, the characteristic roots are $\lambda_1 = \lambda_2 = -2$. Thus, the general solution is

$$y(x) = (c_1 + c_2x)e^{-2x}.$$

The initial conditions imply that

$$1 = y(0) = c_1,\ 3 = y'(0) = -2c_1 + c_2 \ \Rightarrow\ c_1 = 1,\ c_2 = 5.$$

So, the general solution of the IVP is $y(x) = (1 + 5x)e^{-2x}$. ∎

2.4 Higher Order Linear ODEs and Systems

We have learnt how to solve linear homogeneous second order ODEs with constant coefficients. The same method can be used to solve higher order linear homogeneous ODEs with constant coefficients and also a similar method can be used to solve homogeneous systems of linear ODEs with constant coefficients.

A **linear homogeneous ODE with constant coefficients** of order n is of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Its characteristic equation is

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

1. If λ is a simple real root of the characteristic equation, then corresponding to it we take the solution as $e^{\lambda x}$.
2. If $\alpha + i\beta$ and $\alpha - i\beta$ are a pair of complex roots of the characteristic equation, then corresponding to this pair, the two linearly independent complex solutions are $e^{(\alpha+i\beta)x}$ and $e^{(\alpha+i\beta)x}$. The real and imaginary parts of one of them coincides with those of the other. They are $a^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$, which form two linearly independent real solutions.
3. If λ is a root of the characteristic equation having multiplicity $m > 1$, then corresponding to this, we take the m linearly independent solutions as $e^{\lambda x}$, $xe^{\lambda x}$, $\ldots$ and $x^{m-1}e^{\lambda x}$.
4. If a pair of complex roots $\alpha + i\beta$ and $\alpha - i\beta$ are repeated, then we compute the corresponding complex solutions as explained in Step 3, and then take their real and imaginary parts as real solutions.

As earlier, the general solution is obtained by multiplying these solutions with arbitrary constants and adding them together. In other words, we take a *linear combination* of all linearly independent solutions thus obtained to get the general solution.

(2.16) ***Example***

Find the general solution of $y^{(4)} + y^{(3)} - 7y'' - y' + 6y = 0.$

The characteristic equation is $\lambda^4 + \lambda^3 - 7\lambda^2 - \lambda + 6 = 0.$
Its roots are $\lambda = 1,\ -1,\ 2$ and -3. Thus, the general solution is

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-3x}.$$ □

(2.17) ***Example***

Find the solution of the IVP

$$y^{(4)}(x) - y(x) = 0, \quad y(0) = 7/2,\ y'(0) = -4,\ y''(0) = 5/2,\ y^{(3)}(0) = -2.$$

The characteristic equation is $\lambda^4 - 1 = 0$. Its roots are $\lambda = 1,\ -1,\ i$ and $-i$. Thus, the general solution is

$$y(x) = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x.$$

We find that

$$\begin{aligned} y' &= c_1e^x - c^2e^{-x} - c_3 \sin x + c_4 \cos x, \\ y'' &= c_1e^x + c_2e^{-x} - c_3 \cos x - c_4 \sin x, \\ y^{(3)} &= c_1e^x - c_2e^{-x} + c_3 \sin x - c_4 \cos x. \end{aligned}$$

Using the initial values, we get

$$\begin{aligned} 7/2 &= y(0) = c_1 + c_2 + c_3 \\ -4 &= y'(0) = c_1 - c_2 + c_4 \\ 5/2 &= y''(0) = c_1 + c_2 - c_3 \\ -2 &= y^{(3)}(0) = c_1 - c_2 - c_4. \end{aligned}$$

Solving these equations we obtain $c_1 = 0,\ c_2 = 3,\ c_3 = 1/2$ and $c_4 = -1$.
Hence the solution of the IVP is $y(x) = 3e^{-x} + \frac{1}{2}\cos x - \sin x$. ∎

(2.18) ***Example***

Find the general solution of the ODE

$$y^{(5)} - 10y^{(4)} + 54y^{(3)} - 132y'' + 137y' - 50y = 0.$$

The characteristic equation is

$$\lambda^5 - 10\lambda^4 + 54\lambda^3 - 132\lambda^2 + 137\lambda - 50 = 0.$$

Trying 1 and 2 as possible values of λ, we factor the left hand side as follows:

$$\begin{aligned} &\lambda^5 - 10\lambda^4 + 54\lambda^3 - 132\lambda^2 + 137\lambda - 50 = (\lambda - 1)(\lambda^4 - 9\lambda^3 + 45\lambda^2 - 87\lambda + 50) \\ &= (\lambda - 1)(\lambda - 1)(\lambda^3 - 8\lambda^2 + 37\lambda - 50) = (\lambda - 1)^2(\lambda - 2)(\lambda^2 - 6\lambda + 25) \\ &= (\lambda - 1)^2(\lambda - 2)\big((\lambda - 3)^2 + 4^2\big). \end{aligned}$$

Hence, the characteristic equation has a simple root as $\lambda_1 = 2$, a double root as $\lambda_2 = \lambda_3 = 1$, and a pair of complex conjugate roots $\lambda_4 = 3 + 4i$ and $\lambda_5 = 3 - 4i$. The corresponding linearly independent solutions are

$$y_1 = e^{2x},\ y_2 = e^x,\ y_3 = xe^x,\ y_4 = e^{3x}\cos(4x),\ y_5 = e^{3x}\sin(4x).$$

Therefore, the general solution is

$$y(x) = c_1 e^{2x} + (c_2 + c_3 x)e^x + e^{3x}\big(c_4 \cos(4x) + c_5 \sin(4x)\big).$$ ∎

A **homogeneous system of first order linear ODEs with constant coefficients** looks like

$$\begin{aligned} y_1'(x) &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2'(x) &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n'(x) &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n. \end{aligned}$$

Here, each of the n functions $y_1(x), \ldots, y_n(x)$ are the dependent variables and x is the only independent variable; further, a_{ij} are given real numbers. We rewrite the system of ODEs as follows:

$$y' = Ay, \quad \text{where} \quad y = y(x) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad y'(x) = \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{n1} \cdots a_{nn} \end{bmatrix}.$$

We call this $n \times n$ matrix A as the **system matrix**. Our goal is to solve this system, that is, to find functions $y_1(x), \ldots, y_n(x)$, as general as possible, which satisfy the above equations simultaneously. In some cases, it is possible to eliminate $n - 1$ dependent variables and obtain an n-th order ODE in the remaining variable. We first illustrate this method; and then proceed to the general method of solution.

(2.19) ***Example***

Solve the first order linear system of ODEs $y_1' = y_1 + y_2,\ y_2' = 4y_1 + y_2$.

We try to eliminate one of the dependent variables, say, y_2. From the first equation we have

$$y_2 = y_1' - y_1.$$

Substituting this in the second equation we get

$$4y_1 = y_2 - y_2' = y_1' - y_1 - (y_1' - y_1)' = y_1' - y_1 - y_1'' + y_1'.$$

This simplifies to

$$y_1'' - 2y_1' - 3y_1 = 0.$$

Its characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$. The roots are $\lambda = 3,\ -1$. Then, the general solution is

$$y_1(x) = c_1 e^{3x} + c_2 e^{-x}.$$

As $y_2 = y_1' - y_1$, we obtain

$$y_2(x) = y_1' - y_1 = 3c_1 e^{3x} - c_2 e^{-x} - c_1 e^{3x} - c_2 e^{-x} = 2c_1 e^{3x} - 2c_2 e^{-x}.$$

This general solution of the system of ODEs can also be written as

$$y(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3x} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-x}.$$

∎

Notice that in Example 2.19, the system of ODEs can be written as

$$y' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} y, \quad \text{with y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic polynomial of the system matrix is

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = (-\lambda - 1)(3-\lambda).$$

Thus, the eigenvalues of the system matrix are $\lambda = -1,\ 3$. Observe that these eigenvalues match exactly with the roots of the characteristic equation obtained by eliminating one of the variables. Further, we find that

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

That is, the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are eigenvectors associated with the eigenvalues 3 and -1, respectively. These eigenvalues and the eigenvectors can be used now directly to write the general solution in the form as obtained earlier.

It so happens that a general solution of the system $y' = Ay$ can be computed by using the information of eigenvalues and eigenvectors of the system matrix. We describe this *eigenvalue-eigenvector method* of solving a system of ODE as in the following.

Consider computing a general solution of the first order system of linear homogeneous ODEs

$$y' = Ay,$$

where $y = [y_1(x), \ldots, y_n(x)]^t$ and A is an $n \times n$ matrix with real entries. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Notice that there can be repetitions and also there can be complex numbers in this list of eigenvalues.

Case 1: Suppose that λ is a real eigenvalue of A which is never repeated in the list of eigenvalues. Let $v \in \mathbb{R}^{n\times 1}$ be an eigenvector associated with λ. Then, a corresponding solution of the system is given by $e^{\lambda x} v$.

Case 2: Suppose that $\lambda = \alpha + i\beta$ is an eigenvalue of A, where $\beta \neq 0$. Let $u + iv$ be an eigenvector associated with λ, where $u, v \in \mathbb{R}^{n\times 1}$. Notice that $\overline{\lambda}$ is also an eigenvalue of A associated with the eigenvector $u - iv$. Corresponding to this pair of eigenvalues, namely, λ and $\overline{\lambda}$, the two linearly independent solutions of the system are given by

$$e^{(\alpha+i\beta)x}(u+iv) \quad \text{and} \quad e^{(\alpha-i\beta)x}(u-iv).$$

Both the real parts and the imaginary parts of these solutions give the real solutions of the ODE. Notice that the two real solutions obtained from the first one are exactly the same obtained from the second one. So, we may consider only one of the complex solutions and proceed to take its real and imaginary parts. These two linearly independent real solutions of the ODE are

$$e^{\alpha x}\big(\cos(\beta x)u - \sin(\beta x)v\big) \quad \text{and} \quad e^{\alpha x}\big(\sin(\beta x)u + \cos(\beta x)v.$$

Case 3: Suppose that λ is a real eigenvalue of A which is repeated m times. We find the maximum number of linearly independent eigenvectors associated with λ; suppose these are $v_1, \ldots, v_{m1}$. The corresponding m_1 number of linearly independent solutions of the system are given by

$$e^{\lambda x}v_1, \ldots, e^{\lambda x}v_{m_1}.$$

If $m_1 = m$, then the process stops here with m number of linearly independent solutions corresponding to the eigenvalue λ. If $m_1 < m$, then we look for nonzero vectors u which are solutions to

$$(A-\lambda I)^2 u = 0,\ (A-\lambda I)u \neq 0.$$

If there are m_2 number of linearly independent solutions of these equations, say, $u_1, \ldots, u_{m_2}$, then the corresponding m_2 number of linearly independent solutions of the system are

$$e^{\lambda x}\big(u_1 + x(A-\lambda I)u_1\big), \ldots, e^{\lambda x}\big(u_{m_2} + x(A-\lambda I)u_{m_2}\big).$$

Linear Algebra guarantees that if $m_1 < m$, then $m_2 \neq 0$ and $m_1 + m_2 \leq m$. If $m_1 + m_2 = m$, then we have got m linearly independent solutions. Otherwise $m_1 + m_2 < m$. Then, we look for nonzero vectors w satisfying

$$(A-\lambda I)^3 w = 0,\ (A-\lambda I)^2 w \neq 0.$$

If there are m_3 number of linearly independent solutions of these equations, say, $w_1, \ldots, w_{m_3}$, then the corresponding m_3 number of linearly independent solutions of the system are

$$e^{\lambda x}\Big(w_1 + x(A-\lambda I)w_1 + \frac{x^2}{2!}(A-\lambda I)^2 w_1\Big), \ldots, e^{\lambda x}\Big(w_{m_3} + x(A-\lambda I)w_{m_3} + \frac{x^2}{2!}(A-\lambda I)^2 w_{m_3}\Big).$$

The process continues until all m linearly independent solutions are obtained.

Finally, a general solution of the system is obtained by taking a linear combination of all solutions obtained by the above method.

Notice that the process becomes complicated when a complex root is repeated. Of course, the earlier method of taking the real and imaginary parts of complex solutions give us the real solutions.

The vectors $u,\ w,\ldots$ obtained in the above process are called the **generalized eigenvectors** of the matrix A. Essentially, it leads to computing the *Jordan form* of the matrix A, and then computing e^{Ax} from the Jordan form.

We remark that a linear homogeneous nth order ODE with constant coefficients can be converted to a linear system of ODEs in the form

$$y' = Ay \quad \text{with} \quad y_2 = y_1',\ y_3 = y_2', \ldots, y_n = y_{n-1}'.$$

Conversely, not all linear homogeneous systems of the form $y' = Ay$ can be converted to a higher order linear homogeneous ODE. The linear system of ODEs in Example 2.19 is an exception. Thus, we need to familiarize ourselves with this eigenvalue-eigenvector method of solution of $y' = Ay$ as outlined earlier.

(2.20) *Example*

Find the general solution of the system of ODEs $y_1' = y_1 + 12y_2,\ y_2' = 3y_1 + y_2$.
The characteristic polynomial of the system matrix A is

$$\begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 36 = (\lambda - 7)(\lambda + 5).$$

Thus, the eigenvalues are $\lambda = 7,\ -5$. We compute the corresponding eigenvectors. For $\lambda = 7$, we solve the linear system

$$\begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 7 \begin{bmatrix} a \\ b \end{bmatrix}$$

for a nonzero solution. It gives $a + 12b = 7a,\ 3a + b = 7b$, or $a = 2b$. One such solution is the eigenvector $[2\ \ 1]^t$. The corresponding solution is $y(x) = e^{7x}[2\ \ 1]^t$. For $\lambda = -5$, we solve the linear system

$$\begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -5 \begin{bmatrix} a \\ b \end{bmatrix}$$

for a nonzero solution. It gives $a + 12b = -5a,\ 3a + b = -5b$, or $a = -2b$. One such solution is the eigenvector $[-2\ \ 1]^t$. The corresponding solution is $y(x) = e^{-5x}[-2\ \ 1]^t$.

Notice that the two solutions obtained are linearly independent. The general solution is given by a linear combination of these two. That is the general solution of the system of ODEs is

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 e^{7x} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-5x} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Or, $y_1(x) = 2c_1e^{7x} - 2c_2e^{-5x},\ y_2(x) = c_1e^{7x} + c_2e^{-5x}$. ∎

In the above example, it is possible to convert the system to a higher order ODE by eliminating one of the variables. We choose to eliminate y_2. The equations $y_1' = y_1 + 12y_2,\ y_2' = 3y_1 + y_2$ imply that $12y_2 = y_1' - y_1$. Differentiating this we get $12y_2' = y_1'' - y_1'$. Using the second equation, we get

$$y_1'' - y_1' = 12(3y_1 + y_2) = 36y_1 + 12y_2 = 36y_1 + y_1' - y_1.$$

It gives

$$y_1'' - 2y_1' - 35y_1 = 0.$$

Its characteristic polynomial as $\lambda^2 - 2\lambda - 35 = (\lambda - 7)(\lambda + 5)$. The characteristic roots are $\lambda = 7,\ -5$. Then the general solution is

$$y_1(x) = c_1e^{7x} + c_2e^{-5x}.$$

Then, $12y_2 = y_1' - y_1 = 7c_1e^{7x} - 5c_2e^{-5x} - c_1e^{7x} - c_2e^{-5x}$. Or,

$$y_2(x) = \frac{1}{2}c_1e^{7x} - \frac{1}{2}c_2e^{-5x}.$$

We have thus obtained the general solution of the system. To see that it is the same as obtained in Example 2.20, replace c_1 with $2c_1$ and c_2 with $-2c_2$. This can be done since the constants are arbitrary. We then get

$$y_1(x) = 2c_1e^{7x} - 2c_2e^{-5x},\ y_2(x) = c_1e^{7x} + c_2e^{-5x}$$

as obtained by the eigenvalue-eigenvector method in Example 2.20.

(2.21) *Example*

Find the general solution of the system $y_1' = y_1 - y_2,\ y_2' = y_1 + y_2$.
Here, it is possible to convert the system to a higher order ODE by eliminating one of the variables. However, we illustrate the eigenvalues and eigenvector method of finding the general solution. The system of ODEs is in the form

$$y' = Ay, \quad \text{where} \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1.$$

Its roots are $\lambda = 1 \pm i$. We find an eigenvector corresponding to one of the complex conjugate pairs. For $\lambda = 1 + i$, we seek a nonzero complex solution of $A[a\ \ b]^t = \lambda[a\ \ b]^t$. This is the linear system

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (1+i) \begin{bmatrix} a \\ b \end{bmatrix}.$$

Or, $a - b = (1+i)a,\ a + b = (1+i)b \Rightarrow 2a = (1+i)(a+b) = (1+i)^2 b$. One nonzero solution is obtained by taking $b = 1$ and $a = (1+i)^2/2 = i$.

Thus, a complex solution corresponding to this pair of complex conjugate roots is

$$e^{(1+i)x}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^x \begin{bmatrix} i(\cos x + i\sin x) \\ \cos x + i\sin x \end{bmatrix} = e^x \begin{bmatrix} -\sin x + i\cos x \\ \cos x + i\sin x \end{bmatrix} = e^x \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix} + ie^x \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}.$$

The real part and the imaginary part of this complex solution are the linearly independent solutions of the system, which are

$$e^x \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix}, \quad e^x \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}.$$

Hence, the general solution of the system is

$$y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 e^x \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix} + c_2 e^x \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}.$$

It is same as $y_1(x) = e^x(-c_1 \sin x + c_2 \cos x),\ y_2(x) = e^x(c_1 \cos x + c_2 \sin x)$. □

(2.22) ***Example***

Find the general solution of $y_1' = y_2 + y_3,\ y_2' = y_1 + y_3,\ y_3' = y_1 + y_2$.
It does not seem possible to eliminate two of the dependent variables so that the system could be converted to a higher order ODE. We write the system in the form $y' = Ax$ as follows:

$$y' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} y.$$

The characteristic polynomial of the matrix is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda - 2)(\lambda + 1)^2.$$

Thus the eigenvalues of the matrix are $2,\ -1,\ -1$. Notice that -1 is repeated twice. However, the matrix is real symmetric; thus, there are two linearly independent eigenvectors associated with the repeated root -1. We compute the eigenvectors. Fpr $\lambda = 2$, we seek a, b, c not all zero such that

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2 \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} b + c = 2a \\ a + c = 2b. \\ a + b = 2c \end{matrix}$$

Solving the linear equations gives $a = b = c$. One solution is the eigenvector

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}^t.$$

Corresponding to the eigenvalue $\lambda = 2$, we have a solution as

$$e^{2x}\begin{bmatrix}1 & 1 & 1\end{bmatrix}^t.$$

For the eigenvalue -1, we have

$$\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = -1\begin{bmatrix}a\\ b\\ c\end{bmatrix} \Rightarrow \begin{array}{r}b + c = -a\\ a + c = -b.\\ a + b = -c\end{array}$$

Two linearly independent solutions of these equations (actually only one equation $a + b + c = 0$) are

$$\begin{bmatrix}1 & 0 & -1\end{bmatrix}^t, \quad \begin{bmatrix}0 & 1 & -1\end{bmatrix}^t.$$

Thus, the two linearly independent solutions corresponding to the eigenvalue -1 are

$$e^{-x}\begin{bmatrix}1 & 0 & -1\end{bmatrix}^t, \quad e^{-x}\begin{bmatrix}0 & 1 & -1\end{bmatrix}^t.$$

Then, the general solution is obtained by taking a linear combination of all the linearly independent solutions. It is

$$y = \begin{bmatrix}y_1(x)\\ y_2(x)\\ y_3(x)\end{bmatrix} = c_1e^{2x}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} + c_2e^{-x}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix} + c_3e^{-x}\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}.$$

This can be alternatively written as

$$y_1(x) = c_1e^{2x} + c_2e^{-x},\ y_2(x) = c_1e^{2x} + c_3e^{-x},\ y_3(x) = c_1e^{2x} - (c_2 + c_3)e^{-x}. \qquad \Box$$

(2.23) ***Example***

Find the general solution of the system of ODEs:

$$y_1' = 2y_1 + y_2 + 3y_3,\ y_2' = 2y_2 - y_3,\ y_3' = 2y_3.$$

The characteristic polynomial of the system matrix A is

$$|A - \lambda I| = \begin{vmatrix}2-\lambda & 1 & 3\\ 0 & 2-\lambda & -1\\ 0 & 0 & 2-\lambda\end{vmatrix} = (2-\lambda)^3.$$

The only eigenvalue is 2 repeated 3 times. We will get three linearly independent solutions. For the first such solution, we compute the eigenvector of A. So, we solve $(A - 2I)v = 0$. With $v = [a\ \ b\ \ c]^t$, we have the linear equations given by

$$(A - 2I)\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0 & 1 & 3\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

The linear equations are $b + 3c = 0,\ -c = 0$. This gives $b = 0,\ c = 0$ and a is arbitrary. Thus there is only one linearly independent eigenvector. Choosing $a = 1$, we get one such, which is $[1\ \ 0\ \ 0]^t$. The corresponding solution is

$$e^{2x}\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

For other linearly independent solutions, we solve $(A-2I)^2u = 0$ but $(A-2I)u \neq 0$. With $u = [a\ \ b\ \ c]^t$, the equations are

$$(A-2I)^2u = \begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}u = \begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

It gives $c = 0$ and both $a,\ b$ arbitrary. We choose $a = 0$ and $b = 1$ so that the vector $u = [0\ \ 1\ \ 0]^t$ satisfies $(A-2I)^2u = 0$ and $(A-2I)u \neq 0$. The corresponding solution is

$$e^{2x}\begin{bmatrix}0\\1\\0\end{bmatrix} + xe^{2x}\begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = e^{2x}\begin{bmatrix}0\\1\\0\end{bmatrix} + e^{2x}x\begin{bmatrix}1\\0\\0\end{bmatrix} = e^{2x}\begin{bmatrix}x\\1\\0\end{bmatrix}.$$

We have got only two linearly independent solutions. So, we proceed further following the same method. We solve the linear equations $(A-2I)^3w = 0$ with $(A-2I)^2w \neq 0$. As $(A-2I)^3 = 0$, the zero matrix, any nonzero vector w satisfies $(A-2I)^3w = 0$. As $(A-2I)^2w \neq 0$, we choose $w = [0\ \ 0\ \ 1]^t$. Then, the corresponding solution is

$$\begin{aligned}
&e^{2x}\big[w + x(A-2I)w + \frac{x^2}{2}(A-2I)^2w\big]\\
&= e^{2x}\begin{bmatrix}0\\0\\1\end{bmatrix} + xe^{2x}\begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} + \frac{x^2}{2}e^{2x}\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\\
&= e^{2x}\begin{bmatrix}0\\0\\1\end{bmatrix} + xe^{2x}\begin{bmatrix}3\\-1\\0\end{bmatrix} + \frac{x^2}{2}e^{2x}\begin{bmatrix}-1\\0\\0\end{bmatrix} = e^{2x}\begin{bmatrix}3x - x^2/2\\-x\\1\end{bmatrix}.
\end{aligned}$$

Since we have got 3 linearly independent solutions, the general solution is obtained by taking a linear combination of these linearly independent solutions. It is given by

$$y = \begin{bmatrix}y_1(x)\\y_2(x)\\y_3(x)\end{bmatrix} = c_1e^{2x}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2e^{2x}\begin{bmatrix}x\\1\\0\end{bmatrix} + c_3e^{2x}\begin{bmatrix}3x - x^2/2\\-x\\1\end{bmatrix}.$$

We can write the general solution as

$$y_1(x) = \big(c_1 + c_2 x + c_3(3x - x^2/2)\big)e^{2x},\ y_2(x) = \big(c_2 - c_3 x\big)e^{2x},\ y_3(x) = c_3 e^{2x}. \qquad \square$$

2.5 Euler-Cauchy Equation

A particular type of ODE, called the *Euler-Cauchy equations*, do not have constant coefficients but they can be solved by the methods suitable for constant coefficients. The Euler-Cauchy equation is a linear second order ODE of the form

$$x^2 y'' + axy' + by = 0 \quad \text{for } x > 0 \tag{2.5.1}$$

where a and b are constants. Notice that it is a linear second order ODE but not of constant coefficients type.

Substitute $t = \log x$ so that $x = e^t$. Using the chain rule, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{dy}{dt}\, e^{-t}. \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\Big(\frac{dy}{dt}e^{-t}\Big) = \frac{d}{dt}\Big(\frac{dy}{dt}e^{-t}\Big)\frac{dt}{dx} = \Big(\frac{d^y}{dt^2}e^{-t} + \frac{dy}{dt}(-e^{-t})\Big)\frac{1}{x} \\ &= \Big(\frac{d^2y}{dt^2} - \frac{dy}{dt}\Big)\frac{1}{x^2}. \end{aligned}$$

Substituting these in (2.5.1), we obtain

$$0 = x^2y'' + axy' + by = \frac{d^2y}{dt^2} - \frac{dy}{dt} + a\frac{dy}{dt} + by = \frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by.$$

Thus, we have got a linear homogeneous second order ODE with constant coefficients, whose characteristic equation is

$$\lambda^2 + (a-1)\lambda + b = 0.$$

Notice that this equation can also be written as

$$\lambda(\lambda - 1) + a\lambda + b = 0.$$

This equation being the characteristic equation for the ODE with the new variable t, is called the **Auxiliary equation** for the original ODE (2.5.1).

We solve the above ODE with constant coefficients having the independent variable as t, and then substitute back to obtain a solution to (2.5.1) with the independent variable as x. The three ensuing cases are as follows.

Case 1: Suppose $\lambda_1 \neq \lambda_2$ are the two real roots of the auxiliary equation. Then the general solution is given by

$$y(x) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{\lambda_1 \log x} + c_2 e^{\lambda_2 \log x} = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}.$$

Indeed, it is easily verified that $y_1 = x^{\lambda_1}$ and $y_2 = x^{\lambda_2}$ satisfy the ODE (2.5.1).

Case 2: Suppose $\lambda_2 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then the general solution is given by (with $t = \log x$)

$$y(x) = e^{\alpha t}\big[c_1 \cos(\beta t) + c_2 \sin(\beta t)\big] = x^{\alpha}\big[c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)\big].$$

Also, we can directly verify that $y_1 = x^{\alpha}\cos(\beta \log x)$ and $y_2 = x^{\alpha}\sin(\beta \log x)$ are solutions of (2.5.1).

Case 3: Suppose $\lambda_2 = \lambda_1 \in \mathbb{R}$. Then the general solution is given by $(t = \log x)$

$$y(x) = (c_1 + c_2 t)e^{\lambda_1 t} = (c_1 + c_2 \log x)x^{\lambda_1}.$$

Again, this fact can be verified without going through the details of derivation.

Notice that finally, one solution is obtained in the form x^{λ} instead of $e^{\lambda x}$ as used to be for the constant coefficients case. This is also easy to guess since the second order derivative is multiplied with x^2 and the first order derivative is multiplied with x. If we try a solution in the form $y = x^{\lambda}$, then the equation (2.5.1) yields

$$0 = x^2(x^{\lambda})'' + ax(x^{\lambda})' + b(x^{\lambda}) = x^{\lambda}\big(\lambda(\lambda - 1) + a\lambda + b\big).$$

Since x^{λ} is not the zero function, we get the auxiliary equation

$$\lambda(\lambda - 1) + a\lambda + b = 0.$$

This is another heuristic way to solve the Euler-Cauchy equation.

(2.24) *Example*

(1) The ODE $2x^2y'' + 3xy' - y = 0$ is the Euler-Cauchy equation

$$x^2y'' + (3/2)xy' - (1/2)y = 0.$$

Its auxiliary equation $\lambda(\lambda-1)+(3/2)\lambda-(1/2) = 0$ has roots $\lambda_1 = 1/2$ and $\lambda_2 = -1$. As in Case 1, the general solution is

$$y = c_1\sqrt{x} + c_2/x.$$

(2) The ODE $100x^2y'' + 60xy' + 1604y = 0$ is the Euler-Cauchy equation

$$x^2y'' + 0.6xy' + 16.04y = 0.$$

Its auxiliary equation $\lambda(\lambda - 1) + 0.6\lambda + 16.04 = 0$ has roots $\lambda_1 = 0.2 + 4i$ and $\lambda_2 = 0.2 - 4i$. As in Case 2, the general solution is

$$y = x^{0.2}\big[c_1 \cos(4 \log x) + c_2 \sin(4 \log x)\big].$$

(3) The Euler-Cauchy equation

$$x^2 y'' - 5xy + 9y = 0$$

has the auxiliary equation $\lambda(\lambda - 1) - 5\lambda + 9 = 0$. The roots of the auxiliary equation are $\lambda_1 = \lambda_2 = 3$. As in Case 3, the general solution is

$$y = x^3(c_1 + c_2 \log x).$$ □

(2.25) ***Example***
Solve the ODE $4x^2 y'' + 4\alpha x y' + (\alpha - 1)^2 y = 0$ for $x > 0$.

This is an Euler-Cauchy equation with $a = \alpha$ and $b = (\alpha - 1)^2/4$. The auxiliary equation is $\lambda(\lambda - 1) + \alpha\lambda + (\alpha - 1)^2/4 = 0$ or $\lambda^2 + (\alpha - 1)\lambda + (\alpha - 1)^2/4 = 0$. Its roots are $\lambda_1 = \lambda_2 = (1 - \alpha)/2$. Hence, the general solution is

$$y = x^{(1-\alpha)/2}(c_1 + c_2 \log x).$$ □

2.6 Reduction of order

Consider the homogeneous linear second order ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{2.6.1}$$

In this ODE, the coefficients of y and y' are functions of x. The method of taking characteristic equations will not apply to this case. Unfortunately, there is no simple method to solve these “variable coefficients" type of ODEs. However, methods exist to get a second solution if one solution is known so that the two would become linearly independent. This method is a simple adaptation of the same for the "constants coefficients" case.

So, suppose $y_1(x)$ is a solution to (2.6.1). That means

$$y_1'' + p(x)y_1' + q(x)y_1 = 0.$$

We wish to determine a second solution $y_2(x)$ so that y_1 and y_2 are linearly independent. That means, if $y_2(x) = y_1(x)u(x)$, then the function $u(x)$ should not be a constant function. We thus assume that

$$y_2(x) = y_1(x)u(x)$$

for some function $u(x)$ which we wish to find. Now,

$$y_2' = y_1'u + y_1u',\ y_2'' = y_1''u + 2y_1'u' + y_1u''.$$

Then the ODE (2.6.1) becomes

$$\begin{aligned} 0 &= y_2'' + p(x)y_2' + q(x)y_2 \\ &= y_1''u + 2y_1'u' + y_1u'' + p(x)(y_1'u + y_1u') + q(x)y_1u \\ &= y_1u'' + \big(2y_1' + p(x)y_1\big)u' + \big(y_1'' + p(x)y_1' + q(x)y_1\big)u \\ &= y_1u'' + \big(2y_1' + p(x)y_1\big)u' \end{aligned}$$

We see that $y_2(x) = y_1(x)u(x)$ is a solution of (2.6.1) provided $v(x) = u'(x)$ satisfies

$$y_1v' + \big(2y_1' + p(x)y_1\big)v = 0.$$

This is a linear first order equation. Its solution is

$$\begin{aligned} v(x) &= c\exp\left(-\int\left(2\,\frac{y_1'(x)}{y_1(x)} + p(x)\right)dx\right) \\ &= c\exp\left(-\int p(x)\,dx\right)\,\exp\left(-2\int\frac{y_1'(x)}{y_1(x)}\,dx\right) \\ &= \frac{c\exp\left(-\int p(x)\,dx\right)}{y_1^2(x)}. \end{aligned}$$

Since we are interested in only one function $u(x)$, we set the constant $c = 1$ in the above and obtain

$$u'(x) = v(x) = \frac{\exp\left(-\int p(x)\,dx\right)}{y_1^2(x)}.$$

Integrating this and setting the arbitrary constant to 0, we obtain the function $u(x)$. Therefore, the second solution $y_2(x)$ is given by

$$y_2(x) = y_1(x)u(x) = y_1(x)\int v(x)\,dx, \quad v(x) = \frac{\exp\left(-\int p(x)\,dx\right)}{y_1^2(x)}.$$

In this method, we solve a second order equation by solving another first order equations in $v(x)$. This is the reason, the method is named as the *method of reduction of order*. However, it applies when we have already got one solution of the ODE.

(2.26) *Example*

Find the solution of the IVP

$$(1 - x^2)y'' + 2xy' - 2y = 0,\ y(0) = 3,\ y'(0) = -4 \quad \text{for} \ \ -1 < x < 1.$$

We see that $y_1(x) = x$ is a solution of the ODE. To get another solution, we use the method of reduction of order. So, let $y_2(x) = y_1(x)u(x)$. Since in (2.6.1) the coefficient of y'' is taken as 1, we rewrite the ODE as

$$y'' + \frac{2x}{1-x^2}y' - \frac{2}{1-x^2}y = 0.$$

Now, $p(x) = 2x/(1-x^2)$ and $q(x) = 1/(1-x^2)$. Our formula for the second solution gives

$$v(x) = \frac{\exp\Big(-\int p(x)\,dx\Big)}{y_1^2(x)} = \frac{\exp\Big(-\int \frac{2x}{1-x^2}\,dx\Big)}{x^2} = \frac{e^{\log(1-x^2)}}{x^2} = \frac{1-x^2}{x^2}.$$

$$y_2(x) = y_1(x)u(x) = x\int v(x)\,dx = x\int \frac{1-x^2}{x^2}\,dx = -x\Big(\frac{1}{x} + x\Big) = -(1+x^2).$$

Therefore, the general solution is

$$y(x) = c_1 x - c_2(1+x^2).$$

The initial conditions imply that

$$3 = y(0) = -c_2, \quad -4 = y'(0) = c_1.$$

Hence the solution to the IVP is $y(x) = 3 - 4x + 3x^2$. ∎

(2.27) ***Example***

Given that $y_1(x) = e^{ax}$ is a solution of the ODE $xy'' - (1+3x)y' + 3y = 0$ for some $a \in \mathbb{R}$, find the general solution.

Substituting $y = e^{ax}$ in the ODE, we get

$$xa^2e^{ax} - (1+3x)ae^x + 3e^{ax} = 0 \Rightarrow (a-3)(xa-1) = 0.$$

Since $a \in \mathbb{R}$, a constant, we have $a = 3$. So, $y_1(x) = e^{3x}$.

The ODE is re-written as

$$y'' - \Big(\frac{1}{x} + 3\Big)y' + \Big(\frac{3}{x}\Big)y = 0$$

so that $p(x) = -(1/x + 3)$ and $q(x) = 3/x$.

For the second solution, we set $y_2(x) = y_1(x)u(x)$ and $v(x) = u'(x)$. By the method of reduction of order, we have

$$\begin{aligned} v(x) &= \frac{\exp\Big(-\int p(x)\,dx\Big)}{y_1^2(x)} = \frac{\exp\Big(-\int(-1/x+3)\,dx\Big)}{e^{3x}} \\ &= e^{-6x}\exp(\log x - 3x) = xe^{-9x}. \end{aligned}$$

$$\begin{aligned} y_2(x) &= y_1(x)\int v(x)\,dx = e^{3x}\int (xe^{-9x})\,dx = e^{3x}\Big(-\frac{x}{9} + \frac{1}{81}\Big)e^{-9x} \\ &= \frac{1}{81}(1-9x)e^{-6x}. \end{aligned}$$

Then, the general solution is $y(x) = c_1 e^{3x} + c_2(1 - 9x)e^{-6x}$. □

2.7 Non-homogeneous second order linear ODE

Before proceeding to discuss any method to to solve the general non-homogeneous linear second order ODE, we will discuss some properties of such equations. These properties will help us in finding general solutions. We will consider the non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{for } x \in I \tag{2.7.1}$$

where I is an open interval and the right hand side function $r(x)$ is not the zero function. Corresponding to this we will consider the homogeneous ODE on the interval I:

$$y'' + p(x)y' + q(x)y = 0 \quad \text{for } x \in I. \tag{2.7.2}$$

We assume that the functions $p(x)$, $q(x)$ and $r(x)$ are continuous on I so that the corresponding IVPs with initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ have unique solutions for a given $x_0 \in I$.

Relations between solutions of the solutions of (2.7.1-2.7.2) is given by the following results.

(2.28) ***Theorem***

The difference of any two solutions of (2.7.1) is a solution of (2.7.2).

Proof. Let $u_1(x)$ and $u_2(x)$ be two solutions of (2.7.1). Then

$$u_1'' + p(x)u_1' + q(x)u_1 = r(x), \quad u_2'' + p(x)u_2' + q(x)u_2 = r(x).$$

Subtracting the second from the first, we get

$$(u_1 - u_2)'' + p(x)(u_1 - u_2)' + q(x)(u_1 - u_2) = 0.$$

That is, $u_1 - u_2$ is a solution of (2.7.2). ∎

(2.29) ***Theorem***

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation (2.7.2) and let $\phi(x)$ be any one (particular) solution of the non-homogeneous equation (2.7.1). Then every solution of (2.7.1) is in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \phi(x)$$

for some constants c_1 and c_2.

Proof. Let $y(x)$ be any solution of (2.7.1). By (2.28), $y(x) - \phi(x)$ is a solution of (2.7.2). By (2.8), $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of (2.7.2). So, $y(x) - \phi(x) = c_1 y_1(x) + c_2 y_2(x)$ for some constants c_1 and c_2. It then follows that $y(x) = c_1 y_1(x) + c_2 y_2(x) + \phi(x)$. ∎

Notice that this theorem reduces the problem of finding a general solution of the non-homogeneous problem to finding two linearly independent solutions of the homogeneous problem and just one solution of the non-homogeneous problem.

(2.30) ***Example***

Find the general solution of the ODE $y'' + y = x$.

The functions $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are two linearly independent solutions of $y'' + y = 0$. The function $\phi(x) = x$ satisfies the ODE $y'' + y = x$. Hence, the general solution of the given ODE is

$$y(x) = c_1 \cos x + c_2 \sin x + x.$$

□

(2.31) ***Example***

If $\phi_1(x) = x$, $\phi_2(x) = x + e^x$ and $\phi_3(x) = 1 + x + e^x$ are three solutions of a certain non-homogeneous linear second order ODE, then find its general solution.

By (2.28), $\psi_1(x) = \phi_2 - \phi_1 = e^x$ and $\psi_2(x) = \phi_3 - \phi_2 = 1$ satisfy the corresponding homogeneous linear second order ODE. Also, $\psi_1(x)$ and $\psi_2(x)$ are linearly independent. The function $\phi_1(x)$ is a particular solution of the non-homogeneous ODE. By (2.29), the general solution is given by

$$y(x) = c_1\psi_1(x) + c_2\psi_2(x) + \phi_1(x) = c_1 e^x + c_2 + x$$

where c_1, c_2 are arbitrary constants. □

To sum up, we know how to solve a homogeneous linear second order ODE with constant coefficients. For the variable coefficients case, if we already know one solution, then we can find another solution linearly independent with the known one by the method of reduction of order. For the non-homogeneous case, we also need a particular solution.

2.8 Variation of parameters

How do we find a particular solution of a non-homogeneous linear second order ODE? We discuss a method that can compute such a particular solution from the two linearly independent solutions of the corresponding homogeneous ODE.

We consider the non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{for } x \in I \tag{2.8.1}$$

where the functions $p(x)$, $q(x)$ and $r(x)$ are continuous on the open interval I. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \quad \text{for } x \in I. \tag{2.8.2}$$

We know that any function in the form $c_1y_1(x) + c_2y_2(x)$ is a solution of (2.8.2). *Idea* : if we treat the constants c_1, c_2 as functions, then probably we will be able to satisfy (2.8.1). So, we try to determine two functions $u_1(x)$ and $u_2(x)$ so that

$$\phi(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

is a solution of (2.8.1). It looks that the idea is a bogus one, since for determining one function $\phi(x)$ we now need to determine two functions $u_1(x)$ and $u_2(x)$. However, it also implicitly says that we have probably some freedom in choosing these two functions. That is, if necessary we can impose some more conditions suitably so that our work becomes simple. With $\phi(x)$ in the above form, we see that

$$\phi'(x) = \big(u_1y_1 + u_2y_2\big)' = \big(u_1y_1' + u_2y_2'\big) + \big(u_1'y_1 + u_2'y_2\big).$$

We will also require $\phi''(x)$. It will involve the second order derivatives of the unknown functions u_1 and u_2. In order to make our work simple, we impose the condition that

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0.$$

Then, $\phi'(x) = u_1y_1' + u_2y_2'$. As $\phi(x)$ satisfies (2.8.1),we have

$$\begin{aligned} r(x) &= \phi''(x) + p(x)\phi'(x) + q(x)\phi(x) \\ &= \big(u_1y_1' + u_2y_2'\big)' + p(x)\big(u_1y_1' + u_2y_2'\big) + q(x)\big(u_1y_1 + u_2y_2\big) \\ &= u_1'y_1' + u_2'y_2' + u_1\big(y_1'' + py_1' + qy_1\big) + u_2\big(y_2'' + py_2' + qy_2\big) \\ &= u_1'y_1' + u_2'y_2' \end{aligned}$$

The last equality follows since both $y_1(x)$ and $y_2(x)$ are solutions of (2.8.2). To sum up, we see that $\phi(x) = u_1y_1 + u_2y_2$ is a solution of (2.8.1) provided that $u_1(x)$, $u_2(x)$ satisfy

$$y_1u_1' + y_2u_2' = 0, \quad y_1'u_1' + y_2'u_2' = r(x).$$

We need to solve these linear equations in the unknowns u_1' and u_2'. So, multiply the first equation by y_2' and second by y_2, then subtract to get

$$\big(y_1y_2' - y_1'y_2\big)u_1' = -r(x)y_2.$$

Similarly, multiplying the first by y_1' and the second by y_1, then subtracting we get

$$\big(y_1 y_2' - y_1' y_2\big)u_2' = r(x)y_1.$$

Recall that the Wronskian is $W[y_1, y_2](x) = y_1 y_2' - y_1' y_2$. Hence, we obtain

$$u_1'(x) = -\frac{r(x)y_2(x)}{W[y_1, y_2](x)}, \quad u_2'(x) = -\frac{r(x)y_1(x)}{W[y_1, y_2](x)}. \tag{2.8.3}$$

Finally, we get $u_1(x)$ and $u_2(x)$ by integrating these. Of course, we can take any suitable constant of integration to make our choices of u_1, u_2 simple. This method of determining a particular solution for a non-homogeneous equation from a fundamental set of solutions for the corresponding homogeneous equation is called the method of *variation of parameters* due to Lagrange.

(2.32) *Example*

Solve the IVP $y'' + y = \tan t$ for $-\pi/2 < t < \pi/2$; with $y(0) = 1 = y'(0)$.

The corresponding homogeneous equation $y'' + y = 0$ has two linearly independent solutions as $y_1 = \cos x$ and $y_2 = \sin x$. To get a particular solution of the given non-homogeneous equation, we first compute the Wronskian. Now,

$$W[y_1, y_2](x) = y_1 y_2' - y_1' y_2 = \cos x(\cos x) - (-\sin x)\sin x = 1.$$

Using variation of parameters, we seek a particular solution $\phi(x)$ in the form

$$\phi(x) = u_1(x)y_1 + u_2(x)y_2 = u_1(x)\cos x + u_2(x)\sin x$$

where due to (2.8.3),

$$u_1'(x) = -\tan x \sin x, \quad u_2'(x) = \tan x \cos x.$$

Integrating and ignoring the constants of integration, we have

$$\begin{aligned} u_1(x) &= \int \tan x \sin x\, dx = \int \frac{\cos^2 x - 1}{\cos x}\, dx = \sin x - \log(\sec x + \tan x). \\ u_2(x) &= \int \tan x \cos x\, dx = \int \sin x\, dx = -\cos x \\ \phi(x) &= \big[\sin x - \log(\sec x + \tan x)\big]\cos x - \cos x \sin x \\ &= -\cos x \log(\sec x + \tan x). \end{aligned}$$

Then, the general solution of the ODE is given by

$$y(x) = c_1 y_1 + c_2 y_2 + \phi = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x).$$

The initial conditions give

$$1 = y(0) = c_1, \ 1 = y'(0) = c_2 - 1 \ \Rightarrow \ c_1 = 1, \ c_2 = 2.$$

Thus, the solution of the IVP is

$$y(x) = \cos x + 2\sin x - \cos x \log(\sec x + \tan x).$$

□

2.9 Method of undetermined coefficients

The variation of parameters is a very general method to determine a solution of the non-homogeneous equation. If the coefficients of the unknown variable y and its derivatives are constants and the right hand side function involves exponentials, polynomials, or trigonometric functions of certain particular forms, then a particular solution can be determined without resorting to integration.

We consider the non-homogeneous linear second order ODE with constant coefficients:

$$ay'' + by' + cy = r(x), \quad a \neq 0. \tag{2.9.1}$$

Its characteristic equation is

$$a\lambda^2 + b\lambda + c = 0.$$

The method of undetermined coefficients asserts that when $r(x)$ is in certain form, the particular solution $\phi(x)$ of the ODE (2.9.1) is of certain form. These statements, written as 'Rules' below follow from the method of variation of parameters. They are as follows.

Rule 1: Suppose $r(x) = p_n(x)e^{\alpha x}$, where $p_n(x)$ is a polynomial of degree n. Then, the particular solution $\phi(x)$ of (2.9.1) is in the following form, where $u_n(x)$ is some polynomial of degree at most n:

(A) If α is not a root of the characteristic equation, then $\phi(x) = u_n(x)e^{\alpha x}$.

(B) If α is a simple root of the characteristic equation, then $\phi(x) = xu_n(x)e^{\alpha x}$.

(C) If α is a double root of the characteristic equation, then $\phi(x) = x^2u_n(x)e^{\alpha x}$.

A particular case of Rule 1 is worth mentioning. In Rule 1, if $\alpha = 0$, then we get the following.

Rule 2: Suppose $r(x) = p_n(x)$, a polynomial of degree n. Then, the particular solution $\phi(x)$ of (2.9.1) is in the following form:

(A) If $c \neq 0$, then $\phi(x) = u_n(x)$.

(B) If $c = 0,\ b \neq 0$, then $\phi(x) = xu_n(x)$.

(C) If $c = 0 = b$, then $\phi(x) = x^2u_n(x)$.

As earlier, $u_n(x)$ is a polynomial of degree at most n.

Rule 3: Suppose $r(x) = e^{\alpha x}\big[p(x)\cos(\beta x) + q(x)\sin(\beta x)\big]$, where $p(x),\ q(x)$ are polynomials. Then, the particular solution $\phi(x)$ of (2.9.1) is in the following form:

(A) If $\alpha + i\beta$ is not a root of the characteristic polynomial, then

$$\phi(x) = e^{\alpha x}\big[u(x)\cos(\beta x) + v(x)\sin(\beta x)\big].$$

(B) If $\alpha + i\beta$ is a root of the characteristic polynomial, then

$$\phi(x) = xe^{\alpha x}\big[u(x)\cos(\beta x) + v(x)\sin(\beta x)\big].$$

Here, $u(x)$ and $v(x)$ are some polynomials whose degrees are at most the highest degree of the polynomials $p(x)$ and $q(x)$.

We emphasize that if one of the polynomials $p(x)$ or $q(x)$ is equal to 0, then $r(x)$ involves only one of the terms $\cos(\beta x)$ or $\sin(\beta x)$. In that case, $\phi(x)$ may still involve both thee terms $\cos(\beta x)$ and $\sin(\beta x)$.

This rule says that we must try to determine the coefficients in $u_n(x)$ by plugging in this $\phi(x)$ in (2.9.1).

In Rule 3, if $\alpha = 0$ and the polynomials $p(x)$ and $q(x)$ are constants, we get the following important case.

Rule 4: Suppose $r(x) = d_1\cos(\beta x) + d_2\sin(\beta x)$ for some constants d_1 and d_2. Then $\phi(x)$ is in the following form:

(A) If βi is not a root of the characteristic equation, then

$$\phi(x) = A\cos(\beta x) + B\sin(\beta x).$$

(B) If βi is a root of the characteristic equation, then

$$\phi(x) = x\big[A\cos(\beta x) + B\sin(\beta x)\big].$$

We remark that if $r(x)$ is a sum of functions, then their corresponding $\phi(x)$ are to be added.

(2.33) ***Example***

Find a particular solution of the ODE $y'' + y' + y = x^2$.

By Rule 2, a particular solution may be tried in the form $\phi(x) = A + Bx + Cx^2$. As ϕ satisfies the ODE, we obtain

$$\begin{aligned} x^2 &= \phi'' + \phi' + \phi = 2C + (B + 2Cx) + A + Bx + Cx^2 \\ &= (A + B + 2C) + (B + 2C)x + Cx^2 \\ &\Rightarrow A + B + 2C = 0,\ B + 2C = 0,\ C = 1 \ \Rightarrow\ A = 0,\ B = -2,\ C = 1 \end{aligned}$$

Hence, $\phi(x) = -2x + x^2$ is a particular solution. □

(2.34) ***Example***

Find a particular solution of the ODE $y'' - 3y' + 2y = (1 + x)e^{3x}$.

To use Rule 1, we should check whether 3 is a root of the characteristic equation.

The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$, and 3 is not its root. So, by Rule 1(A), $\phi(x) = (A + Bx)e^{3x}$. Then

$$\begin{aligned}(1+x)e^{3x} &= \phi'' - 3\phi' + 2\phi \\ &= e^{3x}\big((9A + 6B + 9Bx) - 3(3A + B + 3Bx) + 2(A + Bx)\big) \\ &= e^{3x}\big((2A + 3B) + 2Bx\big) \\ &\Rightarrow 2B = 1,\ 2A + 3B = 1 \Rightarrow A = -1/4,\ B = 1/2.\end{aligned}$$

Hence, a particular solution is $\phi(x) = (-1/4 + x/2)e^{3x}$. □

(2.35) *Example*

Solve the ODE $y'' - 7y' + 6y = (x - 2)e^x$.

The characteristic equation is $\lambda^2 - 7\lambda + 6 = 0$ whose roots are 6 and 1. Here, the right hand side is in the form $p_1(x)e^x$ and $\alpha = 1$ is a simple root of the characteristic equation. So, we seek a particular solution in the form $\phi(x) = x(A + Bx)e^x$. Plugging in the equation, we obtain

$$\begin{aligned}(x-2)e^x &= \big((Ax + Bx^2) + (2A + 4Bx) + 2B - 7(Ax + Bx^2) \\ &\quad -7(A + 2Bx) + 6(Ax + Bx^2)\big)e^x \\ &= (-5A + 2B - 10Bx)e^x \\ &\Rightarrow -5A + 2B = -2,\ -10B = 1 \Rightarrow A = 9/25,\ B = -1/10.\end{aligned}$$

Hence, $\phi(x) = x(9/25 - x/10)e^x$ is a particular solution. The general solution of the ODE is $y(x) = c_1 e^{6x} + c_2 e^x + x(9/25 - x/10)e^x$. □

(2.36) *Example*

Find a particular solution of the ODE $y'' + 4y = \sin(2x)$.

The characteristic equation $\lambda^2 + 4 = 0$ has roots $\pm 2i$. By Rule 4(B), a particular solution is in the form $\phi(x) = x\big[A\cos(3x) + B\sin(3x)\big]$. Plugging it in the equation, we get

$$\begin{aligned}\sin(2x) &= \phi'' + 4\phi \\ &= x\big[-4A\cos(2x) - 4B\sin(2x)\big] + \big[-2A\sin(2x) + 2B\cos(2x)\big] \\ &\quad + \big[-2A\sin(2x) + 2B\cos(2x)\big] + 4x\big[A\cos(2x) + B\sin(2x)\big] \\ &= -4A\sin(2x) + 4B\cos(2x)\end{aligned}$$

Comparing the left and the right hand sides, we get $A = -1/4$ and $B = 0$. Then, a particular solution is given by $\phi(x) = -(x/4)\cos(2x)$. □

(2.37) *Example*

Solve the IVP $y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x$,
$y(0) = 2.78,\ y'(0) = -0.43$.

The characteristic equation is $\lambda^2 + 2\lambda + 0.75 = 0$ having roots as $\lambda_1 = -1/2$ and $\lambda_2 = -3/2$. Hence, two linearly independent solutions of the homogeneous equation are $y_1 = e^{-x/2}$ and $y_2 = e^{-3x/2}$.

The non-homogeneous term is $r(x) = (2\cos x - 0.25\sin x) + 0.09x$. We first find a particular solution $\phi(x)$ for

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x.$$

Since 1 is not a root of the characteristic equation, by Rule 4(A), we try a particular solution in the form $\phi(x) = A\cos x + B\sin x$. Plugging it in the ODE, we get

$$\begin{aligned}
&2\cos x - 0.25\sin x = \phi'' + 2\ \ ' + 0.75\phi \\
&= (-A\cos x - B\sin x) + 2(-A\sin x + B\cos x) + 0.75(A\cos x + B\sin x) \\
&= (-A + 2B + 0.75)\cos x + (-B - 2A + 0.75)\sin x \\
\Rightarrow\ & -A + 2B + 0.75 = 2,\ -B - 2A + 0.75 = -0.25 \\
\Rightarrow\ & A = 0,\ B = 1.
\end{aligned}$$

So, $\phi(x) = \sin x$. Next, we find a particular solution $\psi(x)$ for

$$y'' + 2y' + 0.75y = 0.99x.$$

By Rule 2, we try $\psi(x) = C + Dx$. As $\psi' = D$ and $\psi'' = 0$, we get

$$0.09x = \psi'' + 2\psi' + 0.75\psi = 2D + 0.75(C + Dx) \ \Rightarrow\ C = -0.32,\ D = 0.12.$$

Hence, $\psi(x) = 0.12x - 0.32$. Therefore, a particular solution of the ODE is

$$\phi + \psi = \sin x + 0.12x - 0.32.$$

That is, the general solution of the ODE is

$$y(x) = c_1 e^{-x/2} + c_2 e^{-3/2} + \sin x + 0.12x - 0.32.$$

The initial conditions imply

$$2.78 = y(0) = c_1 + c_2 - 0.32,\ -0.4 = y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12.$$

Solving it, we obtain $c_1 = 3.1$ and $c_2 = 0$. So, the solution of the IVP is

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32.$$

□

(2.38) ***Example***

Find the general solution of $y'' - 4y' + 4y = (1 + x + x^2 \cdots + x^{25})e^{2x}$.

The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$ of which the roots are $\lambda_1 = \lambda_2 = 2$. Hence, $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ are two linearly independent solutions of the corresponding homogeneous equation. A particular solution $\phi(x)$ is in the form

$$\phi(x) = x^2\big(A_0 + A_1x + \cdots + A_{25}x^{25}\big)e^{2x}$$

It is of course sheer waste of time to plug in such a $\phi(x)$ in the ODE and try to evaluate $A_0, A_1, \ldots, A_{25}$. Following the method of variation of parameters, we rather write

$$\phi(x) = u(x)e^{2x}$$

and plug it in the ODE. It gives

$$\phi'(x) = \big(u'(x) + 2u(x)\big)e^{2x}, \quad \phi''(x) = \big(u''(x) + 4u'(x) + 4u(x)\big)e^{2x}.$$

As $\phi(x)$ satisfies the ODE, we get

$$\phi'' - 4\phi' + 4\phi = u''(x)e^{2x} = \big(1 + x + x^2 \cdots + x^{25}\big)e^{2x}.$$

That is, $u''(x) = 1 + x + x^2 \cdots + x^{25}$. Integrating twice and setting the constants of integration to 0, we have

$$u(x) = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \cdots + \frac{x^{27}}{26\cdot 27}.$$

Hence, the general solution is

$$y(x) = \left(c_1 + c_2x + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \cdots + \frac{x^{27}}{26\cdot 27}\right)e^{2x}.$$ □

3

Series Solutions

3.1 Introduction

To recall, we could solve a linear homogeneous second order ODE with constant coefficients some what satisfactorily. For such an ODE with variable coefficients, we could only get a second solution provided a first solution is already known. How do we get this first solution? We relied on guess work. The main aim of this chapter is to obtain a first solution by using power series. We recall some facts about power series.

A power series about $x = x_0$ is in the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0) \tag{3.1.1}$$

where $a_0, a_1, \ldots$ are constants.

Each power series has an interval of convergence. That is, there exists $r \geq 0$ such that the power series (3.1.1) converges for all x with $|x - x_0| < r$ and diverges for all x with $|x - x_0| > r$. This number r is called the radius of convergence of the power series (3.1.1).

If $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists, then the limit is equal to the radius of convergence of (3.1.1). Also, the radius of convergence of (3.1.1) is equal to $\left(\lim_{n\to\infty} |a_n|^{1/n}\right)^{-1}$ provided this limit exists in $\mathbb{R} \cup \{\infty\}$.

Two power series $\sum_{n=0}^{\infty} a_0(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are equal iff $a_n = b_n$ for each $n = 0, 1, 2, \ldots$. In particular, $\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$ iff $a_n = 0$ for each $n = 0, 1, 2, \ldots$.

Two power series can be added and multiplied the following way:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n + \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n.$$

$$\Big(\sum_{n=0}^{\infty} a_n(x - x_0)^n\Big)\Big(\sum_{n=0}^{\infty} b_n(x - x_0)^n\Big) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

If $b_0 \neq 0$, then the quotient $\dfrac{a_0 + a_1 x + \cdots}{b_0 + b_1 x + \cdots}$ of two power series is a power series.

The power series (3.1.1) can be differentiated and integrated term by term and the resultant series has the same radius of convergence. In particular,

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)' = \big(a_0 + a_1 x + a_2 x^2 + \cdots\big)' = a_1 + 2a_2 x + \cdots = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

A function $f(x)$ is said to be **analytic** at $x = x_0$ iff there exist constants $a_0, a_1, a_2, \ldots$ such that for all x in a neighborhood of x_0,

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

This series is called the Taylor series of the function $f(x)$ at $x = x_0$ and the coefficients satisfy

$$a_n(n!) = f^{(n)}(x_0).$$

When $x_0 = 0$, the Taylor series is called the Maclaurin series. We are familiar with the following Maclaurin series:

$$\begin{aligned}
(1-x)^{-1} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1. \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{for } x \in \mathbb{R}. \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for } x \in \mathbb{R} \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for } x \in \mathbb{R}.
\end{aligned}$$

3.2 Regular and singular points

Our plan is to plug in a power series in place of $y(x)$ in a linear second order homogeneous ODE and try to evaluate the coefficients a_n. We hope that if a solution of the ODE has an analytic solution at a point, then we should be able to find out the coefficients. Before discussing what will be the general case, let us consider an example, and try to execute our ideas.

(3.1) *Example*

Find a power series solution of the first order ODE $y' - y = 0$.

Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution of the ODE. Plugging it in the ODE and using term by term differentiation, we get

$$0 = \sum_{n=1}^{\infty} n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n x^{n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} (n a_n - a_{n-1}) x^{n-1}$$
$$\Rightarrow a_n = \frac{a_{n-1}}{n} \quad \text{for } n \geq 1.$$

We obtain a recurrence relation between the coefficients. It gives

$$a_1 = \frac{a_0}{1},\ a_2 = \frac{a_1}{2} = \frac{a_0}{2!},\ a_3 = \frac{a_2}{3} = \frac{a_0}{3!},\ \ldots,\ a_n = \frac{a_0}{n!}.$$

Notice that the constant a_0 remains arbitrary. Then

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

We see that we have obtained the general solution of the ODE. ∎

We wish to apply the power series method to linear second order ODEs. For this purpose, we consider the following linear second order homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0. \tag{3.2.1}$$

When the coefficient of y'' is 1 as in (3.2.1), we say that the ODE is in its **standard form**. The central fact about such equations is that the nature of solutions depend on the nature of the coefficient functions $p(x)$ and $q(x)$. The ease of obtaining a solution depends on how smooth are the coefficient functions. To demarcate the cases, we will need some terminology.

A point x_0 is said to be an **ordinary point** of the ODE (3.2.1) iff both the functions $p(x)$ and $q(x)$ are analytic at $x = x_0$. An ordinary point is sometimes called a *regular point*.

A point x_0 is called a **singular point** of the ODE iff it is not an ordinary point of the ODE. At a singular point at least one of $p(x)$ or $q(x)$ fails to be analytic.

A singular point x_0 of the ODE (3.2.1) is called a **regular singular point** iff both the functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.

A singular point x_0 of the ODE (3.2.1) is called an **irregular singular point** iff at least one of the functions $(x - x_0)p(x)$ or $(x - x_0)^2 q(x)$ fails to be analytic at x_0.

Roughly speaking at a regular singular point x_0, $p(x)$ is not worse than $(x - x_0)^{-1}$ and $q(x)$ is not worse than $(x - x_0)^{-2}$. The reason for defining regular singular point is that we can still obtain a solution to the ODE which involves a power series at x_0. We will soon see this in the guise of a theorem.

We should take care to bring a given equation to the form of (3.2.1), which is called the *standard form* while deciding about a point being ordinary or singular.

Further, in some ODEs we will require the behavior of a solution as x approaches ∞. Thus, we need to determine whether $x_0 = \infty$ is an ordinary point or a regular singular point or neither. In such a case, we transform the ODE to one by taking $t = 1/x$ and then find what kind of a point $t = 0$ is. So, write $Y(t) = y(x) = y(1/t)$. Then

$$y'(x) = \frac{dY/dt}{dx/dt} = -t^2\frac{dY}{dt}$$
$$y''(x) = -t^2\frac{d}{dt}\Big(-t^2\frac{dY}{dt}\Big) = t^4\frac{d^2Y}{dt^2} + 2t^3\frac{dY}{dt}.$$

Then the ODE (3.2.1) reduces to

$$t^4\frac{d^2Y}{dt^2} + \big(2t^3 - t^2p(1/t)\big)\frac{dY}{dt} + q(1/t)Y(t) = 0.$$

Next, we say that $x = \infty$ is an ordinary, a regular singular , or an irregular singular point of the original ODE according as $t = 0$ is a respective point of the above ODE.

(3.2) *Example*

1. Consider the ODE $xy'' - y = 0$. In the standard form of (3.2.1), $p(x) = 0$ and $q(x) = -1/x$. Thus $x_0 = 1$ is an ordinary point. But $x_0 = 0$ is a singular point. Further, $xp(x) = 0$ and $xq(x) = -1$ are analytic at $x_0 = 0$. Hence $x_0 = 0$ is a regular singular point.
2. The Legendre's equation $(1 - x^2)y'' - 2xy' + 6y = 0$, in standard form, is

   $$y'' - \frac{2x}{1 - x^2}y' + \frac{6}{1 - x^2} = 0.$$

   Here, $p(x) = -2x/(1 - x^2)$ and $q(x) = 6/(1 - x^2)$. The point $x = 0$ is an ordinary point of the ODE. In fact every point other than $x = \pm 1$ is an ordinary point of the ODE. The points $x = \pm 1$ are its singular points. Further, $(x-1)p(x) = 2x/(1+x)$ and $(x-1)^2q(x) = 6(1-x)/(x+1)$ are analytic at $x_0 = 1$. Hence, $x_0 = 1$ is a regular singular point. Similarly, $(x + 1)p(x) = 2x/(x - 1)$ and $(x + 1)^2q(x) = 6(x + 1)/(1 - x)$ are analytic at $x = -1$. So, $x_0 = -1$ is a regular singular point.
3. Consider the ODE $(x + 1)^2y'' + y' - y = 0$. Here, $p(x) = (x + 1)^{-2}$ and $q(x) = -(x + 1)^{-2}$. Any point $x_0 \neq -1$ is an ordinary point. The point $x_0 = -1$ is a singular point. Now, $(x + 1)p(x) = (x + 1)^{-1}$ is not analytic at $x_0 = -1$. Hence, $x_0 = -1$ is an irregular singular point.
4. Consider Airy's equation $y'' - xy = 0$. To classify the point at infinity, we put $t = 1/x$. Here $p(x) = 0$ and $q(x) = -1$. The ODE is reduced to

   $$t^4\frac{d^2Y}{dt^2} + \big(2t^3 - t^2(0)\big)\frac{dY}{dt} + (-1)Y(t) = 0 \quad \text{or} \quad \frac{d^2Y}{dt^2} + \frac{2}{t}\frac{dY}{dt} - \frac{1}{t^3}Y(t) = 0.$$

Here, $p(t) = 2/t$ and $q(t) = -1/t^3$. Since $t^2 q(t) = -1/t$ is not analytic at $t_0 = 0$, we conclude that $t_0 = 0$ is an irregular singular point. Therefore, $x_0 = \infty$ is an irregular singular point of Airy's equation. □

In the following two sections, we discuss how to obtain a series solution of (3.2.1) about an ordinary point, and also about a regular singular point. Unfortunately, we do not have any general method for finding a series solution of a linear homogeneous second order ODE with variable coefficients when the concerned point is an irregular singular point.

3.3 Power series solution at an ordinary point

We assume that the point x_0 is an ordinary point of the ODE (3.2.1). That is, the coefficient functions $p(x)$ and $q(x)$ have power series expansions at x_0. Due to (3.3), we assume that $y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$ is a solution of the ODE. Using term by term differentiation, we get the series expansions of $y'(x)$ and $y''(x)$. Substituting these expressions in to (3.2.1) and comparing the coefficients of powers of x, we determine the coefficients a_n except possibly two. These two constants will remain arbitrary and we would obtain a general solution of (3.2.1).

The following result guarantees that the above method works.

We will use the following result without proof.

(3.3) ***Theorem***

Let x_0 be a regular point of the ODE (3.2.1). Then the following are true:

(1) *There exists a solution $y(x)$ of (3.2.1) which is analytic at x_0.*

(2) *The IVP consisting of the ODE (3.2.1) and the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$ for $y_0, y_0' \in \mathbb{R}$ has a unique solution $y(x)$ which is analytic at x_0.*

(3) *If $p(x)$ and $q(x)$ have Taylor series expansions about $x = x_0$ convergent for all x with $|x - x_0| < \rho$ for some $\rho > 0$, then in both (1)-(2), the radius of convergence of the Taylor series for $y(x)$ is at least ρ.*

(3.4) ***Example***

Solve Legendre's equation $(1 - x^2)y'' - 2xy' + 2y = 0$ by power series method.

Here $p(x) = -2x/(1 - x^2)$ and $q(x) = 2/(1 - x^2)$ are analytic at $x = 0$. By (3.3), there exists a power series solution to the ODE about $x = 0$. So, we assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Plugging it in the ODE, we obtain

$$\begin{aligned}0 &= (1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}[-n(n-1)-2n+2]a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n-1)(n+2)a_n x^n\end{aligned}$$

So, coefficient of each power of x is 0. It gives $a_{n+2} = \dfrac{n-1}{n+1}a_n$ for $n \ge 0$.

This recurrence relation gives $a_3 = 0$, $a_5 = \frac{2}{4}a_3 = 0, \ldots$. That is, all odd coefficients except a_1 are 0. And,

$$a_2 = -a_0,\ a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0,\ a_6 = \frac{3}{5}a_4 = -\frac{1}{3}\frac{3}{5}a_0 = \frac{1}{5}a_0, \ldots$$

The even coefficients are given by $a_{2n} = -\dfrac{1}{2n-1}a_0$. Hence,

$$y(x) = a_1 x + a_0(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots).$$

□

(3.5) ***Example***

Determine two linearly independent solutions of

$$y'' + \frac{3x}{1+x^2}y' + \frac{1}{1+x^2}y = 0.$$

Then, find the solution $y(x)$ of the ODE that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 3$.

We will use the power series method for solving the IVP. The functions $3x/(1+x^2)$ and $1/(1+x^2)$ are analytic at $x = 0$. Due to (3.3), we try a solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Instead of plugging in the expression for y in the ODE, we multiply the ODE with $(1+x^2)$ and then put the series for $y(x)$. This will make our computations simpler. Then

$$\begin{aligned}0 &= (1+x^2)y'' + 3xy' + y\\
&= (1+x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 3x\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n\\
&= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}[n(n-1)+3n+1]a_n x^n\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)^2 a_n x^n.\end{aligned}$$

So, the coefficient of like powers of x is 0; it gives $(n+2)(n+1)a_{n+2}+(n+1)^2a_n = 0$. Hence

$$a_{n+2} = -\frac{(n+1)^2 a_n}{(n+2)(n+1)} = -\frac{(n+1)a_n}{n+2} \quad \text{for } n \geq 0.$$

This is a recurrence relation to express $a_2, a_3, \ldots$ in terms of a_0 and a_1. We choose two simplest cases: (i) $a_0 = 1,\ a_1 = 0$; (ii) $a_0 = 0,\ a_1 = 1$ to obtain two linearly independent solutions.

(i) $a_0 = 1,\ a_1 = 0$. Now, $a_3 = 0$, $a_5 = 0$; in fact, all odd coefficients are 0. The even coefficients are determined from

$$a_2 = -\frac{a_0}{2} = -\frac{1}{2},\ a_4 = -\frac{3a_2}{4} = \frac{1\cdot 3}{2\cdot 4},\ \ldots.$$

Proceeding inductively, we find that

$$a_{2n} = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = (-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n n!}.$$

Thus,

$$y_1(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n n!} = 1 - \frac{t^2}{2} + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots.$$

(ii) $a_0 = 0,\ a_1 = 1$. In this case, all even coefficients are 0, and the odd coefficients are determined from

$$a_3 = -\frac{2a_1}{3} = -\frac{2}{3},\ a_5 = -\frac{4a_3}{5} = \frac{2\cdot 4}{3\cdot 5},\ \ldots.$$

Proceeding inductively, we find that

$$a_{2n+1} = (-1)^n\frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)} = \frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)}.$$

Thus,

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)} x^{2n+1} = x - \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 - \cdots .$$

It is easily verified that both the power series for $y_1(x)$ and $y_2(x)$ converge for $|x| < 1$ and diverge for $|x| > 1$.

Further, observe that by construction, the solutions $y_1(x)$ and $y_2(x)$ satisfy

$$y_1(0) = 1,\ y_1'(0) = 0, \quad y_2(0) = 0,\ y_2'(0) = 1.$$

Therefore, the initial conditions $y(0) = 2$ and $y'(0) = 3$ are satisfied by the solution

$$y(x) = 2y_1(x) + 3y_2(x).$$ □

(3.6) ***Example***

Solve the IVP: $(x^2 - 2x)y'' + 5(x-1)y' + 3y = 0,\ y(1) = 7,\ y'(1) = 3.$

The initial conditions are given at $x = 1$. So, we try a solution as a power series about $x = 1$. Set $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. Plugging it in the ODE, we obtain

$$\begin{aligned}
0 &= (x^2 - 2x)y'' + 5(x-1)y' + 3y \\
&= \big((x-1)^2 - 1\big)\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + 5(x-1)\sum_{n=0}^{\infty} na_n(x-1)^{n-1} + 3\sum_{n=0}^{\infty} a_n(x-1)^n \\
&= -\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty}(5n+3)a_n(x-1)^n \\
&= -\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty}(n^2+4n+3)a_n(x-1)^n.
\end{aligned}$$

So, the coefficient of all powers of $(x-1)$ are 0. It gives

$$a_{n+2} = \frac{n^2+4n+3}{(n+2)(n+1)}a_n = \frac{n+3}{n+2}a_n \quad \text{for } n \ge 0.$$

Now, $a_0 = y(1) = 7$ and $a_1 = y'(1)/1! = 3$. Using the above recurrence relation,

$$\begin{aligned}
a_2 &= \frac{3}{2}a_0 = \frac{3}{2}\cdot 7,\ a_4 = \frac{5}{4}a_2 = \frac{5\cdot 3}{4\cdot 2}\cdot 7,\ a_6 = \frac{7}{6}a_4 = \frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2}\cdot 7,\ \ldots \\
a_3 &= \frac{4}{3}a_1 = \frac{4}{3}\cdot 3,\ a_5 = \frac{6}{5}a_3 = \frac{6\cdot 4}{5\cdot 3}\cdot 3,\ a_7 = \frac{8}{7}a_5 = \frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}\cdot 3,\ \ldots
\end{aligned}$$

Proceeding inductively, we find that

$$a_0 = 7,\ a_{2n} = \frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)}\cdot 7,\ a_1 = 3,\ a_{2n+1} = \frac{4\cdot 6\cdots(2n+2)}{3\cdot 5\cdots(2n+1)}\cdot 3 \quad \text{for } n \ge 1.$$

And, $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$, where a_ns are as shown above. ∎

We remark that in the recurrence relation for the coefficients, it can very well happen that a_{n+2} depends on a_{n-1}, a_n and a_{n+1}. In such a case, we may not be able to write a_{n+2} as an expression in n.

3.4 Series solution about a regular singular point

Suppose $x = 0$ is a regular singular point of the ODE (3.2.1). Then $xp(x)$ and $x^2q(x)$ have Maclaurin series expansion. This means

$$p(x) = \frac{p_0}{x} + p_1 + p_2x + p_3x^2 + \cdots$$
$$q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3x + q_4x^2 + \cdots$$

Moreover, p_0, q_0 and q_1 are nonzero, so that $p(x)$ and $q(x)$ are not analytic at $x = 0$. In this case, we cannot apply (3.3). In fact, for such equations we do not have a power series solution at $x = 0$. The following result shows that in such a case a solution can be obtained in the form of x^r times a power series for some real number r. But this also is guaranteed only under some more restrictions.

(3.7) ***Theorem*** (Frobenius)

Let $x = 0$ be a regular singular point of the ODE (3.2.1) so that the functions $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with power series expansions

$$xp(x) = p_0 + p_1x + p_2x^2 + \cdots, \quad x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots$$

which converge for $|x| < \rho$ for some $\rho > 0$. Let r_1 and r_2 be the roots of the equation (called the indical equation)

$$r(r-1) + p_0r + q_0 = 0.$$

Then the ODE (3.2.1) has two linearly independent solutions in the following form on the interval $0 < x < \rho$:

(a) *If $r_1,\ r_2 \in \mathbb{R}$, $r_1 > r_2$ and $r_1 - r_2$ is neither 0 nor a positive integer, then*

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_nx^n, \quad y_2(x) = x^{r_2}\sum_{n=0}^{\infty} b_nx^n, \quad a_0 \neq 0,\ b_0 \neq 0.$$

(b) *If $r_1,\ r_2 \in \mathbb{R}$ and $r_1 - r_2$ is a positive integer, then*

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = a y_1(x)\log x + x^{r_2}\sum_{n=1}^{\infty} b_n x^n, \quad a_0 \neq 0.$$

Here, the constant a may turn out to be 0.

(c) *If $r_1,\ r_2 \in \mathbb{R}$ and $r_1 = r_2$, then*

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n,\ y_2(x) = y_1(x)\log x + x^{r_1}\sum_{n=1}^{\infty} b_n x^n, \quad a_0 \neq 0.$$

(d) *If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ with $\beta \neq 0$, then*

$$y_1(x) = \mathrm{Re}\big(z(x)\big), \quad y_2(x) = \mathrm{Im}\big(z(x)\big), \quad z(x) = x^{\alpha+i\beta}\sum_{n=0}^{\infty} a_n x^n.$$

The indical equation referred to in the above result comes from trying a solution of the ODE in the form x^r times a power series. In fact, we will use the above theorem to determine the form of the series which could be a solution of the given ODE. Next, we plug in this series in the ODE and setting the coefficients of all powers to 0, we determine the coefficients.

(3.8) *Example*

Find two linearly independent solutions of $2xy'' + y' + xy = 0$ for $x > 0$.

Here, $p(x) = (2x)^{-1}$ and $q(x) = 1/2$. At $x = 0$, $q(x)$ is analytic, but $p(x)$ is not. However, $xp(x) = 1/2$ and $x^2 q(x) = x^2/2$ are analytic at $x = 0$. So, $x = 0$ is a regular singular point of the ODE. We use Frobenius method to get a solution of the ODE. Since $p_0 = 1/2$ and $q_0 = 0$, the indical equation gives

$$r(r-1) + p_0 x + q_0 = r^2 - r + \frac{r}{2} = r^2 + \frac{r}{2} = 0 \Rightarrow r_1 = \frac{1}{2},\ r_2 = 0.$$

Since $r_2 - r_1$ is not an integer, by (3.7)(a) the two linearly independent solutions are in the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1/2}, \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^n, \quad a_0 \neq 0,\ b_0 \neq 0.$$

Instead of determining a_ns and b_ns separately, we take any solution $y(x)$ as

$$y(x) = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{where } a_0 \neq 0$$

and then try to determine the coefficients a_n by considering two cases $r = 0$ or $r = 1/2$ at an appropriate stage. So, plugging it in the ODE, we obtain

$$\begin{aligned}
0 &= 2xy'' + y' + xy \\
&= 2x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r+1} \\
&= \big[2r(r-1)a_0 + ra_0\big]x^{r-1} + \big[2(1+r)ra_1 + (1+r)a_1\big]x^r \\
&\quad + \sum_{n=2}^{\infty} \big[2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2}\big]x^{n+r-1}.
\end{aligned}$$

Setting the coefficient of each power of x to 0, we get

1. $2r(r-1)a_0 + ra_0 = r(2r-1)a_0 = 0 \Rightarrow r = 0$ or $r = 1/2$, as we had got earlier. In fact, this gives the indical equation.
2. $2(r+1)ra_1 + (r+1)a_1 = (r+1)(2r+1)a_1 = 0.$
3. $2(n+r)(n+r-1)a_n + (n+r)a_n = (n+r)\big[2(n+r)-1\big]a_n = -a_{n-2}$ for $n \ge 2$.

(a) $r = 0$. The recurrence formula (3) gives $a_n = \dfrac{-a_{n-2}}{n(2n-1)}$ for $n \ge 2$.
Since $a_1 = 0$, all odd coefficients are 0. The even coefficients are determined from (3) and they are:

$$a_2 = \frac{-a_0}{2\cdot 3},\ a_4 = \frac{-a_2}{4\cdot 7} = \frac{a_0}{2\cdot 4\cdot 3\cdot 7},\ a_6 = \frac{-a_4}{6\cdot 11} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11},\ \ldots.$$

Since we will account for constants later, set $a_0 = 1$ to get one solution as

$$y_1(x) = 1 - \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 4\cdot 3\cdot 7} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! 3\cdot 7\cdots(4n-1)}.$$

It is easily verified that this series, a power series, converges for all $x > 0$.

(b) $r = 1/2$. The recurrence formula (3) gives

$$a_n = \frac{-a_{n-2}}{(n+1/2)\big[2(n+1/2)-1\big]} = \frac{-a_{n-2}}{n(2n+1)} \quad \text{for } n \ge 2.$$

All odd coefficients are 0; and the even coefficients are given by

$$a_2 = \frac{-a_0}{2\cdot 5},\ a_4 = \frac{-a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9},\ a_6 = \frac{-a_4}{6\cdot 13} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13},\ \ldots.$$

Again, setting $a_0 = 1$, we have

$$y_2(x) = x^{1/2}\Big(1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2^n n! 5\cdot 9\cdots(4n+1)}\Big).$$

It is easily verified that the series here converges for all $x > 0$. Clearly, $y_1(x)$ and $y_2(x)$ are linearly independent. Then the general solution of the ODE is $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for $x > 0$. ∎

In the indical equation $r(r-1) + p_0 r + q_0 = 0$, the constants p_0 and q_0 are the constant terms in the Maclaurin series expansions of $xp(x)$ and $x^2 q(x)$, respectively. Thus,

$$p_0 = \lim_{x\to 0} \big[xp(x)\big], \quad q_0 = \lim_{x\to 0} \big[x^2 q(x)\big].$$

Alternatively, the indical equation is obtained from the ODE by substituting the series $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the ODE and then setting the coefficient of the least power in x to 0. In practice, we obtain the indical equation this way.

When the indical equation has a double root or the roots differ by an integer, it is usually extremely difficult to determine the second solution $y_2(x)$. In fact, $y_2(x)$ there has been obtained by using reduction of order. Sometimes, it is easier to get $y_2(x)$ by using the method of reduction of order directly once $y_1(x)$ is already available. If that is also difficult, which is often the case, then one only finds a few terms in the series expansion of $y_2(x)$.

(3.9) ***Example***

Solve the ODE $(x^2 - x)y'' + (3x-1)y' + y = 0$ for $x > 0$ using Frobenius method.

Here, $p(x) = (3x-1)/(x^2 - x)$ is not analytic at $x = 0$. However, $xp(x)$ is analytic at $x = 0$ and $x^2 q(x) = x/(x-1)$ is also analytic at $x = 0$. Hence $x = 0$ is a regular singular point of the ODE. We try a solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0.$$

Substituting this in the ODE, we obtain

$$0 = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + 3\sum_{n=0}^{\infty} (n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Equating the coefficient of the least power of x to 0, we obtain the indical equation as

$$0 = \big[-r(r-1) - r\big]a_0 \Rightarrow r^2 = 0.$$

Since $r = 0$ is a double root, by (3.7)(b), the two linearly independent solutions are of the form:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = y_1(x)\log x + \sum_{n=0}^{\infty} b_n x^n, \quad a_0 \neq 0.$$

To find $y_1(x)$, we take $r = 0$ and equate the coefficient of x^n to 0 in the above to obtain the following recurrence relation:

$$n(n-1)a_n - (n+1)na_{n+1} + 3na_n - (n+1)a_{n+1} + a_n = 0.$$

It gives $a_{n+1} = a_n$. By choosing $a_0 = 1$, we get one solution as

$$y_1(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

For the second solution, we use reduction of order. From § 2.6, we have

$$y_2(x) = y_1(x) \int v(x)\, dx, \quad v(x) = \frac{\exp\Big(-\int p(x)\, dx\Big)}{y_1^2(x)},$$

where the ODE is in standard form, that is, when the coefficient of y'' is 1. For our ODE, (with $x > 0$)

$$-\int p(x)\, dx = -\int \frac{3x-1}{x(x-1)}\, dx = -\int \Big(\frac{2}{x-1} + \frac{1}{x}\Big)\, dx = -2\log|x-1| - \log x.$$

$$v(x) = \frac{1}{y_1^2(x)} \exp(-2\log|x-1| - \log x) = (1-x)^2(x(x-1)^2)^{-1} = \frac{1}{x}.$$

$$y_2(x) = y_1(x) \int v(x)\, dx = \frac{1}{1-x} \int \frac{dx}{x} = \frac{\log x}{1-x}.$$

Hence, the general solution of the ODE is $y(x) = (1-x)^{-1}(c_1 + c_2 \log x)$. ∎

(3.10) ***Example***

Solve the ODE $x^2y'' + 3xy' + (1-x)y = 0$ by Frobenius method.

Here, $p(x) = 3/x$ and $q(x) = (1-x)/x^2$ which are not analytic at $x = 0$ but $xp(x) = 3$ and $x^2q(x) = 1 - x$ are analytic at $x = 0$. Hence, $x = 0$ is a regular singular point of the ODE. Here, $p_0 = 3$ and $q_0 = 1$; so the indical equation is

$$r(r-1) + 3r + 1 = r^2 - r + 2r + 1 = (r+1)^2 = 0 \Rightarrow r_1 = -1,\ r_2 = -1.$$

Since $r = -1$ is a double root, by (3.7)(b), the two lienarly independent solutions of the ODE are in the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}, \quad y_2(x) = y_1(x)\log x + \sum_{n=0}^{\infty} b_n x^{n-1}, \quad a_0 \neq 0.$$

So, let $y(x) = \sum\limits_{n=0}^{\infty} a_n x^{n+r}$ with $a_0 \neq 0$. Putting it in the ODE gives

$$\begin{aligned}
0 &= x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty}[(n+r)(n+r-1+3)+1]a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} \\
&= \sum_{n=1}^{\infty}(n+r+1)^2 a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} \\
&= (r+1)^2 a_0 x^r + \sum_{n=1}^{\infty}\left[(n+r+1)^2 a_n - a_{n-1}\right] x^{n+r}.
\end{aligned}$$

Setting the coefficients of all powers of x to 0, we obtain

$$(r+1)^2 = 0, \quad a_n = \frac{a_{n-1}}{(n+r+1)^2} \quad \text{for } n \geq 1.$$

Since $r = -1$, we have $a_n = a_{n-1}/n^2$. Then

$$a_1 = \frac{a_0}{1^2} = a_0,\ a_2 = \frac{a_1}{2^2} = \frac{a_0}{2^2},\ a_3 = \frac{a_2}{3^2} = \frac{a_0}{2^2 3^2},\ \ldots$$

Proceeding inductively, we obtain $a_n = \dfrac{a_0}{(n!)^2}$. Setting $a_0 = 1$ we have a solution of the ODE as

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n!)^2}.$$

We do not compute the second solution, but remark that after some cumbersome computation, the second solution is found to be

$$y_2(x) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n!)^2}\log x - \frac{2}{x}\Big(1 + \sum_{n=1}^{\infty}\frac{H_n x^n}{(n!)^2}\Big) \text{ where } H_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n}. \qquad \square$$

(3.11) ***Example***

Solve the ODE $(x^2 - x)y'' - xy' + y = 0$ for $x > 1$ by Frobenius method.

Here, $x = 0$ is a regular singular point. Write $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ and substitute in the ODE to get

$$(x^2 - x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Simplifying we get

$$\sum_{n=0}^{\infty}(n+r-1)^2 a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} = 0.$$

The lowest power of x is x^{r-1}. Equating its coefficient to 0, we get the indical equation. It gives

$$r(r-1) = 0 \Rightarrow r_1 = 1,\ r_2 = 0.$$

Since the roots differ by an integer, using (3.7)(3), we compute the first solution $y_1(x)$ as follows.

Taking the $r = r_1 = 1$ and setting the coefficient of x^{n+1} to 0, we get

$$\sum_{n=0}^{\infty}\left[n^2 a_n - (n+2)(n+1)a_{n+1}\right]x^{n+1} = 0.$$

It implies the recurrence relation

$$a_{n+1} = \frac{n^2}{(n+1)(n+2)}a_n \quad \text{for } n \ge 0.$$

For $n = 0$, we have $a_1 = 0$. Consequently, $a_n = 0$ for all $n \ge 1$. Choosing $a_0 = 1$ we get the first solution as $y_1(x) = a_0 x^{r_1} = x$.

For the second solution, we use the method of reduction of order. Here, $p(x) = 1/(1-x)$. Then

$$-\int p(x)\,dx = \int \frac{dx}{x-1} = \log|x-1| = \log(x-1) \quad \text{as } x > 1$$

$$\Rightarrow v(x) = \frac{\exp\left(-\int p(x)\,dx\right)}{y_1^2(x)} = \frac{x-1}{x^2}$$

$$\Rightarrow y_2(x) = y_1(x)\int v(x)\,dx = x\int \frac{x-1}{x^2}\,dx = x\left(\log x + \frac{1}{x}\right) = x\log x + 1.$$

Hence, the general solution of the ODE is $y(x) = c_1 x + c_2(\log x + 1)$. ∎

(3.12) *Example*

Find a series solution of Euler-Cauchy equation $x^2y'' - xy' + 10y = 0$ for $x > 0$.

Here, $p(x) = -1/x$ and $q(x) = 10/x^2$. Thus, $x = 0$ is a regular singular point. We have $p_0 = \lim_{x\to 0} xp(x) = -1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = 10$. So, the indical equation is

$$r(r-1) + (-1)r + 10 = r^2 - 2r + 10 = 0.$$

It has complex roots $1 \pm 3i$. We require now complex solutions of the ODE. We proceed as earlier for the recurrence relations. Substituting

$$z(x) = y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the ODE and setting powers of x to 0, we obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 10 a_n x^{n+r} \\ &= \sum_{n=0}^{\infty}\big[(n+r)(n+r-2)+10\big]a_n x^{n+r} \\ &= (r^2-2r+10)a_0 = 0, \quad (n+r)(n+r+10)a_n = 0 \text{ for } n \geq 1. \end{aligned}$$

The first one gives the indical equation. In the second one, with $r = 1 \pm 3i$, the factor $(n+r)(n+r+10) \neq 0$. Hence $a_n = 0$ for each $n \geq 1$. Thus the complex solutions are given by

$$z(x) = a_0 x^{1+3i} = a_0 x \exp\big(\log(x^{3i})\big) = a_0 x\big(\cos(3\log x) + i\sin(3\log x)\big).$$

Setting the constant $a_0 = 1$, and using (3.7)(4), the two linearly independent solutions are

$$y_1(x) = x\cos(3\log x), \quad y_2(x) = x\sin(3\log x).$$

Thus, the series solution of the ODE is given by

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 x\cos(3\log x) + c_2 x\sin(3\log x)$$

where c_1 and c_2 are arbitrary constants. ∎

4

Special Functions

4.1 Legendre polynomials

In this chapter we discuss some special types of ODEs whose series solutions give rise to the *special functions*. First, we consider the Legendre equation in its general form. It is

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \quad \text{for } |x| < 1. \tag{4.1.1}$$

where p is a constant, often called a *parameter*. So, this equation is actually a family of ODEs. We should not be surprised if the nature of solutions differs for various values of p.

The ODE in (4.1.1) has the standard form

$$y'' - \frac{2x}{1-x^2}\, y' + \frac{p(p+1)}{1-x^2}\, y = 0.$$

The coefficient functions $-2x/(1-x^2)$ and $p(p+1)/(1-x^2)$ are analytic at $x = 0$. That is, $x = 0$ is an ordinary point of the ODE. Thus, the ODE has a power series solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting it in the ODE and setting the coefficients of x^n to 0, we obtain

$$\begin{aligned}
&(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + p(p+1)a_n x^n = 0\\
\Rightarrow\ &(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0\\
\Rightarrow\ &(n+2)(n+1)a_{n+2} = (n^2 - n + 2n - p^2 - p)a_n = (n^2 - p^2 + n - p)a_n\\
\Rightarrow\ &a_{n+2} = -\frac{(n-p)(p+n+1)}{(n+1)(n+2)}\, a_n.
\end{aligned}$$

The recurrence relation is used to compute the coefficients of $a_2, a_3, \ldots$ in terms of

a_0 and a_1, which are left arbitrary. To compute a few,

$$a_2 = -\frac{p(p+1)}{1\cdot 2}a_0,\ a_4 = -\frac{(p-2)(p+3)}{3\cdot 4} = \frac{p(p-2)(p+1)(p+3)}{4!}, \ldots$$
$$a_3 = -\frac{(p-1)(p+2)}{2\cdot 3}a_1,\ a_5 = -\frac{(p-3)(p+4)}{4\cdot 5}a_3 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!}a_1, \ldots$$

We thus get a formal solution $y(x) = a_0 y_1(x) + a_1 y_2(x)$, where

$$y_1(x) = 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - \cdots$$
$$y_2(x) = x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \cdots$$

When p is not an integer, the numerators in the coefficients of powers of x do not vanish. In the series for $y_1(x)$, taking the absolute value of ratio of a term and its preceding term, we find that

$$\left|\frac{a_{2n+2}x^{2n+2}}{a_{2n}x^{2n}}\right| = \left|\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)}\right| \to |x|^2 \text{ as } n \to \infty.$$

Hence, the radius of convergence of the series for $y_1(x)$ is 1. Similarly, it is easy to show that the radius of convergence for the series for $y_2(x)$ is also 1 in case p is not an integer. That is, the formal solution given above is a solution for $-1 < x < 1$. Notice that this is the best we can expect since the coefficient functions $-2x/(1-x^2)$ and $p(p+1)/(1-x^2)$ are not analytic at $x = 1$.

Next, we consider the interesting case when p is a non-negative integer. We consider the cases $p = 0$, p is nonzero even, and p is nonzero odd separately.

Case 1: Suppose $p = 0$. Then $y_1(x) = 1$ and

$$y_2(x) = x - \frac{(-1)(2)}{3!}x^3 + \frac{(-1)(-3)(2)(4)}{5!}x^5 - \cdots$$

Here, $y_1(x)$ is a constant and $y_2(x)$ is a power series.

Case 2: Suppose p is nonzero and even, say, $p = 2k$ for some $k \geq 1$. Then

$$y_1(x) = 1 - \frac{2k(2k+1)}{2!}x^2 + \cdots + (-1)^k\frac{2k(2k-2)\cdots(2)(2k+1)(2k+3)\cdots(2k+2k-1)}{(2k)!}x^{2k}.$$

The next term in the series for $y_1(x)$ has in the numerator the factor $(p-2k) = 0$. All succeeding terms are then 0. Therefore, $y_1(x)$ terminates there, and it is a polynomial. In this case, $y_2(x)$ is a power series.

Case 3: Suppose p is odd, say, $p = 2k+1$ for some $k \geq 0$. Then

$$y_2(x) = x - \frac{(2k)(2k+3)}{3!}x^3 + \cdots + (-1)^k\frac{(2k)(2k-2)\cdots(2)(2k+3)(2k+5)\cdots(2k+2k+1)}{(2k+1)!}x^{2k+1}.$$

The next term in the series for $y_2(x)$ is 0 since the numerator has a factor $(p-(2k+1))=0$. All succeeding terms are then 0. Therefore, $y_2(x)$ terminates there, and it is a polynomial. In this case, $y_1(x)$ is a power series.

We thus find that if p is an integer, then exactly one of $y_1(x)$ or $y_2(x)$ is a polynomial.

When $p=0$, the ODE is $(1-x^2)y''-2xy'=0$. Since $p=0$, the polynomial solution of this ODE is $y_1(x)=1$. This polynomial $y_1(x)$ is of degree 0 with $y_1(1)=1$.

When $p=2$, the ODE is $(1-x^2)y''-2xy'+6y=0$. Since p is even, the polynomial solution of this ODE is (with $p=2k,\ k=1$)

$$y_1(x)=1-\frac{2(3)}{2!}x^2=1-3x^2.$$

This polynomial $y_1(x)$ is of degree 2 with $y_1(1)=1-3=-2$.

It continues this way for even p. Let us look at a few cases when p is odd.

When $p=1$, the ODE is $(1-x^2)y''-2xy'+2y=0$. Since p is odd, the polynomial solution is (with $p=2k+1,\ k=0$)

$$y_2(x)=x.$$

This polynomial $y_2(x)$ is of degree 1 with $y_2(1)=1$.

When $p=3$, the ODE is $(1-x^2)y''-2xy'+12y=0$. The polynomial solution is (with $p=2k+1,\ k=1$)

$$y_2(x)=x-\frac{(2)(2+3)}{3!}x^3=x-\frac{5}{3}x^3.$$

This polynomial $y_2(x)$ is of degree 3 with $y_2(1)=1-5/3=-2/3$.

As we see from the above cases, the polynomials when evaluated at $x=1$ give the values as follows:

Parameter p:	0	1	2	3
Degree of polynomial:	0	1	2	3
Which solution:	y_1	y_2	y_1	y_2
Its value at 1:	1	1	-2	$-2/3$

Notice that since $y_1(x)$ is a solution of an appropriate Legendre equation, any constant multiple of $y_1(x)$ is also a solution of the same Legendre equation. The same is also true for $y_2(x)$. In particular, the polynomials and there constant multiples are also solutions of suitable Legendre equations. Thus, we can choose to multiply an appropriate constant in each case so that the resulting polynomial when evaluated at 1 will give the value 1. Such polynomials are called **Legendre polynomials**.

Thus, the Legendre polynomial of degree n, denoted by $P_n(x)$, is the polynomial of degree n that satisfies the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{with } \; y(1) = 1.$$

We find that if n is even, then $P_n(x) = y_1(x)$ and it does not have any odd power of x; and if n is odd, then $P_n(x) = y_2(x)$ and it does not have any even power of x. Further, these polynomials satisfy $P_n(1) = 1$. Using the above computations, we obtain the following:

$$\begin{aligned}
&P_0(x) = y_1(x) \text{ (with } p = 0) = 1. \\
&P_1(x) = y_2(x) \text{ (with } p = 1) = x. \\
&P_2(x) = y_1(x) \text{ (with } p = 2) = \tfrac{1}{-2}(1-3x^2) = \tfrac{1}{2}(3x^2-1). \\
&P_3(x) = y_2(x) \text{ (with } p = 3) = -\tfrac{3}{2}\big(x - \frac{5}{3}x^3\big) = \tfrac{1}{2}(5x^3-3x).
\end{aligned}$$

There is another way to choose these constants so that the condition $P_n(1) = 1$ is satisfied. This way we may be able to express Legendre's polynomials in close form. The idea is to assume certain nice form of the coefficient of highest power of x in $P_n(x)$. So, suppose a_n is the coefficient of x^n in $P_n(x)$. We choose the constants in such a way that

$$a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \quad \text{for } \; n \geq 0.$$

Using our recurrence relation for the coefficients derived earlier, we have

$$\begin{aligned}
a_{n-2} &= -\frac{n(n-1)}{2(2n-1)}a_n = -\frac{n(n-1)(2n)!}{2(2n-1)(n!)^2} = -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}. \\
a_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}. \\
a_{n-2k} &= (-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} \quad \text{for } \; n \geq 2k.
\end{aligned}$$

Using this, Legendre polynomial of degree n may be written as

$$\begin{aligned}
P_n(x) &= \sum_{k=0}^{m}(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}\, x^{n-2k} \quad \text{where } \; m = [n/2] \\
&= \frac{(2n)!}{2^n(n!)^2}x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!}x^{n-2} + \cdots \qquad (4.1.2)
\end{aligned}$$

To see that it is the same $P_n(x)$ we have defined earlier we need only to check that $P_n(1) = 1$ for each n. We will show it later in (4.1.4).

Though $P_n(x)$ is a polynomial, it is treated as a special function because it has some nice properties and it comes in various disguises. One of its useful form is the following:

$$\textbf{Rodrigue's formula}: \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{4.1.3}$$

To see that the formula is correct, notice that

$$\frac{d^n}{dx^n}x^{2n-2k} = \frac{(2n-2k)!}{(n-2k)!}x^{n-2k} \quad \text{for } 0 \le k \le m = [n/2].$$

Thus, $P_n(x)$ is rewritten as

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{m}(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}\,x^{n-2k} \\ &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{k=0}^{m}(-1)^k\frac{n!}{k!(n-k)!}x^{2n-2k}. \end{aligned}$$

When $k > m = [n/2]$, any term in the sum above is a polynomial of degree less than n so that its nth derivative is 0. Hence, the sum above can be extended from $m+1$ to n without changing the value on the left hand side. So,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}x^{2n-2k} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

The last equality follows from the Binomial expansion of $(x^2-1)^n$.

Various useful properties of Legendre polynomials follow from Rodrigue's formula with the help of Leibniz rule for computing the nth derivative of a product of two functions. Leibniz's rule says that

$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{d^k f}{dx^k}\frac{d^{n-k}g}{dx^{n-k}},$$

where the 0th derivative of a function is taken as the function itself.

In Rodrigue' formula writing $(x^2-1)^n = (x+1)^n(x-1)^n$ and applying Leibniz rule we obtain

$$P_n(x) = \frac{1}{2^n n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{d^k[(x+1)^n]}{dx^k}\frac{d^{n-k}[(x-1)^n]}{dx^{n-k}}.$$

The first term in the above sum is

$$\frac{d^0[(x+1)^n]}{dx^0}\cdot\frac{d^n[(x-1)^n]}{dx^n} = (x+1)^n n!.$$

Each of the remaining terms contains the factor $(x-1)$. Thus, when evaluated at $x=1$, each term except the first in the sum becomes 0. Thus,

$$P_n(1) = \frac{1}{2^n n!}\,(1+1)^n n! = 1. \tag{4.1.4}$$

It is often helpful to get the **generating function** for the Legendre polynomials. We will show that the generating function is $(1-2xt+t^2)^{-1}$. That is,

$$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{4.1.5}$$

To see this, we apply the Binomial theorem on the left hand side expression. Recall that the Binomial theorem asserts that

$$(1+z)^r = \sum_{n=0}^{\infty} \frac{r(r-1)\cdots(r-n+1)}{n!} z^n \quad \text{for } |z|<1.$$

Taking $z = t^2-2xt = t(t-2x)$ and assuming that $|t^2-2xt|<1$, we obtain

$$\begin{aligned}
(1-2xt+t^2)^{-1/2} &= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)}{n!} t^n(t-2x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}[n!]^2} t^n(t-2x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2} t^n\Big(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k(-2x)^{n-k}\Big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(2n)!}{2^{2n}n!k!(n-k)!} t^{n+k}(2x)^{n-k}.
\end{aligned}$$

In general, if $C_{k,n}$ is any expression depending on k and n, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{k,n}t^{n+k} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} C_{k,n-k}t^n.$$

Using this on the above sum, we obtain

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!} t^n x^{n-2k} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

An important property of the Legendre polynomials is that they are **orthogonal** to each other. It means

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad \text{for } m \neq n. \tag{4.1.6}$$

To see this we use the fact that Legendre polynomials are solutions of the Legendre ODE, which can be rewritten as

$$[(1-x^2)y']' + p(p+1)y = 0.$$

Therefore,

$$[(1-x^2)P_m'(x)]' + m(m+1) = 0, \quad [(1-x^2)P_n'(x)]' + n(n+1) = 0.$$

Multiply the first with P_n and the second with P_m, subtract, and integrate to get

$$\int_{-1}^{1} \left(P_n[(1-x^2)P_m']' - P_m[(1-x^2)P_n']'\right) dx - \left[m(m+1) - n(n+1)\right] \int_{-1}^{1} P_m P_n \, dx = 0.$$

Evaluate the first integral by using integration by parts. It gives

$$\begin{aligned}
&\int_{-1}^{1} \left(P_n[(1-x^2)P_m']' - P_m[(1-x^2)P_n']'\right) dx \\
&= \left[P_n(1-x^2)P_m'\right]_{-1}^{1} - \left[P_m(1-x^2)P_n'\right]_{-1}^{1} \\
&\quad - \int_{-1}^{1} \left[P_n'(1-x^2)P_m' - P_m'(1-x^2)P_n'\right] dx = 0.
\end{aligned}$$

Hence, If $m \neq n$, then $\int_{-1}^{1} P_m(x)P_n(x)\, dx = 0$.

What happens when $m = n$? We use Rodrigue's formula and integration by parts as follows:

$$\begin{aligned}
\int_{-1}^{1} \left[P_n(x)\right]^2 dx &= \int_{-1}^{1} P_n(x) \frac{d^n}{dx^n}(x^2-1)^n \, dx \\
&= \frac{1}{2^n n!}\left[P_n(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \frac{1}{2^n n!}\int_{-1}^{1} P_n'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \, dx \\
&= 0 - \frac{1}{2^n n!}\int_{-1}^{1} P_n'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \, dx \\
&\vdots \\
&= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} [P_n(x)]^{(n)} \int_{-1}^{1} P_n'(x)\frac{d^0}{dx^0}(x^2-1)^n \, dx \\
&= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} \frac{(2n)!}{2^n n!}(1-x^2)^n \, dx \\
&= \frac{2(2n)!}{2^{2n}(n!)^2}\int_{0}^{1} (1-x^2)^n \, dx \qquad (\text{put } x = \sin\theta) \\
&= \frac{2(2n)!}{2^{2n}(n!)^2}\int_{0}^{\pi/2} \cos^{2n+1}\theta \, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(2n)!}{2^{2n}(n!)^2}\,\frac{2n}{2n+1}\int_0^{\pi/2}\cos^{2n-1}\theta\,d\theta \\
&\vdots \\
&= \frac{2(2n)!}{2^{2n}(n!)^2}\,\frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\cdots\frac{2}{3}\int_0^{\pi/2}\cos\theta\,d\theta \\
&= \frac{2(2n)!}{2^{2n}(n!)^2}\,\frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\cdots\frac{2}{3} = \frac{2}{2n+1}.
\end{aligned}
$$

Hence,

$$\int_{-1}^{1}\left[P_n(x)\right]^2 dx = \frac{2}{2n+1}. \tag{4.1.7}$$

Many problems in engineering depend on the possibility of expanding a given function in a series of Legendre polynomials. It is easy to see that a polynomial can always be expanded this way. For example, consider a polynomial of degree at most 3, say

$$p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3.$$

With $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x)$, we see that

$$1 = P_0(x),\ x = P_1(x),\ x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x),\ x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Hence,

$$p(x) = \left(b_0 + \frac{b_2}{3}\right)P_0(x) + \left(b_1 + \frac{3b_3}{5}\right)P_1(x) + \frac{2b_2}{3}P_2(x) + \frac{2b_3}{5}P_3(x).$$

Similarly, x^n can be expanded as $\sum_{k=0}^{n} a_k P_k(x)$ for some constants a_k. It looks that if a function has a power series expansion, then it can also be expanded in terms of Legendre polynomials $P_n(x)$. However, some conditions my be required so that the obtained series is convergent. We rather concentrate on how to compute the coefficients in such a series expansion if it exists.

When a function $f(x)$ for $-1 < x < 1$, can be written in the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

we say that $f(x)$ has a **Legendre series** expansion. Our question is, if $f(x)$ has a Legendre series expansion, then how do we compute the coefficients a_n?

We multiply the above with $P_m(x)$, integrate term by term (assuming that this is permissible), and use (4.1.6-4.1.7) to obtain

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2a_m}{2m+1}.$$

Therefore,

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx.$$

Many other properties of Legendre polynomials are included in the exercises. As a convention, when $P_n(x)$ is treated as a function, we assume that $-1 \le x \le 1$.

4.2 Bessel Functions

The linear homogeneous second order ordinary differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \tag{4.2.1}$$

is called the **Bessel's equation** with non-negative parameter ν. (It is nu not vee.) It arises many where in applications. In standard form, the equation is

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

The point $x = 0$ is a regular singular point of the ODE. Hence the ODE has a solution in the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r} \quad \text{with } a_0 \neq 0.$$

Substituting it in (4.2.1), we obtain

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} + \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} - \nu^2 \sum_{k=0}^{\infty} a_k x^{k+r} = 0.$$

Thus coefficients of x^r, x^{r+1} and x^{k+r} for $k \ge 2$, are 0. It follows that

1. $r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0.$
2. $(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0.$
3. $(k+r)(k+r-1)a_k + (k+r)a_k + a_{k-2} - \nu^2 a_k = 0$ for $k \ge 2$.

The first one gives the indical equation as $(r+\nu)(r-\nu) = 0$. The roots are $r_1 = \nu$ and $r_2 = -\nu$. Corresponding to $r = \nu$, the first solution of the ODE is

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+\nu}.$$

We must find the coefficients a_k. For $r = \nu$, the second equation above implies

$$(\nu^2 + \nu + \nu + 1 - \nu^2)a_1 = (2\nu + 1)a_1 = 0 \Rightarrow a_1 = 0.$$

Substituting $r = \nu$ in the third equation, we obtain

$$\begin{aligned}
&(k+\nu)(k+\nu-1)a_k + (k+\nu)a_k + a_{k-2} - \nu^2 a_k = 0 \\
&\Rightarrow \left[(k+\nu)(k+\nu-1+1) - \nu^2\right]a_k + a_{k-2} = 0 \\
&\Rightarrow k(k+2\nu)a_k + a_{k-2} = 0 \\
&\Rightarrow a_k = -\frac{a_{k-2}}{k(k+2\nu)}
\end{aligned}$$

Since $a_1 = 0$ it follows that all odd coefficients are 0. For even coefficients, say, $k = 2m$, the above recurrence looks like

$$a_{2m} = -\frac{a_{2m-2}}{2^2 m(\nu+m)} \quad \text{for } m = 1, 2, 3, \ldots$$

It implies that

$$a_2 = -\frac{a_0}{2^2(\nu+1)}, \quad a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}, \ldots$$

Proceeding inductively, we get

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)} \quad \text{for } m = 1, 2, 3, \ldots$$

By choosing the constant a_0, all even coefficients are evaluated. It is customary to choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

Here,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad \text{for } x \geq 0.$$

Notice that $\Gamma(\nu+1)$ is well defined since ν is non-negative. Some useful properties of the gamma function are as follows:

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(n+1) = n! \text{ for } n = 0, 1, 2, \ldots.$$

It then follows that

$$(x+1)(x+2)\cdots(x+m)\Gamma(x+1) = \Gamma(x+m+1) \quad \text{for } m \in \mathbb{N} \cup \{0\}.$$

With the above choice of a_0, we obtain

$$\begin{aligned} a_{2m} &= \frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)} \\ &= \frac{(-1)^m}{2^{2m} m! 2^\nu \Gamma(\nu+1)(\nu+1)(\nu+2)\cdots(\nu+m)} \\ &= \frac{(-1)^m}{2^{\nu+2m} m! \Gamma(\nu+m+1)} \quad \text{for } m = 1, 2, 3, \ldots. \end{aligned}$$

With these coefficients, the solution $y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+\nu}$ is written as $J_\nu(x)$, and is called the **Bessel function of first kind with order** ν. Thus,

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{\nu+2m} m! \Gamma(\nu+m+1)}. \tag{4.2.2}$$

The absolute value of the ratio of a term to its succeeding term in the seris for $J_\nu(x)$ is given by

$$\left|\frac{a_{2m-2}}{a_{2m}}\right| = \left|\frac{2^2 m(\nu+m)}{x^2}\right| \to \infty \quad \text{for any nonzero } x.$$

The ratio test implies that the series in $J_\nu(x)$ is convergent. Notice that the convergence of the series is fast since factorials are in the denominator. The series obviously converges for $x = 0$. Hence, $J_\nu(x)$ is well defined for all x.

In particular, when $\nu = n \in \mathbb{N} \cup \{0\}$, we have $\Gamma(\nu+1) = \Gamma(n+1) = n!$. Thus,

$$\begin{aligned} a_0 &= \frac{1}{2^n n!} \\ a_{2m} &= \frac{(-1)^m}{2^{\nu+2m} m! \Gamma(\nu+m+1)} = \frac{(-1)^m}{2^{n+2m} m!(n+m)!} \quad \text{for } m = 1, 2, , 3, \ldots. \end{aligned}$$

The odd coefficients are 0 as earlier. Therefore, the first solution $y_1(x)$ of Bessel's equation

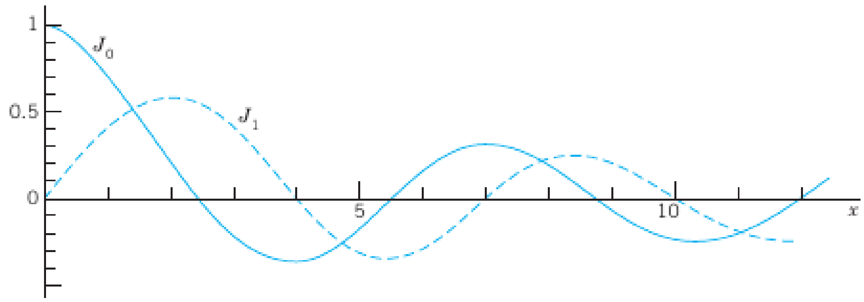
$$x^2 y'' + xy' + (x^2 - n^2)y = 0, \quad n \in \mathbb{N} \cup \{0\}$$

is given by

$$y_1(x) = J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{n+2m} m!(n+m)!} \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{4.2.3}$$

Of course, this expression is directly obtained from (4.2.2) by taking $\nu = n$. For instance, the Bessel functions of first kind and order 0 and 1 are as follows.

$$\begin{aligned} J_0(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots \\ J_1(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} - \frac{x^7}{2^7 3! 4!} + \cdots \end{aligned}$$



Notice that $J_n(0) = 0$ for $n \geq 1$. It can be shown that

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for large } x.$$

For a general solution of Bessel's equation (4.2.1), we consider two cases.

Case 1: Suppose the non-negative parameter ν is not an integer. Then the second solution $y_2(x)$ of Bessel's equation (4.2.1) is given by

$$y_2(x) = J_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-\nu}}{2^{2m-\nu} m! \Gamma(m - \nu + 1)}. \tag{4.2.4}$$

This follows from a derivation similar to that of $J_\nu(x)$. Also, by substituting ν with $-\nu$ in (4.2.2), we obtain this expression for $J_{-\nu}(x)$.

Observe that any power of x in $J_\nu(x)$ is $x^{2m+\nu}$ and any power of x in $J_{-\nu}(x)$ is $x^{2m-\nu}$. Since ν is not an integer, no power of x in $J_\nu(x)$ matches with any power of x in $J_{-\nu}(x)$. Hence $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. Therefore, any solution $y(x)$ of Bessel's equation with non-integral parameter ν is given by

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x) \quad \text{for } \nu \notin \mathbb{Z}.$$

Case 2: Suppose $\nu = n$ is an integer. We know the first solution as $J_n(x)$ for $n \geq 0$. For the second solution, let us look at $J_{-n}(x)$. From (4.2.4) we have

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! (m-n)!}. \tag{4.2.5}$$

We can also get $J_{-n}(x)$ from (4.2.4) another way. In (4.2.4), let ν approach a positive integer n. Then the Gamma function in the first n terms approach ∞ so that the coefficients in the first n terms approach 0. The summation starts from $m = n$ as the Gamma function there is equal to $\Gamma(m - n + 1) = (m - n)!$ for $m \geq n$. Then, shifting the index with $k = m - n$, we obtain

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! (m-n)!} = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} x^{2k+n}}{2^{2k+n} k! (k+n)!}.$$

Comparing the last expression with (4.2.5) we find that it is $(-1)^n J_n(x)$. Therefore,

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{for } n \in \mathbb{Z}. \tag{4.2.6}$$

It implies that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. Thus, we cannot take the second solution $y_2(x)$ as $J_{-n}(x)$. The second solution, denoted by $Y_n(x)$ can be obtained by using reduction of order; it is fairly complicated. We only mention the final result:

$$\begin{aligned} Y_n(x) = \frac{2}{\pi} J_n(x)\Big(\log\frac{x}{2} + \gamma\Big) + \frac{x^n}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n}x^{2m}}{2^{2m+n} m!(m+n)!} \\ -\frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!x^{2m}}{2^{2m-n}m!} \quad \text{for } x > 0 \end{aligned} \tag{4.2.7}$$

where $n = 0, 1, 2, \ldots$, $h_0 = 0$, $h_1 = 1$, $h_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$, and $\gamma = \lim_{k\to\infty}(h_k - \log k)$ is Euler constant. In particular,

$$Y_0(x) = \frac{2}{\pi}\Big[J_0(x)\big(\log(x/2) + \gamma\big) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m x^{2m}}{2^{2m}(m!)^2}\Big].$$

It can be seen that $Y_0(x)$ behaves like $\log x$ for small x and $Y_0(x) \to -\infty$ when $x \to 0$.

In fact, both the cases above can be unified to obtain a function $Y_\nu(x)$ which is a second solution of Bessel's equation. It is as follows:

$$Y_\nu(x) = \operatorname{cosec}(\nu\pi)\big[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\big].$$

With this definition, it can be seen that

$$\lim_{\nu\to n} Y_\nu(x) = Y_n(x).$$

But remember that when ν is not an integer, it does not say that $J_{-\nu}(x)$ is equal to $Y_{-\nu}(x)$. In fact for $\nu \notin \mathbb{Z}$, $Y_{-\nu}(x) = aJ_\nu(x) + bJ_{-\nu}(x)$ for some nonzero a and b. Nonetheless, $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent and $Y_\nu(x)$ is also a solution of Bessel's equation (4.2.1). This function $Y_\nu(x)$ is called **Bessel function of second kind of order** ν. With the help of this function we thus say that the general solution of Bessel's equation (4.2.1) is given by

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

for all values of ν and for $x > 0$.

The complex solutions of Bessel's equation may be given by

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x).$$

These two linearly independent complex solutions of Bessel's equation are called **Bessel functions of third kind of order** ν.

4.3 Properties of J_ν and J_n

In what follows we write J_n to indicate that the parameter ν in J_ν is an integer n. In this section we discuss some well known properties of $J_\nu(x)$ and of $J_n(x)$.

Multiply (4.2.2) by x^ν to get

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2\nu+2m}}{2^{\nu+2m} m! \Gamma(\nu+m+1)}.$$

Differentiate with respect to x, cancel 2, pull out $x^{2\nu-1}$, and use the relation $(\nu+m)\Gamma(\nu+m) = \Gamma(\nu+m+1)$ to obtain

$$\big(x^\nu J_\nu(x)\big)' = \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(\nu+m) x^{2\nu+2m-1}}{2^{\nu+2m} m! \Gamma(\nu+m+1)} = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{\nu+2m-1} m! \Gamma(\nu+m)}.$$

Comparing the last expression with (4.2.2), we find that

$$\big(x^\nu J_\nu(x)\big)' = x^\nu J_{\nu-1}(x). \tag{4.3.1}$$

Multiply (4.2.2) by $x^{-\nu}$, differentiate with respect to x, cancel $2m$, and shift the index by taking $m = k+1$, to obtain

$$\big(x^{-\nu} J_\nu(x)\big)' = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{\nu+2m-1} (m-1)! \Gamma(\nu+m+1)} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{\nu+2k+1} k! \Gamma(\nu+k+2)}.$$

Now, in (4.2.2) take ν as $\nu+1$ and m as k so that you get the last expression as $-x^{-\nu} J_{\nu+1}(x)$. Therefore,

$$\big(x^{-\nu} J_\nu(x)\big)' = -x^{-\nu} J_{\nu+1}(x). \tag{4.3.2}$$

From (4.3.1)-(4.3.2), we get

$$J_{\nu-1}(x) = x^{-\nu}\big(x^\nu J_\nu(x)\big)' = x^{-\nu}\big[x^\nu J_\nu'(x) + \nu x^{\nu-1} J_\nu(x)\big] = J_\nu'(x) + \nu x^{-1} J_\nu(x).$$
$$J_{\nu+1}(x) = -x^\nu\big(x^{-\nu} J_\nu(x)\big)' = -x^\nu\big[x^{-\nu} J_\nu'(x) - \nu x^{-\nu-1} J_\nu(x)\big] = -J_\nu'(x) + \nu x^{-1} J_\nu(x).$$

Subtracting the second from the first, we obtain

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x). \tag{4.3.3}$$

And, adding those two equalities, we get

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x). \tag{4.3.4}$$

This identity can be rewritten as

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x). \tag{4.3.5}$$

Now, we can use it to compute Bessel functions of higher order from lower ones.

Recall that $\Gamma(1/2) = \sqrt{\pi}$. Then,

$$J_{1/2}(x) = \sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+1/2} m! \Gamma(m+3/2)} = \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m! \Gamma(m+1/2)}.$$

However,

$$\begin{aligned} 2^m m! &= 2m(2m-2)\cdots 4\cdot 2. \\ 2^{m+1}\Gamma(m+1/2) &= 2^{m+1}(m+1/2)(m-1/2)\cdots(3/2)\cdot(1/2)\Gamma(1/2) \\ &= (2m+1)(2m-1)\cdots 3\cdot 1\cdot\sqrt{\pi}. \\ 2^{2m+1} m!\Gamma(m+1/2) &= [2^m m!]\,[2^{m+1}\Gamma(m+1/2)] = (2m+1)!\sqrt{\pi}. \end{aligned}$$

Hence,

$$J_{1/2}(x) = \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\, \sin x. \tag{4.3.6}$$

Multiply by $\sqrt{x}$, differentiate with respect to x, and use (4.3.1) with $\nu = 1/2$ to obtain

$$\big(\sqrt{x} J_{1/2}(x)\big)' = \sqrt{\frac{2}{\pi}}\, \cos x, \quad \big(\sqrt{x} J_{1/2}(x)\big)' = \sqrt{x} J_{1/2-1}(x).$$

Therefore,

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x. \tag{4.3.7}$$

Due to (4.3.5) $J_{k/2}(x)$ for any integer k, can be expressed as a product of some rational function and a trigonometric function.

To find a generating function for $J_n(x)$ and $J_{-n}(x)$, let us expand the function $\exp\left[tx/2 - x/(2t)\right]$. We find that

$$\begin{aligned} \exp\left(\frac{tx}{2} - \frac{x}{2t}\right) &= \left(\sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{tx}{2}\right)^r\right) \left(\sum_{s=0}^{\infty} \frac{1}{s!}\left(\frac{x}{2t}\right)^s\right) \\ &= \left(\sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{x}{2}\right)^r t^r\right) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\left(\frac{x}{2}\right)^s t^{-s}\right) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s}{r!s!}\left(\frac{x}{2}\right)^{r+s} t^{r-s} \\ &= \sum_{n=-\infty}^{\infty} \left[\sum_{s=\max\{0,-n\}}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}\right] t^n \end{aligned}$$

For $n \geq 0$, the coefficient of t^n in the above expression is

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{(s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} = x^n \sum_{s=0}^{\infty} \frac{(-1)^s\, x^{2s}}{2^{n+2s}\, s!(n+s)!} = J_n(x).$$

And, for $n \geq 0$, the coefficient of t^{-n} is (shifting the index with $k = s - n$):

$$\sum_{s=n}^{\infty} \frac{(-1)^s}{s!(n-s)!}\left(\frac{x}{2}\right)^{-n+2s} = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(n+k)!k!}\left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x) = J_{-n}(x).$$

We thus conclude that the **generating function** for $J_n(x)$ for $n \in \mathbb{Z}$ is

$$\exp\left(\frac{tx}{2} - \frac{x}{2t}\right).$$

It means

$$\exp\left(\frac{tx}{2} - \frac{x}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{4.3.8}$$

Some more properties of Bessel functions of first kind are to be found in the exercises.

The zeros of Bessel functions of first kind play an important role in modeling of vibrations. It is known that there are infinite number of positive zeros of $J_n(x)$. It is also known that between any two zeros of $J_n(x)$ there exists a unique zero of $J_{n+1}(x)$.

4.4 Sturm-Liouville problems

We have seen that the Legendre polynomials are orthogonal in the sense that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ when $m \neq n$. A similar relation holds for Bessel functions. There is a generalization of all these types of functions that are defined by a second order ODE. We will discuss this generalization here. Later we will conclude many useful properties about Bessel functions using this generalized problem.

Any ODE in the following form is called a **Sturm-Liouville equation**:

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0 \quad \text{for } a < x < b \tag{4.4.1}$$

Here, $\lambda \in \mathbb{R}$ is a parameter.

(4.1) *Example*

1. The simple harmonic motion equation $y'' + n^2 y = 0$ is a Sturm-Liouville equation with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$ and $\lambda = n^2$.

2. The Legendre equation $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ is a Sturm-Liouville equation with $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$ and $\lambda = p(p+1)$.
3. The Bessel's equation

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \quad \text{for } t > 0$$

is a Sturm-Liouville equation. To see this, put $t = kx$ for $k > 0$. We have

$$\frac{dy}{dx} = k\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = k^2\frac{d^2y}{dt^2}.$$

Then the above Bessel equation is reduced to

$$k^2x^2\frac{d^2y}{dt^2} + kx\frac{dy}{dt} + (k^2x^2 - \nu^2)y = 0 \quad \text{or,}$$

$$x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0.$$

However, $x(xy')' = x(xy'' + y') = x^2y'' + xy'$. Hence, the above ODE is rewritten as

$$(xy')' + \Big(-\frac{\nu^2}{x} + \lambda x\Big)y = 0 \quad \text{where } \lambda = k^2.$$

This is a Sturm-Liouville equation with $p(x) = x$, $q(x) = -\nu^2/x$, $r(x) = x$ for $x > 0$.

Notice that $J_n(\lambda x)$ satisfies this ODE. □

With the Sturm-Liouville equation, we associate one of the following conditions:

$$k_1y(a) + k_2y'(a) = 0, \quad \ell_1y(b) + \ell_2y'(b) = 0 \tag{4.4.2}$$

$$p(a) = p(b), \quad y(a) = y(b), \quad y'(a) = y'(b). \tag{4.4.3}$$

$$p(a) = 0, \quad \ell_1y(b) + \ell_2y'(b) = 0, \quad y(x) \text{ remains bounded.} \tag{4.4.4}$$

$$k_1y(a) + k_2y'(a) = 0, \quad p(b) = 0, \quad y(x) \text{ remains bounded.} \tag{4.4.5}$$

Here, k_1, k_2, ℓ_1, ℓ_2 are constants where at least one k is nonzero and at least one ℓ is nonzero, $p(x), p'(x), q(x), r(x)$ are continuous on $a \le x \le b$, and $p(x)$ is a non-zero function. We also assume that either $r(x) > 0$ for all $x \in [a, b]$ or $r(x) < 0$ for all $x \in [a, b]$.

The conditions in (4.4.2)-(4.4.5) are prescribed at two points instead of at one single point; thus these conditions are called **boundary conditions**. Accordingly, the Sturm-Liouville equation (4.4.1) along with one of these boundary conditions is called a **Sturm-Liouville problem**. The names associated with these problems are as follows:

regular Sturm-Liouville problem: (4.4.1) and (4.4.2)

periodic Sturm-Liouville problem: (4.4.1) and (4.4.3)

singular Sturm-Liouville problems: (4.4.1) with any one of (4.4.4) or (4.4.5)

We must remember that if a solution of the BVP exists, then it must be well defined over the interval $[a, b]$.

If the zero function is a solution of a Sturm-Liouville problem, then it is called the **trivial solution**. We are interested in getting non-trivial solutions.

Suppose a Sturm-Liouville problem is given. Corresponding to each value of the parameter λ, there may or may not exist a nontrivial solution of the problem. Those values of λ for which the problem has a non-trivial solution are called **eigenvalues**. Corresponding to an eigenvalue λ, the non-trivial solutions $y(x)$ are called **eigenfunction**.

(4.2) *Example*

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

This is a regular Sturm-Liouville problem with $a = 0,\ b = \pi,\ p(x) = 1,\ q(x) = 0,$ $r(x) = 1$, and $k_1 = \ell_1 = 1,\ k_2 = \ell_2 = 0$. Since the boundary conditions are given at $x = 0$ and at $x = \pi$, the eigenfunctions if exist, must be defined over the interval $[0, \pi]$.

For $\lambda = 0$, the equation is $y'' = 0$ giving the general solution as $y(x) = c_1 + c_2 x$. Now, $y(0) = 0 \Rightarrow c_1 = 0$. So, $y(x) = c_1 x$. Then, $y(\pi) = 0 \Rightarrow c_2 = 0$. So, $y(x) = 0$, the zero function. Thus, $\lambda = 0$ is not an eigenvalue; it means that corresponding to $\lambda = 0$, there does not exist any eigenfunction (non-trivial solution).

Let $\lambda < 0$. Write $\lambda = -\alpha^2$ for nonzero $\alpha \in \mathbb{R}$. The ODE is $y'' = \alpha^2 y$. Its general solution is $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. The boundary conditions imply that $c_1 + c_2 = 0$ and $c_1 e^{\pi} + c_2 e^{-\pi} = 0$. The solution of these two linear equations in c_1, c_2 is unique and it is $c_1 = 0 = c_2$. Consequently, $y(x) = 0$, the zero function. Hence, no negative number is an eigenvalue of this Sturm-Liouville problem.

Let $\lambda > 0$. Write $\beta = \sqrt{\lambda}$. The ODE is $y'' + \beta^2 y = 0$. Its general solution is $y(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. Now, $y(0) = 0$ implies $c_1 = 0$. So, $y(x) = c_2 \sin(\beta x)$. Then $y(\pi) = c_2 \sin(\beta\pi)$. If $c_2 = 0$, then $y(x) = 0$, the zero function. Thus, in order that $y(x)$ be non-trivial, we must have $c_2 \neq 0$. Then, $\sin(\beta x) = 0 \Rightarrow \beta \in \mathbb{Z}$.

Write $\beta = n \in \mathbb{Z}$. Then $\lambda = n^2$ for $n \in \mathbb{Z}$, are the eigenvalues. That is, the eigenvalues are $\lambda = n^2$ for $n = 0, 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $y(x) = \sin(nx)$ defined over the interval $[0, \pi]$. ∎

(4.3) *Example*

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$$

This is again a regular Sturm-Liouville problem. As earlier, we consider three cases.

If $\lambda = 0$, then the general solution is $y(x) = c_1 + c_2 x$. Now, $y(0) = 0 \Rightarrow c_1 = 0$. Then $y(x) = c_2 x \Rightarrow y'(x) = c_2$. Then, $y'(\pi) = 0 \Rightarrow c_2 = 0$. So, $y(x) = 0$. Therefore, 0 is not an eigenvalue.

If $\lambda < 0$, then write $\lambda = -\alpha^2$ for $\alpha > 0$. The general solution is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$ so that $y'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}$. The boundary conditions imply $c_1 + c_2 = 1$ and $c_1 \alpha e^{\alpha\pi} - c_2 \alpha e^{-\alpha\pi} = 0$. It gives $c_1 = c_2 = 0$ so that $y(x) = 0$, the zero function. Hence, negative numbers are not eigenvalues.

So, let $\lambda = \beta^2$ for $\beta > 0$. The general solution is $y(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. Now, $y(0) = 0 \Rightarrow c_1 = 0$ so that $y(x) = c_2 \sin(\beta x)$. Then $y'(x) = c_2 \beta \cos(\beta x)$. The boundary condition $y'(\pi) = 0$ implies $c_2 \beta \cos(\beta\pi) = 0$. Now, $c_2 = 0$ would give only trivial solution. Otherwise, $\beta \cos(\beta\pi) = 0 \Rightarrow \cos(\beta\pi) = 0 \Rightarrow \beta = n + 1/2$ for $n \in \mathbb{Z}$. Thus, the eigenvalues are

$$\lambda_n = \beta^2 = \frac{(2n+1)^2}{4} \quad \text{for } \ n = 0, 1, 2, 3, \ldots$$

Notice that negative values of n give rise to already listed eigenvalues. The corresponding eigenfunctions are

$$y_n(x) = \sin(\beta x) = \sin(n + 1/2)x \quad \text{for } \ n = 0, 1, 2, 3, \ldots.$$

□

(4.4) ***Example***

Find the eigenvalues and eigenfunctions of the periodic Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = y(\ell), \quad y'(0) = y'(\ell)$$

where $\ell > 0$ is given.

If $\lambda = 0$, then the general solution is $y(x) = c_1 + c_2 x$. Now, $y(0) = y(\ell) \Rightarrow c_1 = c_1 + c_2 \ell \Rightarrow c_2 = 0 \Rightarrow y(x) = c_1$, which is a nonzero function for $c_1 \neq 0$. Thus, $\lambda = 0$ is an eigenvalue and $y(x) = 1$ is a corresponding eigenfunction.

If $\lambda < 0$, say, $\lambda = -\alpha^2$ for $\alpha > 0$, then the general solution is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. The boundary conditions imply

$$c_1(1 - e^{\alpha\ell}) = c_2(e^{-\alpha\ell} - 1), \ c_1(1 - e^{\alpha\ell}) = -c_2(e^{-\alpha\ell} - 1).$$

Solving these, we get $c_2 = 0 = c_1$. This leads to the trivial solution. So, no negative number can be an eigenvalue.

If $\lambda > 0$, say, $\lambda = \beta^2$ for $\beta > 0$, then the general solution is $y = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. The boundary conditions give

$$c_1\big(1 - \cos(\beta\ell)\big) = c_2 \sin(\beta\ell), \ c_1\big(1 - \cos(\beta\ell)\big) = -c_1 \sin(\beta\ell).$$

Eliminating c_2, we obtain $2c_1\big(1 - \cos(\beta\ell)\big) = 0$. It implies either $c_1 = 0$ or $\cos(\beta\ell) = 1$.

If $c_1 = 0$, then $c_2 = 0$ so that $y(x)$ is the trivial solution. This does not give any eigenvalue. So, let $\cos(\beta\ell) = 1$. Then, $\beta\ell = 2n\pi$ for $n \in \mathbb{Z}$. Then,

$$\lambda_n = \beta^2 = 4n^2\pi^2/\ell^2 \quad \text{for } n = 0, 1, 2, 3, \ldots$$

are the eigenvalues. The corresponding solutions are

$$y_n(x) = c_1\cos(\beta_n x) + c_2\sin(\beta_n x), \quad \beta_n = 2n\pi/\ell \quad \text{for } n = 0, 1, 2, 3, \ldots$$

Thus, both the functions $\cos(\beta_n x)$ and $\sin(\beta_n x)$ are eigenfunctions associated with the eigenvalue β_n^2. That is, the eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \frac{4n^2\pi^2}{\ell^2}, \quad y_n^1(x) = \cos\left(\frac{2n\pi x}{\ell}\right), \quad y_n^2(x) = \sin\left(\frac{2n\pi x}{\ell}\right)$$

for $n = 0, 1, 2, 3, \ldots$, defined over $[0, \ell]$. ∎

(4.5) *Example*

Find the eigenvalues and eigenfunctions of the regular Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = y'(0), \quad y(1) + y'(1) = 0.$$

Notice that the eigenfunctions must be well defined over $[0, 1]$. As earlier we consider the three cases.

If $\lambda = 0$, then the general solution is $y(x) = c_1 + c_2 x$. The boundary conditions give $c_1 = c_2$, $2c_1 + c_2 = 0 \Rightarrow c_1 = 0 = c_2$. So, $y(x) = 0$. Hence, $\lambda = 0$ is not an eigenvalue.

Let $\lambda < 0$. Write $\lambda = -\alpha^2$ for $\alpha > 0$. The general solution is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. The boundary conditions give

$$c_1(1 - \alpha) = -c_2(1 + \alpha), \quad c_1\big[(1 + \alpha)e^{\alpha} + c_2(1 - \alpha)e^{-\alpha}\big] = 0.$$

If $\alpha = 1$, then the first equation gives $c_2 = 0$; then the second equation gives $c_1 = 0$. It leads to the trivial solution. So, let $\alpha \neq 1$. Eliminating c_2 from the above equations, we get

$$c_1\big[(1 + \alpha)^2 e^{\alpha} - (1 - \alpha)^2 e^{-\alpha}\big] = 0.$$

Since the bracketed term is nonzero, $c_1 = 0$. It then follows that $c_2 = 0$ so that there is no non-trivial solution. In any case, no non-trivial solution exists. So, a negative number cannot be an eigenvalue.

Then, consider $\lambda > 0$. Write $\lambda = \beta^2$ for $\beta > 0$. The general solution is $y(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$. The boundary conditions give

$$c_1 = \beta c_2, \quad c_1\cos\beta + c_2\sin\beta - \beta c_1\sin\beta + \beta c_2\cos\beta = 0.$$

Eliminating c_1, we have $c_2\big[2\beta\cos\beta + (1-\beta^2)\sin\beta\big] = 0$. If $c_2 = 0$, then $c_1 = 0$ and it leads to the trivial solution. For a non-trivial solution, we must have $2\beta\cos\beta + (1-\beta^2)\sin\beta = 0$. It gives

$$\tan\beta = \frac{2\beta}{\beta^2 - 1}.$$

That is, the eigenvalues are β^2, where β satisfies the above equation. The corresponding eigenfunctions are $y^1(x) = \cos(\beta x)$ and $y^2(x) = \sin(\beta x)$. □

(4.6) ***Example***

Find the eigenvalues and eigenfunctions of Bessel's equation

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \quad \text{for } t > 0$$

with the condition that the solution remains bounded on the interval $[0, a]$ and $y(a) = 0$.

See Example 4.1(3); taking $t = kx$ for $k > 0$, the ODE is transformed to the Sturm-Liouville equation

$$(xy')' + \Big(-\frac{\nu^2}{x} + \lambda x\Big)y = 0 \quad \text{where } \lambda = k^2.$$

Notice that $p(0) = 0$ so that this is a singular Sturm-Lioville problem, where $y(a) = 0$.

The linearly independent solutions of the above Bessel equation are $J_n(t)$ and $Y_n(t)$. Hence, the general solution of the Sturm-Liouville equation is

$$y(x) = c_1 J_n(kx) + c_2 Y_n(kx).$$

Recall that $Y_n(kx) \to \infty$ as $x \to 0$. Since we need only bounded solutions, we must set $c_2 = 0$. Thus, the required non-trivial solution is $y(x) = c_1 J_n(kx)$.

Write $R = a/k$. When $t = a$, we have $kx = a \Rightarrow x = R$. The boundary condition says that $J_n(a) = 0$ or

$$J_n(kR) = 0.$$

This condition is satisfied when kR is a zero of $J_n(x)$. Denote the zeros of $J_n(x)$ by $z_{n,r}$ with $r = 1, 2, 3, \ldots$. [It is known that there are infinite number of zeros of $J_n(x)$] Then, the values of k are

$$k = \frac{z_{n,r}}{R} \quad \text{for } r = 1, 2, 3, \ldots$$

As $\lambda = k^2$, the eigenvalues and the corresponding eigenfunctions are

$$\lambda_r = \Big(\frac{z_{n,r}}{R}\Big)^2, \quad y_r(x) = J_n\Big(\frac{z_{n,r}x}{R}\Big) \quad \text{for } r = 1, 2, 3, \ldots$$

where $z_{n,r}$ is the rth positive zero of $J_n(x)$. □

4.5 Orthogonality

The most important property of eigenfunctions of a Sturm-Liouville problem is that the eigenfunctions are orthogonal. For the plane vectors, orthogonality is obtained via the dot product. To generalize this notion, we introduce the so called inner products of funtions.

Let $y_1(x), y_2(x), \ldots$ be functions defined on an interval $[a, b]$. Let $r(x)$ be a positive function defined on $[a, b]$, that is, $r(x) > 0$ for each $x \in [a, b]$. Let $m, n \in \mathbb{N}$. The **inner product** with weight $r(x)$ of $y_m(x)$ and $y_n(x)$ is denoted by $\langle y_m, y_n \rangle$ and is defined as

$$\langle y_m, y_n \rangle := \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx.$$

It follows that when $m = n$, $\langle y_m, y_m \rangle \geq 0$. The **norm** of a function $y_m(x)$ is denoted by $\|y_m\|$ and is defined as

$$\|y_m\| = \sqrt{\langle y_m, y_m \rangle} = \sqrt{\int_a^b r(x) [y_m(x)]^2\, dx}.$$

We say that y_m and y_n are **orthogonal** to each other with weight $r(x)$ iff $\langle y_m, y_n \rangle = 0$. The functions $y_1(x), y_2(x), \ldots$ are called **orthogonal** with weight $r(x)$ iff $y_m(x)$ is orthogonal to $y_n(x)$ for all $m, n \in \mathbb{N}$, $m \neq n$. The functions $y_1(x), y_2(x), \ldots$ are called **orthonormal** with weight $r(x)$ iff they are orthogonal and the norm of each y_j is 1. This happens when for all $m, n \in \mathbb{N}$, we find that

$$\langle y_m, y_n \rangle = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = \delta_{m,n} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

(4.7) ***Example***

The functions $y_j(x) = \sin(jx)$, $j = 1, 2, \ldots$ are orthogonal on the interval $[-\pi, \pi]$ with the weight function $r(x) = 1$. Indeed, if $m \neq n$, then

$$\langle y_m, y_n \rangle = \int_{-\pi}^{\pi} \sin(mx) \sin(nx)\, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x\, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)x\, dx = 0.$$

Also, we find that $\|y_m\|^2 = \langle y_m, y_m \rangle = \displaystyle\int_{-\pi}^{\pi} \sin^2(mx)\, dx = \pi.$

Hence, the functions $\dfrac{\sin x}{\sqrt{\pi}}, \dfrac{\sin(2x)}{\sqrt{\pi}}, \dfrac{\sin(3x)}{\sqrt{\pi}}, \ldots$ are orthonormal. □

In general, if $y_1(x), y_2(x), y_3(x), \ldots$ are orthogonal, then the *normalized functions*

$$\frac{y_1(x)}{\|y_1\|}, \frac{y_2(x)}{\|y_2\|}, \frac{y_3(x)}{\|y_3\|}, \ldots$$

are orthonormal. Similar to the last example, if $m \neq n$, then

$$\int_{-\pi}^{\pi} 1 \cdot \cos(mx)\,dx = 0, \quad \int_{-\pi}^{\pi} 1 \cdot \sin(mx)\,dx = 0, \quad \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0,$$
$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0, \text{ and}$$
$$\int_{-\pi}^{\pi} 1\,dx = 2\pi, \quad \int_{-\pi}^{\pi} \cos^2(mx)\,dx = \pi, \quad \int_{-\pi}^{\pi} \sin^2(mx)\,dx = \pi.$$

Hence, $\dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos(mx)}{\sqrt{\pi}}, \dfrac{\sin(mx)}{\sqrt{\pi}}$ for $m = 1, 2, 3, \ldots$ are orthonormal.

We mention an important fact about Sturm-Liouville problems.

(4.8) *Theorem*

Consider the Sturm-Liouville problem (4.4.1) either with $p(a) = p(b) = 0$ or with one of the boundary conditions in (4.4.2)-(4.4.5). Let $p(x)$, $q(x)$, $r(x)$, $p'(x)$ be continuous and $r(x) > 0$ on $a \leq x \leq b$. Then all eigenvalues are real, and they may be arranged in order as $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, where $\lim_{n\to\infty} \lambda_n = \infty$. Further, if $y_m(x)$ and $y_n(x)$ are eigenfunctions corresponding to distinct eigenvalues λ_m and λ_n, respectively, then y_m and y_n are orthogonal with weight function $r(x)$. That is,

$$\int_a^b r(x)\,y_m(x)\,y_n(x)\,dx = 0 \quad \textit{for } m \neq n.$$

Proof. We prove only the orthogonality of the eigenfunctions corresponding to distinct eigenvalues. So, let $\lambda_m \neq \lambda_n$ be eigenvalues with corresponding eigenfunctions as $y_m(x)$ and $y_n(x)$. These eigenfunctions satisfy the Sturm-Lioville equation. That is,

$$(py_m')' + (q + \lambda r)y_m = 0, \quad (py_n')' + (q + \lambda r)y_n = 0.$$

Multiply the first with y_n, the second with $-y_m$, and add to get

$$(\lambda_m - \lambda_n) r y_m y_n = y_m (py_n')' - y_n (py_m')'.$$

However,

$$\begin{aligned}\big[p(y_n' y_m - y_m' y_n)\big]' &= \big[y_m(py_n') - y_n(py_m')\big]' = \big[y_m(py_n')\big]' - \big[y_n(py_m')\big]' \\ &= y_m'(py_n') + y_m(py_n')' - y_n'(py_m') - y_n(py_m')' = y_m(py_n')' - y_n(py_m')'.\end{aligned}$$

Hence, $(\lambda_m - \lambda_n) r y_m y_n = \big[p(y_n' y_m - y_m' y_n)\big]'$. Integrating from a to b, we obtain

$$\begin{aligned}\phi := (\lambda_m - \lambda_n)\int_a^b r y_m y_n\,dx &= \Big[p(y_n' y_m - y_m' y_n)\Big]_a^b \\ &= p(b)\big[y_n'(b)y_m(b) - y_m'(b)y_n(b)\big] - p(a)\big[y_n'(a)y_m(a) - y_m'(a)y_n(a)\big].\end{aligned}$$

The orthogonality of the eigenfunction is proved if ϕ evaluates to 0. We show that this is the case by breaking into following cases:

Case 1: $p(a) = 0 = p(b)$. Then, $\phi = 0$.

Case 2: $p(a) \neq 0,\ p(b) = 0$. Then $\phi = -p(a)\big[y_n'(a)y_m(a) - y_m'(a)y_n(a)\big]$. The boundary conditions in (4.4.5) are applicable. Since y_m and y_n are solutions of the BVP, we have

$$k_1 y_m(a) + k_2 y_m'(a) = 0 = k_1 y_n(a) + k_2 y_n'(a).$$

At least one of k_1, k_2 is nonzero. Suppose $k_2 \neq 0$. Multiply the first equation by $y_m(a)$, the second by $-y_n(a)$ and add to get

$$k_2\big[y_n'(a)y_m(a) - y_m'(a)y_n(a)\big] = 0.$$

As $k_2 \neq 0$, we get $\big[y_n'(a)y_m(a) - y_m'(a)y_n(a)\big] = 0$ so that $\phi = 0$. A similar proof is given when $k_1 \neq 0$.

Case 3: $p(a) = 0,\ p(b) \neq 0$. This case is similar to Case 2.

Case 4: $p(a) \neq 0,\ p(b) \neq 0,\ p(a) \neq p(b)$. We use both the conditions in (4.4.2) and proceed as in Case 2.

Case 5: $p(a) = p(b)$. The condition (4.4.3) says that $y(a) = y(b)$ and $y'(a) = y'(b)$. These are satisfied for both $y = y_m$ and $y = y_n$. Then, ϕ evaluates to 0. ∎

(4.9) *Example*

Consider the Sturm-Liouville problem of (4.3):

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$$

Here, $p(x) = 1,\ q(x) = 0$ and $r(x) = 1$. We found the eigenvalues and eigenfunctions as

$$\lambda_n = \frac{(2n+1)^2}{4}, \quad y_n(x) = \sin\big[\big(n + \tfrac{1}{2}\big)x\big], \quad n = 0, 1, 2, 3, \ldots$$

By (4.8), we conclude that

$$\int_0^\pi \sin\big[\big(m + \tfrac{1}{2}\big)x\big] \sin\big[\big(n + \tfrac{1}{2}\big)x\big]\, dx = 0 \quad \text{for } m \neq n.$$

Of course, it is easy to verify this directly. □

(4.10) *Example*

Legendre'e equation $(1-x^2)y'' - 2xy' + \rho(\rho+1)y = 0$ is a Sturm-Liouville equation with $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$ and $\lambda = \rho(\rho + 1)$. Here, $p(-1) = p(1) = 0$. Hence, this is a singular Sturm-Liouville problem on the interval $-1 \leq x \leq 1$. We know that $P_n(x)$ is a solution of this equation for $\lambda = n(n+1)$, where $n = 0, 1, 2, \ldots$.

That is, corresponding to the eigenvalue $\lambda_n = n(n+1)$, the eigenfunction is $P_n(x)$. By (4.8), these eigenfunctions are orthogonal with weight $r(x) = 1$. It means that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad \text{for } m \neq n.$$

We have seen that this is the case. ∎

(4.11) ***Example***

As we have seen in (4.6), the Bessel's equation

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \quad \text{for } t > 0$$

with the condition that the solution remains bounded on $[0, a]$ and $y(a) = 0$ is the Sturm-Liouville problem (Take $t = kx$.)

$$(xy')' + \Big(-\frac{\nu^2}{x} + \lambda x\Big)y = 0 \quad \text{where } \lambda = k^2.$$

Here, $p(0) = 0$ so that this is a singular Sturm-Lioville problem, where $y(R) = 0$ with $R = a/k$. Its eigenvalues and eigenfunctions have been found to be

$$\lambda_r = \Big(\frac{z_{n,r}}{R}\Big)^2, \quad y_r(x) = J_n\Big(\frac{z_{n,r}x}{R}\Big) \quad \text{for } r = 1, 2, 3, \ldots$$

where $z_{n,r}$ is the rth positive zero of $J_n(x)$.

By (4.8), the eigenfunctions are orthogonal with weight $r(x) = x$ on the interval $[0, R]$. That is,

$$\int_0^R x\,J_n\Big(\frac{z_{n,m}x}{R}\Big)\,J_n\Big(\frac{z_{n,j}x}{R}\Big)\,dx = 0 \quad \text{for } m \neq j.$$

We see that the permissible values of k in the transformation $t = kx$ are $z_{n,r}/R$. Notice that for fixed n and a fixed $R > 0$, we have infinitely many orthogonal functions $J_n\Big(\frac{z_{n,m}x}{R}\Big)$. The R in this orthogonality can be chosen according to our convenience, but it is to be fixed. ∎

The above example shows that there are infinitely many orthogonal sets of Bessel functions, one for each of $J_0, J_1, J_2, \ldots$ on an interval $0 \le x \le R$ with a fixed positive R of our choice and with the weight function $r(x) = x$.

We have only proved the orthogonality of the Bessel functions. In fact, the norms of those can also be computed from the following result, which is left as an exercise.

$$\Big\|J_n\Big(\frac{z_{n,r}x}{R}\Big)\Big\|^2 = \int_0^R x\,\Big[J_n\Big(\frac{z_{n,r}x}{R}\Big)\Big]^2\,dx = \frac{R^2}{2}\,\Big[J_{n+1}\Big(z_{n,r}\Big)\Big]^2. \tag{4.5.1}$$

Orthogonality helps in expanding functions as series of eigenfunctions just like Fourier series. We have seen in § 4.1 how to express a function defined on $[-1, 1]$ as a series involving the Legendre polynomials. By using orthogonality of Bessel functions, similar series expansion can be obtained.

Fix $n \in \mathbb{N} \cup \{0\}$. Let $f(x)$ be a real valued peicewise smooth function defined on an interval $0 \le x \le R$. A **Fourier-Bessel series** of $f(x)$ using the Bessel function J_n may be written as

$$f(x) = \sum_{m=1}^{\infty} a_m J_n\Big(\frac{z_{n,m}x}{R}\Big) = a_1 J_n\Big(\frac{z_{n,1}x}{R}\Big) + a_2 J_n\Big(\frac{z_{n,2}x}{R}\Big) + \cdots$$

Fix $\ell \in \mathbb{N}$. Multiply the above equation with $xJ_n\Big(\frac{z_{n,\ell}x}{R}\Big)$ and integrate from 0 to R to get

$$\int_0^R xf(x)J_n\Big(\frac{z_{n,\ell}x}{R}\Big)\,dx = \sum_{m=1}^{\infty} a_m \int_0^R xJ_n\Big(\frac{z_{n,m}x}{R}\Big)J_n\Big(\frac{z_{n,\ell}x}{R}\Big)\,dx.$$

Due to orthogonality, the integral in the above summand is 0 when $m \neq \ell$. So, we obtain

$$\int_0^R xf(x)j_n\Big(\frac{z_{n,\ell}x}{R}\Big)\,dx = a_\ell \int_0^R x\Big[J_n\Big(\frac{z_{n,\ell}x}{R}\Big)\Big]^2\,dx = \frac{R^2}{2}J_{n+1}^2(z_{n,\ell}).$$

The last equality follows from (4.5.1). This gives the coefficient a_ℓ for $\ell \in \mathbb{N}$. Thus, the Fourier-Bessel series for $f(x)$ on an interval $[0, R]$ is given as follows:

$$f(x) = \sum_{m=1}^{\infty} a_m J_n\Big(\frac{z_{n,m}x}{R}\Big), \quad \text{where } a_m = \frac{2}{R^2 J_{n+1}^2(z_{n,m})}\int_0^R xf(x)J_n\Big(\frac{z_{n,m}x}{R}\Big)\,dx. \tag{4.5.2}$$

Notice that we have written $f(x)$ is equal to its Fourier-Bessel series for deriving the coefficients. However, the series so obtained may or may not converge to the function $f(x)$. This question of convergence is answered by the following result, which we mention without proof.

(4.12) ***Theorem*** (Convergence of Fourier-Bessel series)
Let $f(x)$ be a piecewise smooth function defined on the interval $0 < x < R$. Then the Fourier-Bessel series (4.5.2) of $f(x)$ converges to $g(x)$, where

$$g(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{1}{2}\big[f(x+) + f(x-)\big] & \text{if } f \text{ is discontinuous at } x. \end{cases}$$

It thus follows that if $f(x)$ is continuous on $0 < x < R$, then its Fourier-Bessel series converges to $f(x)$.

(4.13) ***Example***

Find the Fourier-Bessel series for the function $f(x) = 1$ on $0 < x < 1$.

Here, $R = 1$; we choose $n = 0$. By (4.5.2), the Fourier-Bessel series of $f(x) = 1$ is given by (Write $z_{0,m}$ as z_m.)

$$\sum_{m=1}^{\infty} a_m J_0(z_m x)$$

where z_m for $m = 1, 2, 3, \ldots$ are the positive zeros of $J_0(x)$ and the coefficients a_m are given by

$$a_m = \frac{2}{J_1^2(z_m)} \int_0^1 x J_0(z_m x)\, dx.$$

We use the identity $\left[x J_1(x)\right]' = x J_0(x)$ given in (4.3.1) to evaluate the above integral. Substitute $t = z_m x$. Then, $dt = z_m\, dx$, and when x varies from 0 to 1, t varies from 0 to z_m. Hence,

$$a_m = \frac{2}{z_m^2 J_1^2(z_m)} \int_0^{z_m} t J_0(t)\, dt = \frac{2}{z_m^2 J_1^2(z_m)} t J_1(t)\Big]_0^{z_m} = \frac{2 z_m J_1(z_m)}{z_m^2 J_1^2(z_m)} = \frac{2}{z_m J_1(z_m)}.$$

Since $f(x)$ is continuous everywhere on $(0, 1)$, by the convergence theorem,

$$1 = \sum_{m=1}^{\infty} \frac{2 J_0(z_m x)}{z_m J_1(z_m)}.$$

□

5

Partial Differential Equations

5.1 Introduction

Suppose $u(x, y)$ is a function of two independent variables. Instead of derivatives we now think of its partial derivatives $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. We may also have higher order partial derivatives such as u_{xx}, u_{xy}, u_{yx}, u_{yy}, u_{xxx}, etc.

An equation involving x, y, u and some of its partial derivatives is called a **partial differential equation**, or **PDE** for short. The order of the highest order derivative of u is called the **order** of the PDE.

Usually, we will be concerned with first and second order PDEs. Of course, there can be more than two independent variables. We will be generally taking two or three independent variables and one dependent variable.

The general form of a first order PDE with dependent variable u and two independent variables x, y is

$$F(x, y, u, u_x, u_y) = 0$$

where F is an expression (also a function) involving x, y, u, u_x and u_y. Similarly, a general first order PDE with one dependent variable u and three independent variables x, y, z may be written as

$$F(x, y, z, u, u_x, u_y, u_z) = 0.$$

If such a function F is linear in the dependent variable and its derivatives, then it is called a **linear PDE**. Notice that in a linear PDE, the coefficients of the dependent variable and its derivatives must be functions of x, y only. The general first order linear PDE with two independent variables looks like

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y).$$

When $d(x, y) = 0$, the linear PDE is called **homogeneous**, else, it is called a **non-homogeneous** PDE. For example, the following are first order linear PDEs:

$$xu_x + yu_y - u = 0.$$
$$u_x + (x + y)u_y - 5u = e^x.$$
$$yu_x + xy_u = xy.$$
$$(y - z)u_x + (z - x)u_y + (x - y)u_z = 0.$$

The first and the fourth are homogeneous, whereas the second and the third are non-homogeneous.

A PDE which is not linear is called **nonlinear**. Among the nonlinear PDEs there are some easier classes of problems. A first order PDE is called **semilinear** iff the coefficients of the derivatives of the dependent variable are functions of the independent variables only. A general form of a semilinear first order PDE with two independent varaibles is

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u).$$

A first order PDE is called **quasi-linear** iff the expression $F(\cdots)$ is linear in the derivatives of the dependent variable. It means, the coefficients of the derivatives are now allowed to involve the dependent variable. The general first order quasi-linear PDE with two independent variables looks like

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u).$$

Some examples of quasi-linear PDEs are

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u.$$
$$uu_x + u_y + u^2 = 0.$$
$$(y^2-u^2)y_x - xyu_y = xu.$$

Sometimes it is possible to use the *methods of ordinary differential equations* to solve a PDE. This method is used when all the derivatives can be integrated with respect to some independent variable, or when by substituting a derivative as a new variable an ODE results. Usually, the general solution of an nth order PDE would involve n number of arbitrary functions. See the following examples.

(5.1) *Example*

Solve $u_x(x,y) = x+y$.

Integrating with respect to x, where y is kept constant, we get

$$u(x,y) = \int (x+y)\,dx = \frac{x^2}{2} + xy + f(y).$$

Here, the constant of integration must not depend on x, but it can depend on y. So, we had taken it as $f(y)$, an arbitrary function of the variable y. $\Box$

(5.2) *Example*

Solve $u_{xy}(x,y) = 0$.

Integrating with respect to y, we get (x is kept constant)

$$u_x(x,y) = f(x).$$

Integrating with respect to x, we obtain

$$u(x,y) = \int f(x)\,dx + g(y).$$

Since $f(x)$ is an arbitrary function, we may write its integral as $h(x)$, where this $h(x)$ is also an arbitrary function. Hence, the general solution of the PDE is $u(x,y) = h(x) + g(y)$ for arbitrary functions $h(x)$ of x and $g(y)$ of y. $\square$

There can be initial and boundary conditions along with a PDE, and they are taken care while solving the PDE.

(5.3) ***Example***

Find $u(x,y)$ that satisfies the PDE $u_{xx} = y^2\cos^2 x$ and $u(0,y) = 0 = u(\pi/2, y)$.

Integrating the given equation with respect to x, we get

$$u_x = y^2\Big(\frac{x}{2} + \frac{\sin(2x)}{4}\Big) + f(y).$$

Here, $f(y)$ is an arbitrary function of y alone. Integrating once more with respect to x, we obtain

$$u = y^2\Big(\frac{x^2}{4} - \frac{\cos(2x)}{8}\Big) + f(y)x + g(y).$$

The condition $u(0,y) = 0$ implies

$$0 = y^2\Big(-\frac{1}{8}\Big) + g(y) \;\Rightarrow\; g(y) = \frac{y^2}{8}.$$

Using this expression for $g(y)$ and using the condition $u(\pi/2, y) = 0$ we get

$$0 = y^2\Big(\frac{\pi^2}{16} + \frac{1}{8}\Big) + f(y)\frac{\pi}{2} + \frac{y^2}{8} \;\Rightarrow\; f(y) = -\Big(\frac{\pi}{8} + \frac{1}{2\pi}\Big)y^2.$$

Hence, the solution is

$$u(x,y) = y^2\Big(\frac{x^2}{4} - \frac{\cos(2x)}{8}\Big) - \Big(\frac{\pi}{8} + \frac{1}{2\pi}\Big)xy^2 + \frac{y^2}{8}. \qquad \square$$

Solutions of PDEs with a dependent variable and two independent variables are also called *integral surfaces*. In such a case, e usually write the independent variables as x, y and the dependent variable as z to rhyme with the geometrical language.

5.2 Lagrange method

We will consider the method of characteristics by Lagrange for solving the quasi-linear first order PDE. We assume that the coefficient functions are continuous in the domain of consideration. Also, we assume that they are not simultaneously 0. (Otherwise, the PDE is no more a PDE.) Lagrange's method is encapsulated in the following theorem.

(5.4) ***Theorem*** (Lagrange's method of characteristics)

Suppose $a(x, y, u)$, $b(x, y, u)$ and $c(x, y, u)$ are continuous and they are not simultaneously 0 at any point in a domain. Then, the general solution of the first order quasi-linear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{5.2.1}$$

is given by $f(\phi, \psi) = 0$, where f is an arbitrary function of two variables, and $\phi(x, y, u) = c_1$, $\psi(x, y, u) = c_2$, for arbitrary constants c_1, c_2, are solutions of

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}. \tag{5.2.2}$$

Equations in (5.2.2) are called the **characteristic equations** of the PDE (5.2.1). Their solutions $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are called the **characteristic curves**. Lagrange's method of characteristic reduces the problem of solving the quasi-linear first order PDE to solving two ODEs.

Proof. Suppose $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are solutions of (5.2.2). Since the PDE (5.2.1) has order 1, and $f(\cdot, \cdot)$ is an arbitrary function of two variables, $f(\phi, \psi) = 0$ is the general solution provided it is at all a solution. Now, $f(\phi, \psi) = 0$ is a solution means that if $u(x, y)$ satisfies $f\big(\phi(x, y, u), \psi(x, y, u)\big) = 0$, then $u(x, y)$ also satisfies the PDE (5.2.1). We show that this is the case.

So, suppose $u(x, y)$ satisfies $f(\phi, \psi) = 0$. Computing the differentials of ϕ and ψ we get

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0, \quad d\psi = \psi_x dx + \psi_y dy + \psi_u du = 0.$$

However, $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are solutions of (5.2.2). So,

$$a\phi_x + b\phi_y + c\phi_u = 0, \quad a\psi_x + b\psi_y + c\psi_u = 0.$$

Eliminating a from these two equations, we get $b(\phi_x\psi_y - \phi_y\psi_x) = c(\phi_u\psi_x - \phi_x\psi_u)$. Eliminating b we get $a(\phi_x\psi_y - \phi_y\psi_x) = c(\phi_y\psi_u - \phi_u\psi_y)$. Hence,

$$\frac{a}{\phi_y\psi_u - \phi_u\psi_y} = \frac{b}{\phi_u\psi_x - \phi_x\psi_u} = \frac{c}{\phi_x\psi_y - \phi_y\psi_x}. \tag{5.2.3}$$

Since $f(\phi, \psi) = 0$, differentiating with respect to x and also y, and using the Chain rule, we have

$$f_\phi(\phi_x + \phi_u u_x) + f_\psi(\psi_x + \psi_u u_x) = 0.$$

$$f_\phi(\phi_y + \phi_u u_y) + f_\psi(\psi_y + \psi_u u_y) = 0.$$

Since $f(\phi, \psi)$ is an arbitrary function, f_ϕ and f_ψ are not necessarily the zero functions. Then, the above two linear equations have a non-trivial solution. So, the determinant of the system is 0. That is,

$$(\phi_x + \phi_u u_x)(\psi_y + \psi_u u_y) = (\phi_y + \phi_u u_y)(\psi_x + \psi_u u_x).$$

It simplifies to

$$(\phi_y\psi_u - \phi_u\psi_y)u_x + (\phi_u\psi_x - \phi_x\psi_u)y_y = \phi_x\psi_y - \phi_y\psi_x.$$

By (5.2.3), $au_x + bu_y = c$. ∎

We remark that for more than two independent variables, the statement in (5.4) also holds so that Lagrange's method is still applicable. That is, to solve the quasi-linear PDE

$$a_1u_1 + \cdots + a_nu_n = c$$

where $u = u(x_1, x_2, \ldots, x_n)$, $a_i = a_i(x_1, x_2, \ldots, x_n, u)$, $u_i = u_{x_i}(x_1, x_2, \ldots, x_n)$ and $c = c(x_1, x_2, \ldots, x_n, u)$, we form the characteristic equations

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \cdots = \frac{dx_n}{a_n} = \frac{du}{c}.$$

We get its solution as $\phi_j(x_1, x_2, \ldots, x_n) = c_j$ for $j = 1, 2, \ldots, n$. Then, the general solution of the PDE is given implicitly by $f(\phi_1, \phi_2, \ldots, \phi_n) = 0$ for an arbitrary function f of n arguments.

If $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are solutions of the characteristic equations in (5.2.2), then the general solution may also be written by assuming certain dependence of these two constants. That is, we may write the general solution as $\phi(x, y, u) = g(\psi(x, y, u))$ for an arbitrary function $g(\cdot)$. Notice that this is an explicit way of writing the same general solution $f(\phi, \psi) = 0$. The implicit way of writing is more general than the explicit way. However, if one of the characteristic curves is $u = c_1$, then the explicit way of writing is as general as the implicit way of writing.

(5.5) *Example*

Find the general solution of the PDE $u_x + u_y = 1$.

The characteristic equations are $dx = dy = du$. Taking them in pairs and integrating, we have

$$dx - dy = 0 \Rightarrow x - y = c_1.$$

$$dy - du = 0 \Rightarrow y - u = c_2.$$

Thus, the general solution is $f(x - y, y - u) = 0$ for an arbitrary function f of two arguments. We may also write the general solution as $y - u = g(x - y)$ or $u = y - g(x - y)$ for an arbitrary function g of one variable. □

(5.6) ***Example***

Find the general solution of the PDE $xu_x + yu_y = u$.

The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

Taking in pairs and integrating we obtain

$$\frac{y}{x} = c_1, \quad \frac{u}{x} = c_2.$$

Thus, the general solution is $f(y/x, u/x) = 0$ for an arbitrary function $f(\cdot, \cdot)$. We may also write the general solution as $u/x = g(y/x)$ or $u = xg(y/x)$. ∎

(5.7) ***Example***

Find the general solution of the PDE $x^2u_x + y^2u_y = (x + y)u$.

The characteristic equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x + y)u}.$$

First two equations give $x^{-1} - y^{-1} = c_1$. To get another solution, we subtract the first two and find that

$$\frac{dx - dy}{x^2 - y^2} = \frac{du}{(x + y)u} \Rightarrow \frac{d(x - y)}{x - y} = \frac{du}{u}.$$

Integrating, we get $(x - y)/u = c_2$. Thus, the general solution is given by $f\big(x^{-1} - y^{-1}, (x - y)/u\big) = 0$. Since $x^{-1} - y^{-1}$ is a constant and $(x - y)/u$ is a constant, it follows that xy/u is a constant. Thus, we can also write the general solution as $g(xy/u, (x - y)/u) = 0$. ∎

(5.8) ***Example***

Find the general solution of $(y - z)u_x + (z - x)u_y + (x - y)u_z = 0$.

The characteristic equations are

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} = \frac{du}{0}$$

For ease in integration, instead of pairs of equations, we consider the following equivalent ones:

$$du = 0, \; dx + dy + dz = 0, \quad xdx + ydy + zdz = 0.$$

The solutions are $u = c_1$, $x + y + z = c_2$, $x^2 + y^2 + z^2 = c_3$. The general solution can be written as $u = g(x + y + z, x^2 + y^2 + z^2)$ for an arbitrary function $g(\cdot, \cdot)$. ∎

(5.9) *Example*

Find a function $u(x, y)$ that satisfies $xu_y = yu_x$ and $u(0, y) = y^2$.

The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{du}{0}.$$

Taking in pairs and integrating, we get

$$xdx + ydy = 0 = du \Rightarrow x^2 + y^2 = c_1,\ u = c_2.$$

So, the general solution of the PDE is $f(x^2 + y^2, u) = 0$. Since $u(0, y) = y^2$, we have $f(0^2 + y^2, y^2) = 0$. One such f is $f(v, w) = v - w$. Thus, $u = x^2 + y^2$ is a general solution satisfying the given condition.

If we write the general solution in an explicit way, it is given by $c_2 = g(c_1)$ or, $u = g(x^2 + y^2)$ for an arbitrary function $g(\cdot)$. The associated condition $u(0, y) = y^2$ implies that $g(0^2 + y^2) = y^2$. One such g is $g(u) = u$. Then a general solution satisfying the given condition is $u = x^2 + y^2$ as earlier. ∎

(5.10) *Example*

Find a solution of the PDE $xu_y = yu_x$ which contains the circle $u = 1, x^2 + y^2 = 4$.

From the last example, we see that the general solution of the PDE is $f(x^2+y^2, u) = 0$. Since it contains the given curve, we have $f(4, 1) = 0$.

One such f is $f(v, w) = v - 4w$ in which case, a solution is given by $x^2 + y^2 = 4u$. Another f is $f(v, w) = v - w - 3$, in which case a solution is $x^2 + y^2 - 3 = u$. One more is $f(v, w) = v + w^2 - 5$ in which case a solution is $x^2 + y^2 + u^2 = 5$. In fact, there are infinitely many such solutions. ∎

(5.11) *Example*

Find a solution of $u(x + y)u_x + u(x - y)u_y = x^2 + y^2$, where $u = 0$ on the line $y = 2x$.

The characteristic equations are

$$\frac{dx}{u(x + y)} = \frac{dy}{u(x - y)} = \frac{du}{x^2 + y^2}.$$

The equations imply (We require two equations.)

$$ydx + xdy - udu = 0, \quad xdx - ydy - udu = 0.$$

Writing as differentials, these are

$$d\Big(xy - \frac{u^2}{2}\Big) = 0, \quad d\Big(\frac{x^2 - y^2 - u^2}{2}\Big) = 0.$$

Integrating we get $2xy - u^2 = c_1$ and $x^2 - y^2 - u^2 = c_2$. We write the general solution in the form $c_2 = f(c_1)$, that is,

$$x^2 - y^2 - u^2 = f(2xy - u^2)$$

for an arbitrary function $f(\cdot)$. Since $u(x, y)$ also satisfies the given condition, we substitute $y = 2x$ and $u = 0$ simultaneously to get

$$x^2 - 4x^2 = f(4x^2) \Rightarrow f(4x^2) = -3x^2.$$

We may take $f(u) = -\frac{3}{4}u$ which satisfies this condition. So, one solution is given by $x^2 - y^2 - u^2 = -\frac{3}{4}(2xy - u^2)$ or $4(x^2 - y^2 - u^2) + 3(2xy - u^2) = 0$ or, $7u^2 = 6xy + 4(x^2 - y^2)$. □

For nonlinear PDEs of first order, there does not exist any such general method as Lagrange's. However, numerical techniques exist to solve nonlinear PDEs, which you will learn elsewhere.

5.3 Second order linear PDEs

A general second order linear PDE with two independent variables is given by

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{5.3.1}$$

where $a, \ldots, g$ are functions of x and y that do not vanish simultaneously at any point of the domain of definition of $u(x, y)$. We also assume that these functions and the function u have continuous second order partial derivatives on this domain. Some examples are:

$u_{tt} = k^2 u_{xx}$	One-dimensional wave equation
$u_t = k^2 u_{xx}$	One-dimensional heat equation
$u_{xx} + u_{yy} = 0$	Two-dimensional Laplace equation
$u_{xx} + u_{yy} = f(x, y)$	Two-dimensional Poisson equation
$u_{tt} = k^2(u_{xx} + u_{yy})$	Two-dimensional wave equation

The linear second order PDE (5.3.1) is called **homogeneous** iff $g(x, y)$ is the zero function; else it is called **non-homogeneous**. Just like ODEs, if $u_1(x, y)$ and $u_2(x, y)$ are two solutions of a homogeneous linear second order PDE, then their linear combination $u = c_1u_1 + c_2u_2$, for constants $c_1,\ c_2$, is also a solution of the same homogeneous PDE. Sometimes we can use the method of ODEs to solve these PDEs if it is so possible.

(5.12) ***Example***

1. Solve the PDE $u_{xx}(x, y) - u(x, y) = 0$.

   Since derivatives are taken with respect to x only, we can use ODE methods. Integrating with respect to x, we get $u(x, y) = \phi(y)e^x + \psi(y)e^{-x}$. Observe that the constants of integration are now functions of y. The functions $\phi(y)$ and $\psi(y)$ are arbitrary. In a second order PDE, it is expected that there will be two arbitrary functions.

2. Solve $u_{xy} + u_x = 0$.

   We assume that u is a function of x and y. Let $u_x = v$. Then, the equation is $v_y = -v$ whose solution is $v = \phi_1(x)e^{-y}$. Observe that since integration is with respect to y, the constant of integration can be a function of x, in general. Now, $u_x = v = \phi_1(x)e^{-y}$ gives

   $$u = e^{-y} \int \phi_1(x)\, dx + \psi(y).$$

   Since $\phi_1(x)$ is an arbitrary function, so is its integral, which we then write as $\phi(x)$. Hence, the general solution of the PDE is $u(x, y) = e^{-y}\phi(x) + \psi(y)$, where $\phi(x)$ and $\psi(y)$ are arbitrary functions of $x,\ y$, respectively. ▯

The ODE methods suggest that we try to determine certain transformations so that a linear second order PDE may take one of the following forms:

$$u_{xx} = \phi(x, y, u, u_x, u_y), \quad u_{xy} = \phi(x, y, u, u_x, u_u), \quad u_{yy} = \phi(x, y, u, u_x, u_y).$$

Here, $\phi(x, y, u, u_x, u_y)$ is an expression which is linear in $u,\ u_x$ and u_y. That is, $\phi(x, y, u, u_x, u_y) = f_1(x, y) + f_2(x, y)u + f_3(x, y)u_x + f_4(x, y)u_y$ for some functions f_1, f_2, f_3 and f_4.

However, all linear second order PDEs cannot be transformed to these two forms. The ones which cannot be transformed to one of the above two forms can be transformed to the forms

$$u_{xx} + u_{yy} = \phi(x, y, u, u_x, u_y), \quad u_{xx} - u_{yy} = \phi(x, y, u, u_x, u_y).$$

Further, we can show that any PDE in the form $u_{xx} - u_{yy} = \phi(x, y, u, u_x, u_y)$ can also be transformed to the form

$$u_{xy} = \phi(x, y, u, u_x, u_y).$$

These forms of linear second order PDEs are called **standard forms** or **canonical forms**.

To find out which types of PDEs can be transformed to which form, we look at the **discriminant** $b^2 - 4ac$ of the PDE (5.3.1). Depending on the sign of the discriminant, we classify the linear PDEs. We say that the linear second order PDE (5.3.1) is

hyperbolic iff $b^2 - 4ac > 0$,
parabolic iff $b^2 - 4ac = 0$, and
elliptic iff $b^2 - 4ac < 0$.

Notice that $b^2 - 4ac$ is a function of x and y. Its sign is required to be same thorough out the domain of interest. It is quite possible that a linear second order PDE is of one type in some domain and of another type in another domain.

We see that the discriminant concerns the coefficients of $u_{xx},\ u_{x,y}$ and u_{yy} only. Let us look at the signs of the discriminants of the PDEs in standard form. They are as follows:

Canonical form	a, b, c	$b^2 - 4ac$	Type
$u_{xy} = \phi$	$a = 0, b = 1, c = 0$	> 0	Hyperbolic
$u_{xx} - u_{yy} = \phi$	$a = 1, b = 0, c = -1$	> 0	Hyperbolic
$u_{xx} = \phi$	$a = 1, b = 0, c = 0$	$= 0$	Parabolic
$u_{xx} + u_{yy} = \phi$	$a = 1, b = 0, c = 1$	< 0	Elliptic

As you may be surmising the type of the PDE should remain the same while transforming one to its standard form. It means that the sign of the discriminant will not change when we change the independent variables. We show this key fact below.

Reduction to Standard Form: To transform (5.3.1) to its standard form, we change the independent variables, say,

$$\xi = \xi(x, y), \quad \eta = \eta(x, y).$$

We assume that the functions ξ and η have continuous second order partial derivatives and the Jacobian

$$J = \xi_x \eta_y - \xi_y \eta_x \neq 0$$

in the concerned region. This assumption $J \neq 0$ guarantees that x and y can be determined from given ξ and η. To change the variables, we compute the derivatives as follows:

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}.
\end{aligned}$$

Substituting these in (5.3.1) and grouping together terms, we obtain

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + Fu = G \tag{5.3.2}$$

where $A, \ldots, G$ are functions of ξ and η and they are given by

$$\begin{aligned}
A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 \\
B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \\
C &= a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2 \\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y \\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y \\
F &= f(x(\xi,\eta), y(\xi,\eta)) \\
G &= g(x(\xi,\eta), y(\xi,\eta)).
\end{aligned} \tag{5.3.3}$$

Notice that on the right side, $a, \ldots, e$ should first be expressed in terms of ξ and η so that $A, \ldots, E$ are also expressed in terms of ξ and η. And, there is no change in F and G; they are now expressed in terms of ξ and η.

Computing the *discriminant* $B^2 - 4AC$ for the new equations, we find that

$$B^2 - 4AC = \big(\xi_x\eta_y - \xi_y\eta_x\big)^2(b^2 - 4ac).$$

Since $J = \xi_x\eta_y - \xi_y\eta_x \neq 0$, the sign of the discriminant remains invariant. Hence, the type of the PDE remains same under such a general transformation. We thus need to choose particular ξ and η for reducing a PDE to its standard form. Our choice will depend on the type of the problem. Observe that if $a \neq 0$, then A and C in (5.3.3) can be factored as follows:

$$\begin{aligned}
A &= (4a)^{-1}\big[2a\xi_x + \big(b + \sqrt{b^2 - 4ac}\,\big)\xi_y\big]\big[2a\xi_x + \big(b - \sqrt{b^2 - 4ac}\,\big)\xi_y\big] \\
C &= (4a)^{-1}\big[2a\eta_x + \big(b + \sqrt{b^2 - 4ac}\,\big)\eta_y\big]\big[2a\eta_x + \big(b - \sqrt{b^2 - 4ac}\,\big)\eta_y\big].
\end{aligned} \tag{5.3.4}$$

Hyperbolic type: Suppose the PDE (5.3.1) is hyperbolic; that is, $b^2 - 4ac > 0$ in the region of interest. If both $a = 0 = c$, then the PDE is already in its standard form. Else, assume that a is nonzero. To bring the PDE to its standard form, we put $A = C = 0$. To obtain two different solutions, we take different factors in the factorizations of A and C in (5.3.4). That is, we set

$$2a\xi_x + \big(b + \sqrt{b^2 - 4ac}\,\big)\xi_y = 0, \quad 2a\eta_x + \big(b - \sqrt{b^2 - 4ac}\,\big)\eta_y = 0.$$

Solving these first order PDEs by Lagrange's method, we have the characteristic equations as

$$\frac{dx}{2a} = \frac{dy}{b + \sqrt{b^2 - 4ac}}, \quad \frac{dx}{2a} = \frac{dy}{\big(b - \sqrt{b^2 - 4ac}\big)}. \tag{5.3.5}$$

If the solutions of the characteristics are respectively $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$, then we take the transformation as

$$\xi = \phi(x, y), \quad \eta = \psi(x, y).$$

As we see, this will make $A = 0 = C,\ B \neq 0$ in (5.3.2-5.3.3) so that the PDE (5.3.1) is transformed to its standard form

$$u_{\xi\eta} = \frac{1}{B}\big(G - Du_\xi - Eu_\eta - Fu\big). \tag{5.3.6}$$

This is called the *first standard form of a hyperbolic PDE.* Notice that by this choice of ξ and η, their Jacobian remains nonzero.

By taking new independent variables as $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, the above standard form is again transformed to

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{B}\big(G - (D+E)u_\alpha - (D-E)u_\beta - Fu\big) \tag{5.3.7}$$

where $B, \ldots, G$ are expressed in terms of α and β. That is, we replace $\xi = (\alpha+\beta)/2$ and $\eta = (\alpha - \beta)/2$ in the earlier expressions of $B, \ldots, G$ to express those in terms of α and β, and use the resulting expressions here. This standard form is called the *second standard form of a hyperbolic PDE.*

If $a = 0$, then c is nonzero, and we switch the roles of x and y. That is, we interchange x and y, proceed as above. Notice that the standard form will involve ξ and η. Since $u_{\xi\eta} = u_{\eta\xi}$, interchanging x and y there will have no effect. But this interchange will affect the transformations ξ and η. See (5.15) below.

(5.13) *Example*

Reduce the PDE $u_{xx} + 8u_{xy} + 7u_{yy} + u_x + 2u_y + 3u + y = 0$ to its standard form.

As per the notation in (5.3.1), $a = 1$, $b = 8$, $c = 7$, $d = 1$, $e = 2$, $f = 3$ and $g = y$ so that the discriminant $b^2 - 4ac = 8^2 - 28 = 36 > 0$. The PDE is hyperbolic on the whole of $\mathbb{R}^2$. Now, $b \pm \sqrt{b^2 - 4ac} = 8 \pm 6 = 14,\ 2$. By (5.3.5), the characteristic equations are

$$\frac{dx}{2} = \frac{dy}{14}, \quad \frac{dx}{2} = \frac{dy}{2} \Rightarrow dy - 7dx = 0, \quad dy - dx = 0.$$

Its solutions are $y - 7x = c_1$ and $y - x = c_2$. Thus, we take

$$\xi(x,y) = y - 7x, \quad \eta(x,y) = y - x.$$

One can proceed directly from this place to get the derivatives and substitute in the PDE to get one in standard form. We use the formula given in (5.3.6) as in the following.

$$\xi_x = -7, \quad \xi_y = 1, \quad \eta_x = -1, \quad \eta_y = 1, \quad x = (\eta - \xi)/6, \quad y = (7\eta - \xi)/6.$$

$$\begin{aligned} B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y \\ &= 2(-7)(-1) + 8\big((-7)(1) + (1)(-1)\big) + 2(7)(1)(1) = -36. \\ D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = -7 + 2 = -5. \\ E &= a\eta_{xx} + b\eta_{xy} + c\eta yy + d\eta_x + e\eta_y = -1 + 2 = 1. \\ F &= 3, \quad G = -y = (\xi - 7\eta)/6. \end{aligned}$$

Then the PDE is transformed to its first standard form:

$$\begin{aligned} u_{\xi\eta} &= \frac{1}{B}\big(G - Du_\xi - Eu_\eta - Fu\big). \\ &= \frac{1}{-36}\Big(\frac{\xi - 7\eta}{6} - (-5)u_\xi - (1)u_\eta - 3u\Big) \\ &= \frac{1}{36}\Big(-5u_\xi + u_\eta + 3u + \frac{1}{6}(7\eta - \xi)\Big). \end{aligned}$$

For the second standard form, we take $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. Thus, $\xi = (\alpha + \beta)/2$ and $\eta = (\alpha - \beta)/2$. Except G, all other coefficients in (5.3.7) are constants. Now,

$$G = \frac{\xi - 7\eta}{6} = \frac{1}{6}\Big(\frac{\alpha + \beta}{2} - 7\frac{\alpha - \beta}{2}\Big) = \frac{4\beta - 3\alpha}{6}.$$

By (5.3.7), the transformed PDE with independent variables α, β is,

$$\begin{aligned} u_{\alpha\alpha} - u_{\beta\beta} &= \frac{1}{B}\big(G - (D + E)u_\alpha - (D - E)u_\beta - Fu\big) \\ &= \frac{1}{-36}\Big(\frac{4\beta - 3\alpha}{6} - (-5 + 1)u_\alpha - (-5 - 1)u_\beta - 3u\Big) \\ &= \frac{1}{-36}\Big(\frac{4\beta - 3\alpha}{6} + 4u_\alpha + 6u_\beta - 3u\Big). \end{aligned}$$

□

(5.14) *Example*

Transform the PDE $y^2u_{xx} - x^2u_{yy} = 0$ for $xy \neq 0$, to its standard form.

Here, $a = y^2$, $b = 0$, $c = -x^2$, $d = 0$, $e = 0$, $f = 0$ and $g = 0$. Now, $b^2 - 4ac = 4x^2y^2 > 0$ since $xy \neq 0$. And, $b \pm \sqrt{b^2 - 4ac} = \pm 2xy$. The characteristic equations are

$$\frac{dx}{2y^2} = \frac{dy}{2xy}, \quad \frac{dx}{2y^2} = \frac{dy}{-4x^2y^2} \;\Rightarrow\; ydy - xdx = 0, \quad ydy + xdx = 0.$$

Their general solutions are $(y^2 - x^2)/2 = c_1$ and $(y^2 + x^2)/2 = c_2$, respectively. We use the transformation

$$\xi = \frac{y^2 - x^2}{2}, \quad \eta = \frac{y^2 + x^2}{2}.$$

Instead of using the formula, let us compute the derivative directly. We have

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = -x u_\xi + x u_\eta.\\
u_y &= u_\xi \xi_y + u_\eta \eta_y = y u_\xi + y u_\eta.\\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}\\
&= x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta.\\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}\\
&= y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta.
\end{aligned}$$

Substituting these in the given PDE and simplifying we obtain the standard form:

$$u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)} u_\xi - \frac{\xi}{2(\xi^2 - \eta^2)} u_\eta. \qquad \square$$

(5.15) ***Example***

Reduce the PDE $4u_{xy} + u_{yy} + u_y = 0$ to its standard form.

This is a hyperbolic PDE with the coefficient of u_{xx} as 0. We interchange the variables x and y to get

$$u_{xx} + 4u_{xy} + u_x = 0.$$

Here, $a = 1$, $b = 4$, $c = 0$, $d = 1$ and $e = f = g = 0$. Now, $b \pm \sqrt{b^2 - 4ac} = 4 \pm 4 = 8,\ 0$. By (5.3.5) the characteristics are

$$\frac{dx}{2} = \frac{dy}{8}, \quad \frac{dx}{2} = \frac{dy}{0} \Rightarrow dy - 4dx = 0, \quad dy = 0.$$

The solutions are $y - 4x = c_1$ and $y = c_2$. We take the transformations as $\xi = y - 4x$ and $\eta = y$. Then $\xi_x = -4$, $\xi_y = 1$, $\eta_x = 0$ and $\eta_y = 1$. By (5.3.3),

$$\begin{aligned}
B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 4(-4) = -16.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = -4\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = 0.
\end{aligned}$$

The first standard form is

$$u_{\xi\eta} = \frac{G - Du_\xi - Eu_\eta - Fu}{B} = \frac{4u_\xi}{-16} \Rightarrow u_{\xi\eta} + \frac{1}{4}u_\xi = 0.$$

Interchanging x and y retains the above standard form. But the transformations change to $\xi = x - 4y$ and $\eta = x$. You can verify that if we take this transformation directly, then the given PDE reduces to $u_{\xi\eta} + \frac{1}{4}u_\xi = 0$ as earlier. $\square$

Parabolic type: Suppose the PDE (5.3.1) is parabolic; that is, $b^2 - 4ac = 0$ in the region of interest. From (5.3.4), we obtain

$$\begin{aligned} A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 &= \frac{1}{4a}\left(4a^2\xi_x^2 + 4ab\xi_x\xi_y + 4ac\xi_y^2\right) \\ &= \frac{1}{4a}\left(4a^2\xi_x^2 + 4ab\xi_x\xi_y + b^2\xi_y^2\right) = \frac{1}{4a}(2a\xi_x + b\xi_y)^2. \end{aligned}$$

Computing similarly for C, we find that

$$C = \frac{1}{4a}(2a\eta_x + b\eta_y)^2.$$

Now, both $2a\xi_x + b\xi_y = 0$ and $2a\eta_x + b\eta_y$ give the same characteristic $\dfrac{dx}{2a} = \dfrac{dy}{b}$ or,

$$b\,dx - 2a\,dy = 0. \tag{5.3.8}$$

It says that parabolic equations have only one characteristic curve. Suppose the general solution of this characteristic is $\phi(x, y) = c_1$. We choose $\eta = \phi(x, y)$. This will make $C = 0$. Since $B^2 - 4AC = 0$, it will force $B = 0$. The only nonzero term is the remaining $u_{\xi\xi}$ so that the reduced PDE will be in the standard form. Recall that this computation assumes that the Jacobian is nonzero. Hence, after choosing η we choose ξ in such a manner that the Jacobian

$$J = \xi_x\eta_y - \xi_y\eta_x \neq 0.$$

We thus have $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ and the reduced PDE is

$$Au_{\xi\xi} = G - Du_\xi - Eu_\eta - Fu. \tag{5.3.9}$$

Here again, $A, \dots, G$ in (5.3.7) are expressed in terms of ξ and η by using $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$.

(5.16) *Example*

Reduce the PDE $u_{xx} + 4u_{xy} + 4u_{yy} + u_x + 3x = 0$ to its standard form.

Here, $a = 1$, $b = 4$, $c = 4$, $d = 1$, $e = 0$, $f = 0$ and $g = -3x$. The discriminant $b^2 - 4ac = 0$. So, it is a parabolic PDE with $a \neq 0$ and $c \neq 0$. The characteristic curve is, by (5.3.8),

$$b\,dx - 2a\,dy = 0 \Rightarrow 4\,dx - 2\,dy = 0 \Rightarrow y - 2x = c_1.$$

Thus, we take $\eta = y - 2x$. Here, $\eta_x = -2$ and $\eta_y = 1$. We choose $\xi = x$ so that $\xi_x = 1$ and $\xi_y = 0$. This makes Jacobian

$$J = \xi_x\eta_y - \xi_y\eta_x = 1 \cdot 1 - 0 \cdot (-2) = 1 \neq 0.$$

From (5.3.3) we get

$$\begin{aligned}
A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 = 1.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + E\xi_y = 1.\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = -2.\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0.\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = -3\xi.
\end{aligned}$$

By (5.3.9), the PDE is transformed to the standard form

$$Au_{\xi\xi} = G - Du_\xi - Eu_\eta - Fu \;\Rightarrow\; u_{\xi\xi} = -3\xi - u_\xi + 2u_\eta. \qquad \square$$

(5.17) ***Example***

Reduce the PDE $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ for $x > 0$ to its standard form.

Here, $a = x^2$, $b = -2xy$, $c = y^2$, $d = x$, $e = y$, $f = g = 0$ so that $b^2 - 4ac = 4x^2y^2 - 4x^2y^2 = 0$. It is a parabolic PDE. By (5.3.8), the characteristic is

$$b\,dx - 2a\,dy = 0 \;\Rightarrow\; -2xy\,dx - 2x^2\,dy = 0 \;\Rightarrow\; ydx + xdy = 0 \;\Rightarrow\; xy = c_1.$$

Thus, $\eta = xy$. Then $\eta_x = y$ and $\eta_y = x$. We choose $\xi = x$ so that $\xi_x = 1$, $\xi_y = 0$ and the Jacobian $J = \xi_x\eta_y - \xi_y\eta_x = x$ is nonzero. Also, $x = \xi$ and $y = \eta/x = \eta/\xi$. By (5.3.3),

$$\begin{aligned}
A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 = a = x^2 = \xi^2.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = d = x = \xi.\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = b + dy + ex = -2xy + xy + xy = 0.\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0.\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = 0.
\end{aligned}$$

By (5.3.9), the PDE is transformed to the standard form

$$Au_{\xi\xi} = G - Du_\xi - Eu_\eta - Fu \;\Rightarrow\; \xi^2u_{\xi\xi} = -\xi u_\xi \;\Rightarrow\; u_{\xi\xi} + \tfrac{1}{\xi}u_\xi = 0. \qquad \square$$

Elliptic type: Suppose that the PDE (5.3.1) is elliptic; that is, $b^2 - 4ac < 0$ in a region of interest. The factors of A and C in (5.3.4) are now complex. Thus, elliptic PDEs have no characteristics. The standard form of an elliptic PDE have the coefficient of $u_{\xi\eta}$ as 0 and the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ are equal. It means, in (5.3.2), we must have $A - C = B = 0$. That is, using (5.3.3), we have

$$\begin{aligned}
A - C &= a(\xi_x^2 - \eta_x^2) + b(\xi_x\xi_y - \eta_x\eta_y) + c(\xi_y^2 - \eta_y^2) = 0\\
B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0.
\end{aligned}$$

Multiply the second with i, add to the first, and write $\phi = \xi + i\eta$ to obtain

$$\begin{aligned}
0 &= (A - C) + iB \\
&= a(\xi_x^2 - \eta_x^2) + i\big(2a\xi_x\eta_x\big) + b(\xi_x\xi_y - \eta_x\eta_y) + ib(\xi_x\eta_y + \xi_y\eta_x) + c(\xi_y^2 - \eta_y^2) + ic\xi_y\eta_y \\
&= a(\xi_x + i\eta_x)^2 + b(\xi_x + i\eta_x)(\xi_y + i\eta_y) + c(\xi_y + i\eta_y)^2 \\
&= a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2.
\end{aligned}$$

Since $a \neq 0$, we can factor the last equation as

$$\frac{1}{4a}\left[2a\phi_x + \left(b + i\sqrt{4ac - b^2}\right)\phi_y\right]\left[2a\phi_x + \left(b - i\sqrt{4ac - b^2}\right)\phi_y\right] = 0.$$

We are interested in real solutions, and each of these factors will give rise to same pair of real solutions as their real and imaginary parts. So, we consider the first factor:

$$2a\phi_x + \left(b + i\sqrt{4ac - b^2}\right)\phi_y = 0.$$

Using Lagrange's method, we set its corresponding ODE:

$$\frac{dx}{2a} = \frac{dy}{b + i\sqrt{4ac - b^2}}.$$

We rewrite it as follows and refer to it by telling the *complex characteristic* :

$$\left(b + i\sqrt{4ac - b^2}\right)dx - 2a\,dy = 0. \tag{5.3.10}$$

Suppose $\phi(x, y) = c_1$ is the general solution of (5.3.11). Then, we use the change of variables as $\xi = \mathrm{Re}(\phi)$ and $\eta = \mathrm{Im}(\phi)$. In this case, it can be shown that the Jacobian is nonzero so that we will be able to uniquely determine $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$. This change of variables will make $A = C$ and $B = 0$. Hence, the given elliptic PDE (5.3.1) is reduced to

$$Au_{\xi\xi} + Au_{\eta\eta} + Du_\xi + Eu_\eta + Fu = G \tag{5.3.11}$$

where the coefficients A, D, E, F, G are as in (5.3.3) expressed in terms of ξ and η.

(5.18) *Example*

Reduce the PDE $5u_{xx} - 2u_{xy} + 2u_{yy} + 2u_y + 4y = 0$ to its standard form.

As per the notation in (5.3.1), $a = 5$, $b = -2$, $c = 2$, $d = 0$, $e = 2$, $f = 0$ and $g = -4y$. The discriminant $b^2 - 4ac = 4 - 40 = -36 < 0$; so the PDE is elliptic on the whole $\mathbb{R}^2$. By (5.3.10), the complex characteristic is

$$\left(b + i\sqrt{4ac - b^2}\right)dx - 2a\,dy = 0 \implies (-2 + 6i)\,dx - 10\,dy = 0.$$

Its general solution is $(-2+6i)x - 10y = c_1$ or $\phi(x,y) = (x+5y) - i(3x) = c_2$. Thus, the change of variable is

$$\xi = \mathrm{Re}(\phi) = x + 5y, \quad \eta = \mathrm{Im}(\phi) = 3x.$$

Then, we find that $x = \eta/3$, $y = (3\xi - \eta)/15$, $\xi_x = 1$, $\xi_y = 5$, $\eta_x = 3$ and $\eta_y = 0$. By (5.3.3), we have

$$\begin{aligned}
A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 = 45.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = 10.\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0.\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0.\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = -4(3\xi - \eta)/15.
\end{aligned}$$

By (5.3.11), the PDE has the standard form $A(u_{\xi\xi} + u_{\eta\eta}) + Du_\xi + Eu_\eta + Fu = G$ which gives

$$u_{\xi\xi} + u_{\eta\eta} + \frac{2}{9}u_\xi + \frac{4}{225}\xi - \frac{4}{675}\eta = 0. \qquad \Box$$

(5.19) ***Example***

Reduce the PDE $u_{xx} + xu_{yy} = 0$ for $x > 0$, to its standard form.

Here, $a = 1$, $b = 0$, $c = x$, $d = e = f = g = 0$ so that $b^2 - 4ac = -4x < 0$ for $x > 0$. Hence it is an elliptic PDE on the given region. By (5.3.10), the complex characteristic is

$$\left(b + i\sqrt{4ac - b^2}\right)dx - 2a\,dy = 0.$$

It gives $i(2\sqrt{x})\,dx - 2\,dy = 0 \Rightarrow i\frac{4}{3}x^{3/2} - 2y = c_1$ or, $x^{3/2} + i\frac{3}{2}y = c_2$ With $\phi = x^{3/2} + i\frac{3}{2}y$, the transformation is given by

$$\xi = \mathrm{Re}(\phi) = x^{3/2}, \quad \eta = \mathrm{Im}(\phi) = \frac{3}{2}y.$$

Then, $x = \xi^{2/3}$, $y = \frac{2}{3}\eta$, $\xi_x = \frac{3}{2}x^{1/2}$, $\eta_y = \frac{3}{2}$, and by (5.3.3),

$$\begin{aligned}
A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 = \frac{9}{4}x = \frac{9}{4}\xi^{2/3}.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = \frac{3}{4}x^{-1/2} = \frac{3}{4}\xi^{-1/3}.\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0.\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0.\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = 0.
\end{aligned}$$

By (5.3.11), the PDE has the standard form $A(u_{\xi\xi} + u_{\eta\eta}) + Du_\xi + Eu_\eta + Fu = G$ which gives

$$\frac{9}{4}\xi^{2/3}\left(u_{\xi\xi} + u_{\eta\eta}\right) + \frac{3}{4}\xi^{-1/3}u_\xi = 0 \Rightarrow u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\xi}u_\xi = 0. \qquad \Box$$

Reduction of linear second order PDEs to standard forms helps in solving the PDE, at least in hyperbolic and parabolic cases. We illustrate this idea in the following examples.

(5.20) *Example*

Obtain the general solution of the PDE $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$.

Here, $a = 3,\ b = 10,\ c = 3, d = e = f = g = 0$ so that $b^2 - 4ac = 64 > 0$ implies that the PDE is hyperbolic on $\mathbb{R}^2$. Now, $b \pm \sqrt{b^2 - 4ac} = 10 \pm 8 = 18,\ 2$. By (5.3.5), the characteristics are given by

$$\frac{dx}{6} = \frac{dy}{18}, \quad \frac{dx}{6} = \frac{dy}{2} \Rightarrow dy - 3x = 0, \quad dy - \frac{dx}{3} = 0.$$

Their solutions are $y - 3x = c_1$ and $y - x/3 = c_2$. Thus, the transformation is

$$\xi = y - 3x, \quad \eta = y - \frac{x}{3}.$$

We have $\xi_x = -3$, $\xi_y = 1$, $\eta_x = -1/3$, $\eta_y = 1$, and from (5.3.3),

$$\begin{aligned}
B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = -73/3.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = 0.\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0.\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0.\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = 0.
\end{aligned}$$

By (5.3.9), the first standard form is

$$u_{\xi\eta} = \frac{G - Du_\xi - Eu_\eta - Fu}{B} = -\frac{3}{73} \times 0 = 0.$$

Its general solution is $u(\xi,\eta) = h_1(\xi) + h_2(\eta)$. In terms of the original variables, the general solution may be given by

$$u(x,y) = h_1(y - 3x) + h_2\left(y - \frac{x}{3}\right)$$

where h_1 and h_2 are arbitrary functions of one argument each. ∎

(5.21) *Example*

Reduce the PDE $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ for $y \neq 0$, to its standard form and then find its general solution.

Here, $a = x^2$, $b = 2xy$, $c = y^2$, $d = e = f = g = 0$ so that $b^2 - 4ac = 0$. So, it is a parabolic PDE on the whole plane. By (5.3.8), the characteristic is given by

$$b\,dx - 2a\,dy = 0 \Rightarrow 2xy\,dx - 2x^2\,dy = 0 \Rightarrow y\,dx - x\,dy = 0 \Rightarrow \frac{x}{y} = c_1.$$

We thus take $\eta = x/y$. Now, $\eta_x = 1/y$ and $\eta_y = -x/y^2$. Choose $\xi = y$ so that $\xi_x = 0$ and $\xi_y = 1$. Then the Jacobian

$$J = \xi_x \eta_y - \xi_y \eta_x = -\frac{1}{y} \neq 0.$$

With this choice of the change of variables $\xi = y$ and $\eta = x/y$, we have $x = \xi\eta$, $y = \xi$, and

$$\begin{aligned}
A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 = y^2 = \xi^2.\\
D &= a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = 0.\\
E &= a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0.\\
F &= f(x(\xi,\eta), y(\xi,\eta)) = 0.\\
G &= g(x(\xi,\eta), y(\xi,\eta)) = 0.
\end{aligned}$$

By (5.3.9), the PDE is transformed to the standard form

$$Au_{\xi\xi} = G - Du_\xi - Eu_\eta - Fu \;\Rightarrow\; \xi^2 u_{\xi\xi} = 0.$$

The domain is $y > 0$, that is, $\xi > 0$. Hence, the reduced PDE is $u_{\xi\xi} = 0$. Integrating the equation with respect to ξ, we have

$$u_\xi = h_1(\eta) \;\Rightarrow\; u(\xi,\eta) = h_1(\eta)\xi + h_2(\eta),$$

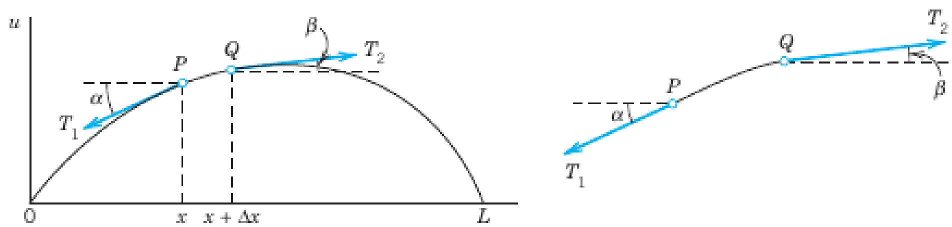
where $h_1(\eta)$ and $h_2(\eta)$ are arbitrary functions of η. Substituting the expressions for ξ and η the general solution is written as $u(x,y) = h_1\big(x/y\big)y + h_2\big(x/y\big)$. □

6

Separation of Variables

6.1 Modeling wave

In most engineering problems, we need to model and solve wave propagation and heat distribution. We start with a very brief introduction to modeling wave in a vibrating string. An elastic string is fixed at two ends, say at $x = 0$ and $x = L$. It is distorted at some instant of time, say $t = 0$ and is released to vibrate. The problem is to determine its deflection $u(x, t)$ at any point $x \in [0, L]$ and time $t > 0$.



For a simple model we assume the following:

1. The string is perfectly elastic; it does not resist to bend.
2. It is homogeneous, i.e., mass of the string per unit length is constant, denote it by ρ.
3. The string has been fastened by stretching it and the tension due to the stretching is so high that the action of gravitation on it is negligible.
4. Every particle of the string moves strictly vertically so that the deflection and the slope st every point on it remains small in absolute value.

We consider the forces acting on a small portion Δx of the string. Due to the above assumptions, the tension on the string is tangential to the initial shape (we distorted it) of the string at each point. Let T_1 and T_2 be the tension at the points P (point x) and Q (point $x + \Delta x$) of that portion. There is no horizontal motion, i.e., the horizontal components of tension is constant. See the figure. It means

$$T_1 \cos \alpha = T_2 \cos \beta = T = \text{constant}.$$

The vertical component at P is downward and at Q is upward; so they are $-T_1 \sin \alpha$ and $T_2 \sin \beta$. By Newton's second law, the resultant of these forces is equal to the mass $\rho\Delta x$ times the acceleration u_{tt} evaluated at some point $x = x^*$ between x and $x + \Delta x$. Hence,

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\, \Delta x\, u_{tt}(x^*, t).$$

Dividing by T and using the previous equation, we get

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T}\, u_{tt}(x^*, t) \Rightarrow \frac{\rho\Delta x}{T}\, u_{tt}(x^*, t) = \tan\beta - \tan\alpha.$$

However, $\tan\alpha$ is the slope of the (distorted) string at the point x. Similarly, $\tan\beta$ is the slope at the point $x + \Delta x$. That is,

$$\tan\alpha = u_x(x, t), \quad \tan\beta = u_x(x + \Delta x, t).$$

Hence,

$$u_{tt}(x^*, t) = \frac{T}{\rho}\, \frac{u_x(x + \Delta x) - u_x(x)}{\Delta x}.$$

Write $T/\rho = c^2$ since it is positive. Take limit of both sides as $\Delta x \to 0$. Then, $x + \Delta x \to x$ and $x^* \to x$ so that we obtain

$$u_{tt} = c^2 u_{xx} \quad \text{where } c > 0. \tag{6.1.1}$$

This is called the **one-dimensional wave equation**. It is a linear homogeneous second order PDE.

6.2 D' Alembert's solution of wave equation

We consider solving the wave equation in (6.1.1):

$$u_{tt} - c^2 u_{xx} = 0.$$

Notice that $u = u(x, t)$, a function of x and t. As a linear second order PDE, comparing it with (5.3.1) with y there as t here, we find that $a = -c^2$, $b = 0$, $c(x, t) = 1$, $d = e = f = g = 0$. The discriminant is $b^2 - 4ac = 4c^2 > 0$. So, it is a hyperbolic PDE. By (5.3.5), the characteristics are

$$\frac{dx}{2a} = \frac{dt}{b + \sqrt{b^2 - 4ac}} \Rightarrow \frac{dx}{-c^2} = \frac{dt}{-2c} \Rightarrow x - ct = c_1.$$
$$\frac{dx}{2a} = \frac{dt}{\left(b - \sqrt{b^2 - 4ac}\right)} \Rightarrow \frac{dx}{-2c^2} = \frac{dt}{2c} \Rightarrow x + ct = c_2.$$

Thus, the transformation is given by

$$\xi = x + ct, \quad \eta = x - ct.$$

We find that $x = (\xi + \eta)/2$, $t = (\xi - \eta)/(2c)$, $\xi_x = 1$, $\xi_t = c$, $\eta_x = 1$ and $\eta_t = -c$. By (5.3.3) with the variable y as t, the new coefficients are given by

$$\begin{aligned} B &= 2a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c(x,1)\xi_t\eta_t = -3c^2 \\ D &= a\xi_{xx} + b\xi_{xt} + c\xi_{tt} + d\xi_x + e\xi_t = 0 \\ E &= a\eta_{xx} + b\eta_{xt} + c\eta_{tt} + d\eta_x + e\eta_t = 0 \\ F &= f(x(\xi,\eta), t(\xi,\eta)) = 0 \\ G &= g(x(\xi,\eta), t(\xi,\eta)) = 0. \end{aligned}$$

By (5.3.6), the PDE is reduced to its standard form

$$u_{\xi\eta} = \frac{1}{B}(G - Du_\xi - Eu_\eta - Fu) = 0.$$

You can also directly compute u_{tt} and u_{xx} using the Chain rule and substitute to get the same equation $u_{\xi\eta} = 0$.

Integrating the above equation with respect to η, we get

$$u_\xi = f_1(\xi)$$

for an arbitrary function f_1 of ξ. Integrating this equation with respect to ξ, we get

$$u(\xi,\eta) = \int f_1(\xi)\, d\xi + f_2(\eta).$$

Since $f_1(\xi)$ ia an arbitrary function, we may write $\int f_1(\xi)\, d\xi$ as another arbitrary function, say $f_3(\xi)$. Hence, the general solution of the above equation is $u(\xi,\eta) = f_3(\xi) + f_2(\eta)$. Going back to the variables x and t, we obtain the general solution of the wave equation (6.1.1) as

$$u(x,t) = \phi(x + ct) + \psi(x - ct) \tag{6.2.1}$$

where ϕ and ψ are arbitrary functions of x and t. This solution is known as the **D' Alembert's solution** of the wave equation.

Suppose the initial distortion of the string is given as a function of x, say, $f(x)$, and the initial velocity, when we leave the string to vibrate is given by a function of x, say, $g(x)$. In our notation, the wave equation (6.1.1) now comes with two initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x). \tag{6.2.2}$$

To get a solution of the initial value problem (6.1.1) and (6.2.2), we start with D' Alembert's solution and try to determine the arbitrary functions ϕ and ψ. From (6.2.1), we get

$$u_t(x,t) = \phi'(x + ct)\frac{\partial(x + ct)}{\partial t} - \psi'(x - ct)\frac{\partial(x - ct)}{\partial t} = c\phi'(x + ct) - c\phi'(x - ct).$$

The initial condition imply that

$$u(x,0) = \phi(x) + \psi(x) = f(x), \quad u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x).$$

Taking the definite integral of the second equation with respect to x varying from any fixed x_0 to any y in the range of values of the variable x, we obtain

$$c\big[\phi(y) - \psi(y)\big] - c\big[\phi(x_0) - \psi(x_0)\big] = \int_{x_0}^{y} g(s)\,ds.$$

So, we have now $\phi(y) + \psi(y) = f(y)$ and $\phi(y) - \psi(y)$ from the above. Then,

$$\phi(y) = \frac{1}{2}f(y) + \frac{1}{2c}\int_{x_0}^{y} g(s)\,ds + \frac{1}{2}\big[\phi(x_0) - \psi(x_0)\big].$$
$$\psi(y) = \frac{1}{2}f(y) - \frac{1}{2c}\int_{x_0}^{y} g(s)\,ds - \frac{1}{2}\big[\phi(x_0) - \psi(x_0)\big].$$

Replacing y by $x + ct$ in the first and $x - ct$ in the second, we obtain

$$\begin{aligned} u(x,t) &= \phi(x+ct) + \psi(x-ct) \\ &= \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds \\ &= \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \frac{1}{2c}\int_{x-ct}^{x_0} g(s)\,ds \\ &= \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \end{aligned}$$

We observe that two initial conditions as given in (6.2.2) determine the solution of the wave equation (6.1.1) uniquely.

In particular, when the initial velocity is 0, the function $g(x)$ is the zero function. We see that the solution is $u(x,t) = [f(x+ct) + f(x-ct)]/2$.

6.3 Series solution of the wave equation

Physically, the string has two fixed end-points, which we have not considered while discussing D' Alembert's solution. The end-points are fixed at $x = 0$ and $x = L$; it means that the deflection is 0 for all time to come. This translates to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for } t \geq 0. \tag{6.3.1}$$

We still have the same initial conditions that initial deflection is $f(x)$ and initial velocity is $g(x)$, but now, it is valid only for $0 \le x \le L$. That is,

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad \text{for } 0 \le x \le L. \tag{6.3.2}$$

Notice that D' Alembert's solution is Uniquely determined when the wave equation is given only the initial conditions. So, that may not satisfy the boundary conditions (6.3.1). Indeed, D' Alembert's solution now involves $f(-ct)$ which does not mean anything physically. This solution is valid for all x and not only for $0 \le x \le L$. Potentially, this solution applies to a string that is elongated from $-\infty$ to ∞. Thus, it leaves open the case that when x is restricted to the interval $[0, L]$, there may or may not exist solutions which will also satisfy the initial conditions.

We will describe the simple and powerful *method of separating the variables* for obtaining such a solution. In this method, we use the heuristic that possibly there is a solution of the wave equation in the form

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x)\, G_n(t)$$

which also satisfies the initial conditions and the boundary conditions. However, we do not directly plug it in the wave equation so as to satisfy the initial and boundary conditions. We rather think of $u_n(x,t) = F_n(x)\, G_n(t)$ to satisfy the wave equation and the boundary conditions only. The series would then be required when we try to satisfy the initial conditions.

So, we start with $u(x,t) = F(x)\, G(t)$ initially. We plug it in the wave equation to obtain two ODEs, one for $F(x)$ and the other for $G(t)$. This constitutes *Step 1* of the method. In *Step 2*, we determine (nonzero) solutions of these ODEs that satisfy the boundary conditions in (6.3.1) thereby obtaining possible $u_n(x,t)$. In *Step 3*, we use a series $\sum a_n u_n(x,t)$ to compose the solutions found in Step 2 so that the series solution satisfies the initial conditions. We execute the plan as in the following.

Step 1: Suppose $u(x,t) = F(x)\, G(t)$. Differentiating, we get

$$u_{tt} = F\, \ddot{G}, \quad u_{xx} = F''G.$$

Here, the dot denotes derivative with respect to t and prime denotes derivative with respect to x. Then the wave equation $u_{tt} = c^2 u_{xx}$ in (6.1.1) takes the form

$$F\ddot{G} = c^2 F'' G \;\Rightarrow\; \frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

The left side is independent of x and the right side is independent of t. So, both are independent of x and t, that is, it is a constant, say, k. Of course, the constant k is yet unknown. We then have

$$F'' - kF = 0, \quad \ddot{G} - c^2 kG = 0. \tag{6.3.3}$$

Step 2: We are interested in nonzero solutions. The boundary conditions in (6.3.1) take the form

$$u(0,t) = F(0)G(t) = 0 \;\Rightarrow\; F(0) = 0, \quad u(L,t) = F(L)G(t) = 0 \;\Rightarrow\; F(L) = 0.$$

These are conditions on $F(x)$ only. Again, $F(x) = 0$ satisfies these conditions but we require nonzero solutions. The ODE $F'' - kF = 0$ for F in (6.3.3) involves an unknown constant k.

If $k = 0$, then $F = ax + b$ for constants a and b. Now, $F(0) = 0 \Rightarrow b = 0$. So, $F(x) = ax$. And, $F(L) = 0 \Rightarrow aL = 0 \Rightarrow a = 0$. So, $F(x)$ is the zero function, which we do not require.

If $k > 0$, then $F(x) = ae^{\sqrt{k}x} + be^{-\sqrt{k}x}$. The conditions $F(0) = 0 = F(L)$ imply that

$$a + b = 0, \quad ae^{\sqrt{k}L} + be^{-\sqrt{k}L} = 0 \;\Rightarrow\; a = 0 = b.$$

So, $F(x)$ is the zero function, which we do not require.

So, $k < 0$; and we write $k = -p^2$ for $p > 0$. Notice that p is yet to be determined. Now, the equation of $F(x)$ is $F'' + p^2F = 0$. Its general solution is

$$F(x) = a\cos(px) + b\sin(px).$$

Now, $F(0) = 0 \Rightarrow a = 0$. So, $F(x) = b\sin(px)$. Then, $F(L) = 0 \Rightarrow b\sin(pL) = 0$. By taking $b = 0$, we get only trivial solution. So, we take the other alternative $\sin(pL) = 0$. It gives

$$pL = n\pi \;\Rightarrow\; p = \frac{n\pi}{L} \quad \text{for } n = 1, 2, 3, \ldots.$$

Corresponding to each value of p, we obtain a solution. These are:

$$F_n(x) = b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 1, 2, 3, \ldots. \tag{6.3.4}$$

Now that the possible values for p has been obtained, we use these values to solve the equation for $G(t)$ in (6.3.3). Notice that $k = -p^2 = -\big(n\pi/L\big)^2 \Rightarrow c^2k = -(cn\pi/L)^2$. The equation $\ddot{G} - c^2kG = 0$ for $G(t)$ now reads as

$$\ddot{G} + \lambda_n^2 G = 0, \quad \lambda_n = \frac{cn\pi}{L} \quad \text{for } n = 1, 2, 3, \ldots.$$

Its general solution is

$$G_n(t) = c_n\cos(\lambda_n t) + d_n\sin(\lambda_n t) \quad \text{for } n = 1, 2, 3, \ldots.$$

Then $u_n(x,t) = F_nG_n = b_n\sin(n\pi x/L)\big[c_n\cos(\lambda_n t) + d_n\sin(\lambda_n t)\big]$. However, we do not expect any of these u_ns to satisfy the initial conditions, in general. So, we will be taking a series $u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t)$ and try to satisfy the initial conditions.

In that case, notice that we do not require so many constants like $a_n,\ b_n,\ c_n$ and d_n. Only c_n and d_n will suffice. It is enough to consider $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, where

$$u_n(x,t) = \big[c_n \cos(\lambda_n t) + d_n \sin(\lambda_n t)\big] \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } \ n = 1, 2, 3, \ldots. \tag{6.3.5}$$

The numbers $\lambda_n = cn\pi/L$ are called the **eigenvalues** and the corresponding functions $u_n(x,t)$ above are called the **eigenfunctions** of the vibrating string. The set $\{\lambda_1, \lambda_2, \ldots\}$ of eigenvalues is called the **spectrum**.

Observe that each u_n represents a harmonic motion with frequency as $\lambda_n/(2\pi)$ cycles per unit time. This motion is called the **normal mode** of the string. The first mode, corresponding to $n = 1$, is called the **fundamental mode** and the others are called the **overtones**. Since $\sin(n\pi x/L) = 0$ for $x = L/n,\ 2L/n,\ \ldots,\ (n-1)L/n$, the nth normal mode has $n-1$ **nodes**. Like the end-points, the string does not move at the nodes. This is expected due to the wave-like movement of the string, from which the name for the equation in (6.1.1) comes.

Step 3: We have seen that the eigenfunctions in (6.3.5) satisfy the wave equation and the boundary conditions. We do not expect a single $u_n(x,t)$ to satisfy the initial conditions. As discussed earlier, we set

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \big[c_n \cos(\lambda_n t) + d_n \sin(\lambda_n t)\big] \sin\left(\frac{n\pi x}{L}\right). \tag{6.3.6}$$

With this $u(x,t)$, the first initial condition in (6.3.2) gives

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \text{for } \ 0 \le x \le L.$$

It says that $f(x)$ has been expanded in its Fourier sine series. Thus,

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{for } \ n = 1, 2, 3, \ldots. \tag{6.3.7}$$

For the second initial condition, we first differentiate $u(x,t)$ in (6.3.6), evaluate it at $t = 0$ to get

$$\begin{aligned} u_t(x,0) &= \left[\sum_{n=1}^{\infty} \big[-c_n\lambda_n \sin(\lambda_n t) + d_n\lambda_n \cos(\lambda_n t)\big] \sin\left(\frac{n\pi x}{L}\right)\right]_{t=0} \\ &= \sum_{n=1}^{\infty} d_n\lambda_n \sin\left(\frac{n\pi x}{L}\right) = g(x). \end{aligned}$$

Hence, $g(x)$ is expanded in its Fourier sine series. Thus,

$$d_n\lambda_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Putting back the value of $\lambda_n = cn\pi/L$ we get

$$d_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n = 1, 2, 3, \ldots. \tag{6.3.8}$$

To summarize, the solution of the wave equation (6.1.1) with boundary conditions in (6.3.1) and initial conditions in (6.3.2) is given by (6.3.6) with $\lambda_n = cn\pi/L$, where c_n and d_n are as in (6.3.7-6.3.8).

It can be shown that the series in (6.3.6) is convergent for $0 \le x \le L$ and all $t \ge 0$. Further, the solution $u(x,t)$ in the above series form is a solution of the wave equation with the initial and boundary conditions if $f(x)$ is twice differentiable on $0 < x < L$, and it has one-sided second derivatives at the end-points $x = 0$ and $x = L$, which are equal to 0.

(6.1) ***Example***

Find the solution of the wave equation $u_{tt} = c^2 u_{xx}$ satisfying $u(0,t) = 0 = u(L,t)$, $u_t(x,0) = 0$ and $u(x,0) = 2kx/L$ for $0 \le x \le L/2$, $u(x,0) = 2k(L-x)/L$ for $L/2 < x \le L$.

According to (6.3.6), the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[c_n\cos(\lambda_n t) + d_n\sin(\lambda_n t)\right]\sin\left(\frac{n\pi x}{L}\right)$$

where $\lambda_n = cn\pi/L$ and by (6.3.7-6.3.8),

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\int_0^{L/2}\frac{2k}{L}x\,\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_0^L\frac{2k}{L}(L-x)sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \\
d_n &= \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0.
\end{aligned}$$

Since $\sin(n\pi/2)$ is 0 for even n, 1 for $n = 4m+1$, and -1 for $n = 4m+3$, we find that

$$u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L} + \cdots\right]. \qquad \Box$$

(6.2) ***Example***

Suppose the vibration of a stretched string of length 1 unit is clamped at each end and starts from rest with the initial shape $u(x,0) = kx(1-x)$. Here, $k > 0$ is such that the maximum transverse displacement is small. Find the vibration $u(x,t)$.

The function $u(x,t)$ satisfies $u_{tt} = c^2 u_{xx}$ for some constant c depending on the material of the string, the boundary conditions are $u(0,t) = 0 = u(1,t)$, and the initial conditions are $u(x,0) = kx(1-x)$ and $u_t(x,0) = 0$. Here, $L = 1$, $f(x) = kx(1-x)$ and $g(x) = 0$. By (6.3.6),

$$u(x,t) = \sum_{n=1}^{\infty} \left[c_n \cos(\lambda_n t) + d_n \sin(\lambda_n t)\right] \sin\left(\frac{n\pi x}{L}\right)$$

where $\lambda_n = cn\pi/L = cn\pi$, and

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad d_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Since $g(x) = 0$, we have $d_n = 0$. And,

$$c_n = 2\int_0^1 kx(1-x)\sin(n\pi x)\,dx = 2\,\frac{2k}{n^3\pi^3}\big(1-\cos(n\pi)\big) = \begin{cases} 0 & \text{if } n \text{ even} \\ 8k/(n^3\pi^3) & \text{if } n \text{ odd.} \end{cases}$$

Since only odd terms remain, we write $n = 2m+1$ for $m = 0, 1, 2, 3, \ldots$. Then

$$u(x,t) = 8k\pi^{-3}\sum_{m=0}^{\infty}(2m+1)^{-3}\sin\big((2m+1)\pi x\big)\cos\big((2m+1)c\pi t\big). \qquad \square$$

In this section we have discussed how to use the method of separation of variables for solving the wave equation. The same method can be used to solve first order PDEs. You can work out the details by solving the exercises.

6.4 One-dimensional heat flow

Consider the temperature in a long thin metal wire of constant cross sectional area. Assume that it is perfectly insulated so that heat flows in one direction only. Call the direction of flow as the x-axis. Write the temperature as $u(x,t)$, where t is time. Write K for the thermal conductivity, c for the thermal diffusivity, σ for specific heat, and ρ for the density of the wire. Then $c^2 = K/\rho\sigma$ and the heat flow is governed by the heat equation

$$u_t = c^2 u_{xx}. \tag{6.4.1}$$

Suppose that the wire is of length L and its ends prescribed by $x = 0$ and $x = L$ are kept at zero temperature. This gives the boundary conditions

$$u(0,t) = 0, \quad u(L,0) = 0 \quad \text{for } t \geq 0. \tag{6.4.2}$$

In particular, $u(0,0) = u(L,0) = 0$. Further, assume that the initial temperature on the wire at time $t = 0$ is given as a function of x; say, $f(x)$. Then

$$u(x,0) = f(x) \quad \text{for } 0 \le x \le L. \tag{6.4.3}$$

Notice that due to the boundary conditions, the function $f(x)$ cannot be arbitrary, but it must satisfy $f(0) = f(L) = 0$.

We will use the method of separation of variables to get a series solution of (6.4.1) satisfying (6.4.2) and (6.4.3).

Step 1: Let $u(x,t) = F(x)G(t)$. Substitute in (6.4.1) to get

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F}.$$

The left side is independent of x and the right side is independent of t. So, each is equal to a constant. As in the case of wave equation, if this constant is 0 or positive, we would get only the trivial solution $u(x,t) = 0$. So, suppose that each ratio in the above equation is negative, that is, it is equal to $-p^2$ for $p > 0$. Then, we get two ODEs

$$F'' + p^2 F = 0, \quad \dot{G} + c^2 p^2 G = 0.$$

Step 2: Solving the equation for F we get

$$F(x) = a\cos(px) + b\sin(px).$$

From the boundary condition (6.4.2), we have

$$u(0,t) = F(0)G(t) = 0, \quad u(L,t) = F(L)G(t) = 0.$$

We do not take $G(t) = 0$ since it leads to the trivial solution $u(x,t) = 0$. So, $F(0) = 0$ and $F(L) = 0$. Now, $F(0) = 0 \Rightarrow a = 0 \Rightarrow F(x) = b\sin(px)$. Then, $F(L) = 0 \Rightarrow b\sin(pL) = 0$. Again, $b = 0 \Rightarrow F(x) = 0$ which leads to the trivial solution. So, $\sin(pL) = 0$. Since $p > 0$, it gives

$$p = \frac{n\pi}{L} \quad \text{for } n = 1, 2, 3, \ldots.$$

The corresponding solutions for $F(x)$ are given by

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 1, 2, 3, \ldots.$$

For $p = n\pi/L$, the equation for $G(t)$ becomes

$$\dot{G} + \lambda_n^2 G = 0 \quad \text{where } \lambda_n = \frac{cn\pi}{L}.$$

Corresponding to each n, the general solution is $b_n \exp(-\lambda_n^2 t)$. Since constants will be accommodated later, we set $b_n = 1$ to obtain

$$G_n(t) = \exp(-\lambda_n^2 t)$$

as possible non-trivial solution for $G(t)$ corresponding to the value $n\pi/L$ of p. Then,

$$u_n(x,t) = F_n(x)G_n(t) = \sin\left(\frac{n\pi x}{L}\right)\exp(-\lambda_n^2 t) \quad \text{for } n = 1, 2, 3, \ldots$$

is a possible solution corresponding to the value $n\pi/L$ of p. This function $u_n(x,t)$ is called an **eigenfunction** with respect to the **eigenvalue** $\lambda_n = cn\pi/L$, as earlier.

Step 3: None of the u_ns may satisfy the initial condition. So, we propose to have our solution as a series of eigenfunctions. So, let

$$u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\exp(-\lambda_n^2 t), \quad \text{where } \lambda_n = \frac{cn\pi}{L}. \tag{6.4.4}$$

The initial condition (6.4.3) now gives

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

So, a_ns are the Fourier coefficients of the Fourier sine series for $f(x)$. Thus,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n = 1, 2, 3, \ldots. \tag{6.4.5}$$

It can be verified that $u(x,t)$ of (6.4.4) in series form is a solution of the heat equation (6.4.1) satisfying the conditions in (6.4.2)-(6.4.3) if $f(x)$ is piecewise continuous on $0 \le x \le L$, and has one-sided derivatives at all points of discontinuity.

(6.3) *Example*

Find the temperature $u(x,t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100\sin(\pi x/80)$°C. Assume the following physical data for the bar: density is $8.92\ \text{g/cm}^3$, specific heat is $0.992\ \text{cal/(g°C)}$, thermal conductivity is $0.95\ \text{cal/(cm sec°C)}$. How long it will take for the maximum temperature in the bar to drop to 50°C?

Here, $L = 80$, $f(x) = 100\sin(\pi x/80)$, $c^2 = K/(\rho\sigma) = 0.95/(0.092 \times 8.92) = 1.158\ \text{cm}^2/\text{sec°C}$. Computing the coefficients from (6.4.5), we find that

$$a_1 = \frac{2}{80}\int_0^{80} 100\sin^2\left(\frac{n\pi x}{80}\right)dx = 100, \quad a_n = 0 \text{ for } n \ge 1.$$

Thus, we need only λ_1^2 which equals $1.158 \times 9.870/80^2 = 0.001785[\text{sec}^{-1}]$. Hence, the solution is given by

$$u(x,t) = 100 \sin\left(\frac{\pi x}{80}\right) e^{-0.001785\, t}.$$

The maximum temperature in the bar is achieved when $\sin(\pi x/80) = 1$. It drops to 50 implies $100e^{-0.001785\, t} = 50 \;\Rightarrow\; t = \log(0.5)/(-0.001785) = 388\,[\text{sec}]$. □

(6.4) *Example*

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0 assuming that the initial temperature is $f(x) = x$ for $0 \le x \le L/2$ and $f(x) = L - x$ for $L/2 < x \le L$.

We compute the coefficients from (6.4.5) as follows:

$$a_n = \frac{2}{L}\left(\int_0^{L/2} s \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\,dx\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4L}{n^2\pi^2} & \text{if } n = 4m+1 \\ -\frac{4L}{n^2\pi^2} & \text{if } n = 4m+3. \end{cases}$$

Hence, the solution is

$$u(x,t) = \frac{4L}{\pi^2}\Big[\sin\frac{\pi x}{L}\exp\big[-\big(c\pi/L\big)^2 t\big] - \frac{1}{9}\sin\frac{3\pi x}{L}\exp\big[-(3c\pi/L)^2 t\big] + \cdots\Big].$$

Notice that this is a decreasing function of t. Physically this happens because the ends are kept in zero temperature. □

(6.5) *Example*

Find the solution $u(x,t)$ of the heat equation $u_t = c^2 u_{xx}$ satisfying the conditions $u_x(0,t) = u_x(L,t) = 0$ for all t, and $u(x,0) = f(x)$ for $0 \le x \le L$.

We set $u(x,t) = F(x)G(t)$. As earlier we reach at

$$F(x) = a\cos(px) + B\sin(px), \quad \dot{G} + c^2p^2G = 0.$$

Then

$$F'(x) = -ap\sin(px) + bp\cos(px) \;\Rightarrow\; F'(0) = bp, \quad F'(L) = -ap\sin(pL).$$

The boundary conditions give

$$u_x(0,t) = F'(0)G(t) = bp = 0, \quad u_x(L,t) = F'(L)G(t) = -ap\sin(pL) = 0.$$

Since we need a non-zero solution, we assume that $G(t) \neq 0$ and at least one of a or b is equal to 0. To obtain a series solution, we take $b = 0$ and $a \neq 0$. Further, constants will get accommodated in a series. So, we take $a = 1$. Then we have $p = 0$ or $\sin(pL) = 0$. It implies the possibilities for p as

$$p = p_n = \frac{n\pi}{L} \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Neglecting the coefficients, we get $F_n(x) = \cos(n\pi x/L)$. This does not disturb G_ns. That is, as earlier, $G_n(t) = \exp(-\lambda_n^2 t)$, where $\lambda_n = cn\pi/L$. Hence, the eigenfunctions are

$$u_n(x,t) = F_n(x)G_n(t) = \cos\frac{n\pi x}{L}\exp(\lambda_n^2 t) \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Notice that comparing these eigenfunctions with those in (6.4.4), we have an extra eigenvalue, namely $\lambda_0 = 0$ and corresponding to it the extra eigenfunction $u_0 =$constant. Notice that this is also a solution of the problem when $f(x)$ is a constant function.

As earlier, we have the solution as

$$u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t}, \quad \text{where } \lambda_n = \frac{cn\pi}{L}.$$

The coefficients are obtained from the initial condition $u(x,0) = f(x)$. However, $u(x,0) = \sum_{n=0}^{\infty} a_n \cos(n\pi x/L)$. Thus, a_ns are the Fourier coefficients of the Fourier cosine series of $f(x)$. That is,

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 1, 2, 3, \ldots. \qquad \square$$

When the two ends of the wire are kept in constant temperatures, we get the boundary conditions as $u(0,t) = A$ and $u(L,t) = B$. We try a solution in the form $u(x,t) = A + \dfrac{B-A}{L}x + v(x,t)$. Then, $v(x,t)$ will satisfy the heat equation with homogeneous boundary conditions. We use the method of separation of variables for determining $v(x,t)$. You may need this trick to solve some problems from the exercises.

6.5 Laplace equation

Instead of a metal rod, consider heat distribution on a metal plate. We may approach the problem of modeling in a way similar to the derivation of one-dimensional wave and heat equations. We would arrive at the two-dimensional heat equation

$$u_t = c^2(u_{xx} + u_{yy}).$$

When the steady state is achieved, we find that $u_t = 0$ and it yields the **Laplace equation**

$$u_{xx} + u_{yy} = 0.$$

When the metal plate is rectangular, the Cartesian coordinates system is suitable. Similarly, if the plate is circular, it may be easier to use the polar coordinates. We need to express the **Laplacian** $u_{xx} + u_{yy}$ in polar coordinates.

The relation between Cartesian and the polar coordinates is expressed by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Suppose $u = u(x, y, t)$ is a function of x, y and t. We are interested in computing $u_{xx} + u_{yy}$ in r, θ form. By the chain rule,

$$u_x = u_r r_x + u_\theta \theta_x.$$

Differentiating again, we obtain

$$\begin{aligned}
u_{xx} &= (u_r\, r_x)_x + (u_\theta\, \theta_x)_x \\
&= (u_r)_x\, r_x + u_r\, r_{xx} + (u_\theta)_x\, \theta_x + u_\theta\, \theta_{xx} \\
&= (u_{rr}\, r_x + u_{r\theta}\, \theta_x)\, r_x + u_r\, r_{xx} + (u_{\theta r}\, r_x + u_{\theta\theta}\, \theta_x)\, \theta_x + u_\theta\, \theta_{xx}.
\end{aligned}$$

Using the expressions for r and θ in terms of x, y, we obtain

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \quad \theta_x = \frac{1}{1 + (y/x)^2} \times \frac{-y}{x^2} = -\frac{y}{r^2}.$$

$$r_{xx} = \frac{r - x\, r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}, \quad \theta_{xx} = -y \times \frac{-2}{r^3}\, r_x = \frac{2xy}{r^4}.$$

Assuming that u is two times continuously differentiable with respect to r and θ, we get $u_{r\theta} = u_{\theta r}$. Substituting the expressions above into that of u_{xx} leads to

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta.$$

Similarly,

$$u_{yy} = \frac{y^2}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r - \frac{2xy}{r^4} u_\theta.$$

Adding the two above and suing the fact that $x^2 + y^2 = r^2$, we obtain the expression for the Laplacian in polar coordinates as follows:

$$u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}.$$

Using these and the method of separation of variables, we solve some problems on heat distribution on rectangular and circular plates.

(6.6) ***Example***

Find the steady state temperature distribution $u(x, y)$ on the rectangular region $0 \le x \le \pi,\ 0 \le y \le 2$, given that on the side $y = 0$, $0 \le x \le \pi$, $u(x, 0) = x \sin x$, and the temperature on the other three sides are maintained at $u = 0$.

The steady state temperature $u(x, y)$ satisfies the Laplacian

$$u_{xx} + u_{yy} = 0.$$

We try $u(x, y) = F(x)G(y)$. Substituting it in the equation and simplifying, we get

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2}.$$

The left side is independent of y and the right side is independent of x; so each is a constant, say, c. It then follows that

$$\frac{d^2F}{dx^2} = cF, \quad \frac{d^2G}{dy^2} + cG = 0.$$

The boundary conditions $u(0, y) = 0 = u(\pi, y)$ imply that $F(0) = F(\pi) = 0$. When $c = 0$, $F = a + bx$. These condition on F imply that $F(x) = 0$, leading to the trivial solution $u(x, y) = 0$ which is not the case. If $c > 0$, then $F(x) = ae^{\sqrt{c}x} + be^{-\sqrt{c}x}$. Again, the conditions $F(0) = F(\pi) = 0$ lead to $F(x) = 0$. So, $c < 0$; then let $c = -\lambda^2$ for $\lambda > 0$. We now have the equations as

$$\frac{d^2F}{dx^2} + \lambda^2 F = 0, \quad \frac{d^2G}{dy^2} - \lambda^2 G = 0.$$

Then $F(x) = a\cos(\lambda x) + b\sin(\lambda x)$. $F(0) = 0 \Rightarrow a = 0$. So, $F(x) = b\sin(\lambda x)$. $X(\pi) = 0 \Rightarrow \sin(\lambda\pi) = 0$ as $b = 0$ leads to the trivial solution. Hence, the eigenvalues are

$$\lambda_n = n \quad \text{for } n = 1, 2, 3, \ldots.$$

The corresponding eigenfunctions are (we take $b_n = 1$.)

$$F_n(x) = \sin(nx) \quad \text{for } n = 1, 2, 3, \ldots.$$

Now the equation for G reads as

$$\frac{d^2G}{dy^2} - n^2 G = 0.$$

To show dependence of G on the parameter n, we write its solution as $G_n(y)$. Then, $G_n(y) = c\cosh(ny) + d\sinh(ny)$. The boundary condition $u(x,2) = 0$ gives $F(x)G(2) = 0 \Rightarrow G(2) = 0$. Or,

$$c\cosh(2n) + d\sinh(2n) = 0 \Rightarrow d = -c\,\coth(2n).$$

Using this in the expression for $G_n(y)$ and setting $c = 1$, we obtain

$$G_n(y) = \cosh(ny) - \coth(2n)\sinh(ny) = \operatorname{cosech}(2n)\sinh(ny - 2n).$$

Since constants will be determined later from a series, we choose $c_n = \sinh(2n)$ in obtaining u_n. Then,

$$u_n(x,y) = \sin(nx)\ \sinh(ny - 2n) \quad \text{for } n = 1, 2, 3, \ldots.$$

To satisfy the other conditions, we set

$$u(x,y) = \sum_{n=1}^{\infty} a_n u_n = \sum_{n=1}^{\infty} a_n \sin(nx)\ \sinh(ny - 2n).$$

Now, $u(x,0) = x\sin x$ implies

$$x\sin x = \sum_{n=1}^{\infty} a_n \sin(nx)\sinh(-2n).$$

Multiply $\sin(mx)$ and integrate over $0 \le x \le \pi$ to obtain

$$\int_0^\pi x\sin x\sin(mx)\,dx = -a_m\sinh(2n)\int_0^\pi \sin(nx)\sin(mx)\,dx.$$

Using the orthogonality property of $\{\sin(nx)\}$ as in evaluating the Fourier coefficients, we get

$$a_1 = -\frac{\pi}{2\sinh 2}, \quad a_n = \frac{4n\big(1 + (-1)^n\big)}{(n^2-1)^2\pi\sinh(2n)} \quad \text{for } n = 2, 3, 4, \ldots.$$

Substituting these values of a_n in the series, we obtain

$$u(x,y) = -\frac{\pi\sin x\sinh(y-2)}{2\sinh 2} + \sum_{n=2}^{\infty}\frac{4n\big(1 + (-1)^n\big)}{(n^2-1)^2\pi\sinh(2n)}\sin(nx)\ \sinh(ny - 2n). \quad \square$$

(6.7) ***Example***

Find the temperature distribution $u(r,\theta,t)$ in a thin (negligible thickness) semicircular metal plate $0 \le r \le 1$, $0 \le \theta \le \pi$ given that its plane faces are insulated to prevent heat loss through them, the straight edge of the plate formed by the diameter $0 \le r \le 1$, $\theta = 0$ and $\theta = \pi$ is insulated, the semicircular boundary is maintained at zero temperature, and the initial temperature distribution is $u(r,\theta,0) = (1-r)\cos\theta$.

Using the Laplacian in polar coordinates, the heat equation on the plate is

$$u_t = c^2\big(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}\big).$$

The bounding diameter is insulated and semicircular boundary is kept at zero temperature. This means that

$$u(r,0,t) = 0, \quad u_\theta(r,\pi,t) = 0, \quad u(1,\theta,t) = 0.$$

To use the separation of variables, we take $u(r,\theta,t) = E(r)F(\theta)G(t)$. Substituting in the heat equation above we get

$$\frac{\dot{G}}{G} = c^2\Big(\frac{E''}{E} + \frac{1}{r}\frac{E'}{E} + \frac{1}{r^2}\frac{F^{(2)}}{F}\Big).$$

Here $E' = dE/dr$, $E'' = d^2E/dr^2$ and $F^{(n)}$ means $d^nF/d\theta^n$. The left side is independent of r and θ, and the right side is independent of t. Hence, all of them are independent of r, θ, t so that they are equal to a constant. Further, the temperature decreases with time; so the constant must be negative. We may also consider three cases of this constant, and verify that non-negative values of this constant lead to the trivial solution.

Now that each side is equal to some negative constant, say, $-\lambda^2$ with $\lambda > 0$, we obtain two equations:

$$\dot{G} + c^2\lambda^2 G = 0, \quad r^2\frac{E''}{E} + r\frac{E'}{E} + \lambda^2 r^2 = -\frac{F^{(2)}}{F}.$$

Again, the second equation has a left side independent of θ and the right side independent of r. Hence, each is a constant. We may verify that for negative values of this constant, we get only the trivial solution. So, we assume that this constant is non-negative. We write it as $q \ge 0$. Then, the second equation gives two equations:

$$F^{(2)} + qF = 0, \quad r^2E'' + rE' + (\lambda^2 r^2 - q)E = 0.$$

The general solution for $F(\theta)$ is

$$F(\theta) = A\cos(\sqrt{q}\,\theta) + B\sin(\sqrt{q}\,\theta).$$

The boundary condition $u_\theta(r,0,t) = 0$ implies $F^{(1)}(0) = 0$ and the condition $u_\theta(r,\pi,t) = 0$ implies $F^{(1)}(\pi) = 0$. The first condition yields $B = 0$ and the second leads to $\sin(\sqrt{q}\,\pi) = 0$. Hence, $\sqrt{q} = m$ for $m = 0, 1, 2, 3, \ldots$. Setting the arbitrary constant A to 1, we get

$$F(\theta) = \cos(n\theta) \quad \text{for } m = 0, 1, 2, 3, \ldots$$

The equation for $E(r)$ now becomes

$$r^2E'' + rE' + (\lambda^2 r - m^2)E = 0.$$

We recognize this as the Bessel's equation, whose general solution is

$$E_m(r) = aJ_m(\lambda r) + bY_m(\lambda r).$$

Recall that $Y_m(\lambda r) \to \infty$ as $r \to 0$. However, the temperature on the plate remains finite. Hence, $b = 0$. Further, we will be getting a series solution finally; so, we set $a = 1$ and continue with the solutions

$$E_m(r) = J_m(\lambda r) \quad \text{for } m = 0, 1, 2, 3, \ldots$$

For the boundary condition $u(1,\theta,t) = 0$, we must have $E(1) = 0$. It means, $J_m(\lambda) = 0$. Hence, the eigenvalues λs are the positive zeros of J_m, the Bessel function. So, we take $\lambda_n = z_{m,n}$, the nth positive zero of J_m.

Using these λs in the equation for $G(t)$, which was $\dot{G} + c^2\lambda^2 G = 0$, we have

$$G_{m,n}(t) = b_{m,n}\exp(-z_{m,n}^2 c^2 t).$$

Combining the results for $E(r), F(\theta)$ and $G(t)$, we obtain

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} b_{m,n}J_m(z_{m,n}r)\cos(m\theta)\exp(-z_{m,n}^2 c^2 t).$$

When $t = 0$, the initial condition $u(r,\theta,0) = (1-r)\cos\theta$ gives

$$(1-r)\cos\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} b_{m,n}J_m(z_{m,n}r)\cos(m\theta).$$

This is the Fourier-Bessel series of the left side function. We multiply $\cos\theta$ and integrate over $0 \le \theta \le \pi$. Every term on the right hand side vanishes except those corresponding to $m = 1$. Thus,

$$(1-r)\cos\theta = \sum_{n=1}^{\infty} b_{1,n}J_1(z_{1,n}r)\cos\theta \;\Rightarrow\; 1 - r = \sum_{n=1}^{\infty} b_{1,n}J_1(z_{1,n}r).$$

Multiply the last expression by $rJ_1(z_{1,s}r)$ and integrate over $0 \leq r \leq 1$. Using the orthogonality of the Bessel functions, we get

$$\int_0^1 r(1-r)J_1(z_{1,s}r)\,dr = b_{1,s}\,\frac{1}{2}J_2^2(z_{1,s}).$$

This gives $b_{1,s}$. We write in terms of n:

$$b_{1,n} = 2\big[J_2(z_{1,n})\big]^{-2}\int_0^1 (r-r^2)J_1(z_{1,n}r)\,dr \quad \text{for } n = 1, 2, 3, \ldots.$$

Then the required solution is

$$u(r,\theta,t) = \sum_{n=1}^{\infty} b_{1,n}J_1(z_{1,n}r)\cos\theta\exp(-z_{1,n}^2c^2t).$$

To obtain numerical values for specific tuples (r,θ,t) one must use the tables for the zeros of Bessel functions. $\square$

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