# MA1101, Functions of Several Variables
# Lecture 1

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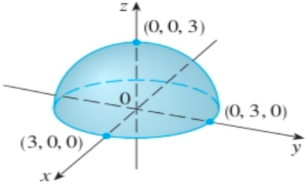
# Graph of functions on subsets of $\mathbb{R}^2$

- Let $D$ be a region in the plane, and let $f : D \to \mathbb{R}$ be a function.
- The graph of $f$ is given by

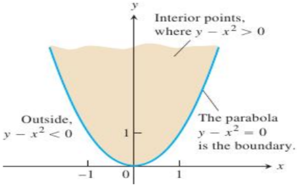
$$\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } z = f(x,y)\}.$$

It is the surface $z = f(x,y)$ in $\mathbb{R}^3$.
For example:

- $f_1(x,y) = \sqrt{9 - x^2 - y^2}$
  - natural domain : $\overline{B}_3(\mathbf{0},0) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$
  - range : $[0,3]$.



- $f_2(x, y) = \sqrt{y - x^2}$
  - ★ Domain : $\{(x, y) \in \mathbb{R}^2 : x^2 \le y\}$
  - ★ range : $\mathbb{R}^+ \cup \{0\}$.




- In general, graphs of functions of two variables are difficult to draw (by hand); try, e.g., drawing the graph of $f_2(x, y) = \frac{x+y}{x-y}$, $x \neq y$.
- But one can get an idea by looking at
  - Counter curves and
  - Level curves.

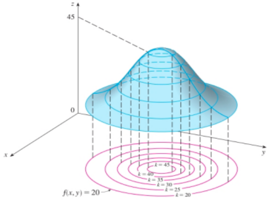
Let $f(x, y)$ be a function of two variables. That is, $f : D \to \mathbb{R}$, where $D$ is a region in $\mathbb{R}^2$.

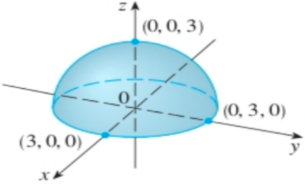
# Contour curves and level curves

- Let $c \in \mathbb{R}$. Then the set $\{(x, y, c) \in \mathbb{R}^3 : (x, y) \in D \text{ and } f(x, y) = c\}$ is called a contour curve of f. It is the intersection of the graph of $f$ by the horizontal plane $z = c$ in $\mathbb{R}^3$.
- The union of all contour curves is the surface $Z = f(x, y)$; it is also the graph of $f$.
- Also, $\{(x, y) \in D : f(x, y) = c\} \subset \mathbb{R}^2$ is called a level curve of $f$. It is the projection of the corresponding contour line on the plane $z = 0$ in $\mathbb{R}^3$.
- Contour curves and level curves help us visualize the behaviour of a function $f$ on its domain in $\mathbb{R}^2$. Such a visualization is not possible for functions defined on subsets of $\mathbb{R}^3$.

For example:

The picture below on the left gives a general illustration of contour curves and level curves.



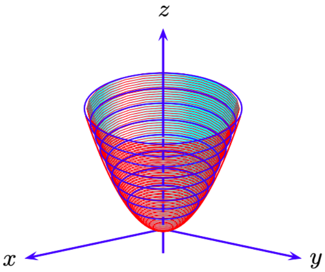




Examples. (i) Let $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$, and let

$$f(x,y) := \sqrt{9 - x^2 - y^2} \text{ for } (x,y) \in D.$$

The graph of $f$ is the upper hemisphere of radius $3$ centered at the origin. This is depicted by the picture on the right above. It is easy to visualize in this case that what are the counter curves and the level curves !

(ii) Let $D := \mathbb{R}^2$, and let $f(x,y) := x^2 + y^2$ for $(x,y) \in D$. The graph of $f$ is a paraboloid.



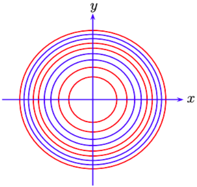




Figure: Contour curves and the corresponding level curves

(iii) Let $D := \{(x,y) \in \mathbb{R}^2 : x \neq y\}$, and let

$$f(x,y) := (x+y)/(x-y) \text{ for } (x,y) \in D.$$

Then the contour curve corresponding to $c = 2$ is $\{(x, x/3, 2) : x \in \mathbb{R} \setminus \{0\}\}$. Level curves are lines through the origin in $\mathbb{R}^2$.

Similarly, for a function $f(x, y, z)$ of three variables, the **level surfaces** are the set of points $(x, y, z)$ such that $f(x, y, z) = c$ for values $c$ in the range of $f$.