MA1101, Functions of Several Variables
Lecture 2

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Limits of Functions of Two Variables

Recall limits of functions of a real variable:

- Let $D \subset \mathbb{R}$ and let $c \in \mathbb{R}$. Let $f : D \to \mathbb{R}$ and let $\ell \in \mathbb{R}$.
- Suppose the values of the function f are as close to the real number ℓ as we like for all x close enough to c (on either side of c), except possibly when $x = c$, then we say the the limit of $f(x)$, as x approaches c, is equal to $\ell \in \mathbb{R}$.

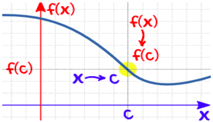


Figure: as x approaches c (from left) then $f(x)$ approaches $f(c)$

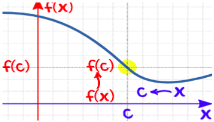
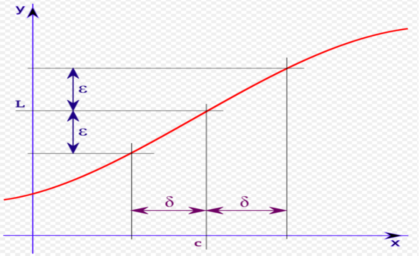


Figure: as x approaches c (from right) then $f(x)$ approaches $f(c)$

If we get different values from left and right (a "jump"), then the limit does not exist!

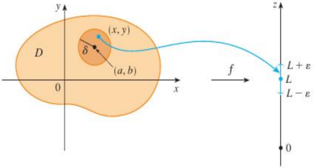
Recall the equivalent condition for the existence of a limit

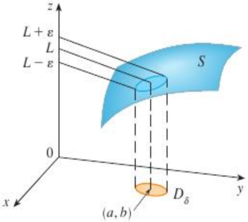
- Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that there is $r > 0$ with $(c-r,c) \cup (c,c+r) \subset D$.
- Let $\ell \in \mathbb{R}$.
- Then $\lim_{x \to c} f(x) = \ell$ if and only if the following $\epsilon - \delta$ condition holds.
  For every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$ with
  $$0 < |x-c| < \delta \Rightarrow |f(x) - \ell| < \epsilon.$$
- If for no real number ℓ, the above happens, then limit of f at c does not exist.



Limits of Functions of Two Variables

- Let D is a region in the plane and $f : D \to \mathbb{R}$ be a function. Suppose $(a,b) \in \mathbb{R}^2$ is such that $B((a,b),r) \setminus \{(a,b)\} \subset D$ for some $r > 0$ [or suppose, more generally, that (a,b) is a limit point of D, which means that every neighborhood of (a,b) contains a point of D other than (a,b)].
- The limit of $f(x,y)$ as (x,y) approaches (a,b) is L if and only if the following $\epsilon - \delta$ condition holds.
  For every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $(x,y) \in D$ with
  $$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon.$$
- In this case, we write $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.
- We also say that L is the limit of f at (a,b).
- If for no real number L, the above happens, then limit of f at (a,b) does not exist.





The intuitive understanding of the notion of limit is as follows:

- The distance between $f(x, y)$ and L can be made arbitrarily small by making the distance between (x, y) and (a, b) sufficiently small but not necessarily zero.
- It is often difficult to show that limit of a function does not exist at a point. We will come back to this question soon.
- When limit exists, we write it in many alternative ways:
  The limit of $f(x, y)$ as (x, y) approaches (a, b) is L or $f(x, y) \to L$ as $(x, y) \to (a, b)$ or $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ or $\lim\limits_{x\to a, y\to b} f(x, y) = L$.

Example. Determine if $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ exists.

Observations:

- Suppose we have obtained a δ corresponding to some ϵ.
- If we take ϵ_1 which is larger than the earlier ϵ, then the same δ will satisfy the requirement in the definition of the limit.
- Thus while showing that the limit of a function is such and such at a point, we are free to choose a pre-assigned upper bound for our ϵ.

Similarly

- Suppose for some ϵ, we have already obtained a δ such that the limit requirement is satisfied.
- If we choose another δ, say δ_1, which is smaller than δ, then the limit requirement is also satisfied.
- Thus, we are free to choose a pre-assigned upper bound for our δ provided it is convenient to us and it works.

Example: Find limit at $(0,0)$ of $f(x,y) = \sqrt{1-x^2-y^2}$, where $D = \{(x,y) : x^2+y^2 \leq 1\}$.

Uniqueness of limit. Let $f(x,y)$ be a real valued function defined on a region $D \subseteq \mathbb{R}^2$. Let $(a,b) \in \overline{D}$. If limit of $f(x,y)$ as (x,y) approaches (a,b) exists, then it is unique.

- For a function of one variable, there are only two directions for approaching a point; from left and from right.
- Whereas for a function of two variables, there are infinitely many directions, and infinite number of paths on which one can approach a point.
- The limit refers only to the distance between (x,y) and (a,b). It does not refer to any specific direction of approach to (a,b).
- If the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b).
- Thus, if we can find two different paths of approach along which the function $f(x,y)$ has different limits, then it follows that limit of $f(x,y)$ as (x,y) approaches (a,b) does not exist.

Theorem (Important): Suppose that $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2. If $L_1 \neq L_2$, then the limit of $f(x,y)$ as $(x,y) \to (a,b)$ does not exist.

Examples:

- Consider $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$. What is its limit at $(0,0)$?
- Consider $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$. What is its limit at $(0,0)$?
- Consider $f(x,y) = \frac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$. What is its limit at $(0,0)$?

Question. Are the following same:

(a) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$; (b) $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y)$; (c) $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$.

Example: Let $f(x,y) = \frac{(y-x)(1+x)}{(y+x)(1+y)}$ for $x+y \neq 0, -1 < x, y < 1$.

- $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} \frac{y}{y(1+y)} = 1$.
- $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} \frac{-x(1+x)}{x} = -1$.
- Along $y = mx$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{x(m-1)(1+x)}{x(1+m)(1+mx)} = \frac{m-1}{m+1}$.

  For different values of m, we get the last limit value different.

So, limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist. But the two iterated limits exist and they are not equal.

Example: Let $f(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}$ for $x \neq 0, y \neq 0$.

Then $\lim\limits_{y\to 0} x\sin\frac{1}{y}$ and $\lim\limits_{x\to 0} y\sin\frac{1}{x}$ do not exist. Hence neither $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ nor $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exists. However

- $|f(x,y) - 0| \leq |x| + |y| = \sqrt{x^2} + \sqrt{y^2} \leq 2\sqrt{(x-0)^2 + (y-0)^2}$.
- Take $\delta = \epsilon/2$.
- Hence, if $|(x,y) - (0,0)| < \delta = \epsilon/2$, then $|f(x,y) - 0| < \epsilon$.
- Therefore $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

That is, the two iterated limits do not exist, but the limit exists.

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Limit theorems for functions of two real variables

Let $\ell_f, \ell_g, c \in \mathbb{R}$. Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = \ell_f$ and $\lim_{(x,y)\to(a,b)} g(x,y) = \ell_g$.

(i) $\lim_{(x,y)\to(a,b)} cf(x,y) = c\ell_f$ (constant multiple)

(ii) $\lim_{(x,y)\to(a,b)} (f \pm g)(x,y) = \ell_f \pm \ell_g$ resp. (sum/subtract)

(iii) $\lim_{(x,y)\to(a,b)} (f \cdot g)(x,y) = \ell_f \cdot \ell_g$ (product)

(iv) If $\ell_g \neq 0$ and $g(x,y) \neq 0$ in an open disk around the point (a,b), then $\lim_{(x,y)\to(a,b)} \left(\frac{f}{g}\right)(x,y) = \frac{\ell_f}{\ell_g}$ (quotient)

(iv) If $r \in \mathbb{R}$, $\ell_f^r \in \mathbb{R}$ and $\lim_{(x,y)\to(a,b)} f(x,y) = \ell_f$, then $\lim_{(x,y)\to(a,b)} (f(x,y))^r = \ell_f^r$ (power).

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Continuity

Let $f(x,y)$ be a real valued function defined on a subset D of $\mathbb{R}^2$. We say that $f(x,y)$ is continuous at a point $(a,b) \in D$ if and only if the following $\epsilon - \delta$ condition holds.

For every $\epsilon > 0$, there exists a $\delta > 0$ such that for all points $(x,y) \in D$ with

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon.$$

- Observe that if (a,b) is an isolated point of D, then f is continuous at (a,b).
- When D is a region, (a,b) is not an isolated point of D; and then f is continuous at $(a,b) \in D$ iff the following are satisfied:
  1. $f(a,b)$ is well defined, that is, $(a,b) \in D$;
  2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists; and
  3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

- The function $f(x, y)$ is said to be continuous on a subset of D iff $f(x, y)$ is continuous at all points in the subset.
- Therefore, constant multiples, sum, difference, product, quotient, and rational powers of continuous functions are continuous whenever they are well defined.
- Polynomials in two variables are continuous functions.
- Rational functions, i.e., ratios of polynomials, are continuous functions provided they are well defined.

Examples.

1. Following two functions $f(x,y)$ are continuous on $\mathbb{R}^2$. Take $f(0,0)=0$.
   - $f(x,y)=\frac{3x^2y}{x^2+y^2}$ if $(x,y)\neq(0,0)$.
   - $f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}$ if $(x,y)\neq(0,0)$.
2. The function $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$, is not continuous at $(0,0)$.

Composition.

As in the single variable case, composition of continuous functions is continuous.

Let $f : D \to \mathbb{R}$ be continuous at (a,b) with $f(a,b) = c$. Let $g : I \to \mathbb{R}$ be continuous at $c \in I$ for some interval I in $\mathbb{R}$. Then $g(f(x,y))$ from D to $\mathbb{R}$ is continuous at (a,b).

Example: $\cos \frac{xy}{1+x^2}$ and $\log(1 + x^2 + y^2)$ are continuous on $\mathbb{R}^2$.

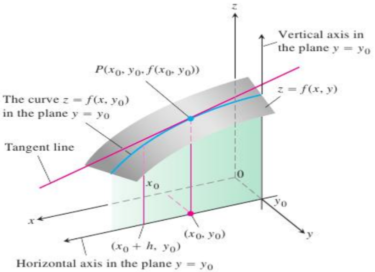
- At what points are $\tan^{-1}(y/x)$ continuous?
  - the function y/x is continuous everywhere except when $x = 0$.
  - The function $\tan^{-1}$ is continuous everywhere on $\mathbb{R}$.
  - Hence, $\tan^{-1}(y/x)$ is continuous everywhere except when $x = 0$.

The function $\frac{1}{x^2+y^2+z^2-1}$ is continuous everywhere except on the sphere $x^2 + y^2 + z^2 = 1$, where it is not defined.

Partial Differentiation (Partial Derivatives)

Recall: Let $D \subset \mathbb{R}$, and let c be an interior point of D. A function $f : D \to \mathbb{R}$ is said to have a **derivative** at c if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists, and then it is denoted by $f'(c)$.

Let $f(x, y)$ be a real valued function defined on a region $D \subseteq \mathbb{R}^2$. Let $(x_0, y_0) \in D$.



If C is the curve of intersection of the surface $z = f(x, y)$ with the plane $y = y_0$, then the slope of the tangent line to C at $(x_0, y_0, f(x_0, y_0))$ is the partial derivative of $f(x, y)$ with respect to x at (x_0, y_0).

More formally, we can define:

Definition

A function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** x at the point (x_0, y_0) if

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

exists, and then it is denoted by $f_x(x_0, y_0)$ or $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ or $\left.\frac{df(x,y_0)}{dx}\right|_{x=x_0}$.

Note: $f(x, y_0)$ must be continuous at $x = x_0$.

Similarly, a function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** y at (x_0, y_0) if

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

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Again, $f(x_0, y)$ must be continuous at $y = y_0$.

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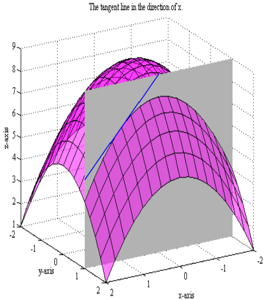
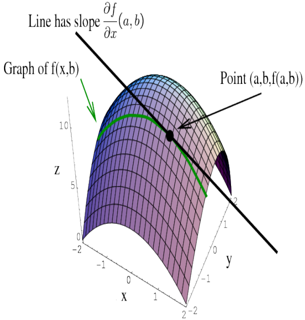

The tangent line in the direction of x.
z-axis
y-axis
x-axis


Line has slope $\frac{\partial f}{\partial x}(a,b)$
Graph of f(x,b)
Point (a,b,f(a,b))
z
x
y

Geometric interpretation

Let C denote the curve obtained by intersecting the graph of f by the horizontal plane $y = y_0$. The partial derivative of f with respect to x at (x_0, y_0) equals the slope of the tangent to the curve C at (x_0, y_0). It can be interpreted as the rate of change in f along the x-axis at (x_0, y_0).

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Partial derivatives of sums, products, quotients and compositions of functions of two variables can be found in exactly the same manner as the derivatives of a function of one variable.

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If the partial derivatives of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ of f exist at (x_0, y_0), then

$$\text{grad } f|_{(x_0,y_0)} = (\nabla f)(x_0, y_0) := (f_x(x_0, y_0), f_y(x_0, y_0)) \in \mathbb{R}^2$$

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Examples.

- Find f_x and f_y, where $f(x,y) = y\sin(xy)$.
  Solution. Treating y as a constant and differentiating with respect to x, we get f_x. Similarly f_y.

$$f_x(x,y) = y\cos(xy)y, \quad f_y(x,y) = yx\cos(xy) + \sin(xy).$$

- Find out $\partial z/\partial x$ and $\partial z/\partial y$ where $z = f(x,y)$ is defined by $x^3 + y^3 + z^3 - 6xyz = 1$.
- The plane $x = 1$ intersects the surface $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at the point $(1,2,5)$.
  Hint: The asked slope is $\partial z/\partial y$ at $(1,2)$. It is $\frac{\partial(x^2+y^2)}{\partial y}\Big|_{(1,2)} = 4.$

Partial Differentiation: Examples

- Let $f(x,y) := \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$. Let $(x_0,y_0) \neq (0,0)$. Then

$$f_x(x_0,y_0) = \frac{x_0}{\sqrt{x_0^2+y_0^2}} \quad \text{and} \quad f_y(x_0,y_0) = \frac{y_0}{\sqrt{x_0^2+y_0^2}}.$$

  But f does not have either partial derivative at $(0,0)$ since $\lim_{h\to 0} |h|/h$ does not exist. Note: f is continuous at $(0,0)$.

- Let $f(x,y) := xy/(x^2+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) = 0$. It is easy to see that $f_x(0,0) = 0 = f_y(0,0)$. We have already seen that f is not continuous at $(0,0)$.

Note that it is still true in higher dimensions that differentiability at a point implies continuity.

What last example suggests that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives.

After few slides, we shall define differentiability for functions of two variables and its connectivity with continuity.

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Higher Order Partial Derivatives

Let $D \subset \mathbb{R}^2$, and $f : D \to \mathbb{R}$. Suppose $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to x at (x_0, y_0), then it is denoted by $f_{xx}(x_0, y_0)$ or $(f_x)_x(x_0, y_0)$ or $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0)$ or $\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial x}(x_0, y_0)$.

Also, if $f_x : D \to \mathbb{R}$ has a partial derivative with respect to y at (x_0, y_0), then it is denoted by $f_{xy}(x_0, y_0)$ or $(f_x)_y(x_0, y_0)$ or $\dfrac{\partial f_x}{\partial y}(x_0, y_0)$ or $\dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ or $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}(x_0, y_0)$.

Similarly, we define $f_{yy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$, or $\dfrac{\partial^2 f}{\partial y^2}(x_0, y_0)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$.

In general, the **mixed partial derivatives** $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ may not be equal.

Mixed Partials

Example: Let $f(x,y) := x\,y\dfrac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$.
Then $f_x(0,y_0) = -y_0$ for $y_0 \in \mathbb{R}$, and $f_y(x_0,0) = x_0$ for $x_0 \in \mathbb{R}$. Hence $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

Theorem (Mixed Partials Theorem/ Mixed Derivative Theorem)

Let D be a region in $\mathbb{R}^2$. Let the function $f : D \to \mathbb{R}$ have continuous first and second order partial derivatives on D. Then $f_{xy} = f_{yx}$.

Example. Find $\partial^2 w/\partial x \partial y$ if $w = xy + \frac{e^y}{y^2+1}$.

- The symbol $\partial^2 w/\partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x. However, we interchange the order of differentiation and differentiate first w.r.t. x, we get answer quickly.
- In two steps: $\frac{\partial w}{\partial x} = y$ and $\frac{\partial^2 w}{\partial y \partial x} = 1$.
- If we differentiate first w.r.t. y, then also we obtain same result as the conditions of above theorem hold for w al all points (x_0, y_0).

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Still Higher Order.

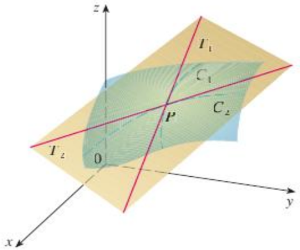
- Although we will deal mostly with first and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist.
- Thus, we get third- and fourth-order derivatives denoted by symbols like $f_{xxy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^3 f}{\partial y \partial x \partial x}$.

Example: Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution. $f_y = -4xyz + x^2$; $f_{yx} = -4yz + 2x$;
$f_{yxy} = -4z$; $f_{yxyz} = -4$.

Increment Theorem.

In order to see the connection between continuity of a function and the partial derivatives, the following geometry may help.



- Let S be the surface $z = f(x,y)$, where f_x, f_y are continuous on the region D, the domain of f.
- Let $(a,b) \in D$. Let C_1 and C_2 be the curves of intersection of the planes $x = a$ and of $y = b$ with S.
- Let T_1 and T_2 be tangent lines to the curves C_1 and C_2 at the point $P(a,b,f(a,b))$.

- The tangent plane to the surface S at P is the plane containing T_1 and T_2.
- The tangent plane to S at P consists of all possible tangent lines at P to the curves C that lie on S and pass through P.
- This plane approximates S at P most closely.
- Write the z-coordinate of P as c. Then $P = (a, b, c)$.
- Equation of any plane passing through P is $z - c = A(x - a) + B(y - b)$.
- When $y = b$, the tangent plane represents the tangent to the intersected curve at P.
- Thus, $A = f_x(a, b)$, the slope of the tangent line. Similarly, $B = f_y(a, b)$.
- Hence, the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, c)$ on S is

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

provided that $f_x,\ f_y$ are continuous at (a, b).

If we take $g(x,y,z) = f(x,y) - z = 0$, then equation of the tangent plane is given by

$$\nabla g \cdot \vec{u} = 0$$

where $\nabla g = (f_x, f_y, -1)$ is a gradient vector and u is the vector from (x,y,z) to (a,b,c), i.e., $(x-a, y-b, z-c)$.

Example. Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1,1,3)$.

Solution: Here $z_x = 4x, z_y = 2y$. So, $z_x(1,1) = 4, z_y(1,1) = 2$. The the equation of the tangent plane is $z - 3 = 4(x-1) + 2(y-1)$
$\Rightarrow z = 4x + 2y - 3$.

The tangent plane gives a linear approximation to the surface at that point. Why?

Write the equation $f(x,y)-f(a,b)=f_x(a,b)(x-a)+f_y(a,b)(y-b)$. Then

$$f(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b).$$

This formula holds true for all points $(x,y,f(x,y))$ on the tangent plane at $(a,b,f(a,b))$.
For approximating $f(x,y)$ for (x,y) close to (a,b), we may take

$$f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b).$$

The RHS is called the standard linear approximation of $f(x,y,z)$. By considering $x=a+\Delta x$, $y=b+\Delta y$ and using Mean value theorem, we obtain:
Increment Theorem. Let D be a region in $\mathbb{R}^2$. Let the function $f:D\to\mathbb{R}$ have continuous first order partial derivatives on D. Then $f(x,y)$ is continuous on D and the total increment
$\Delta f=f(a+\Delta x,b+\Delta y)-f(a,b)$ can be written as

$$\Delta f=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y,$$

where $\epsilon_1\to 0$ and $\epsilon_2\to 0$ as both $\Delta x\to 0$ and $\Delta y\to 0$.

Recall that for a function g of one variable, its differential is defined as $dg = g'(t)dt$.

Let $f(x, y)$ be a given function. The differential of f, also called the total differential, is

$$df = f_x(x, y)dx + f_y(x, y)dy.$$

Here, $dx = \Delta x$ and $dy = \Delta y$ are the increments in x and y, respectively. Then df is a linear approximation to the total increment Δf.

Example. The dimensions of a rectangular box are measured to be 75cm, 60cm, and 40 cm, and each measurement is correct to within 0.2cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

- The volume of the box is $V = xyz$ with $x = 75, y = 60, z = 40$
- So, $dV = V_x(x,y,z)dx + V_y(x,y,z)dy + V_z(x,y,z)dz.$
- Note that $V_x = yz, V_y = xz, V_z = xy.$
- Given that $|\Delta x|, |\Delta y|, |\Delta z| \leq 0.2$cm.
- Hence the largest error in cubic cm is

$$|\Delta V| \approx |dV| = 60 \times 40 \times 0.2 + 40 \times 75 \times 0.2 + 75 \times 60 \times 0.2 = 1980.$$

- Notice that the relative error is $1980/(75 \times 60 \times 40)$ which is about 1.1%.

Differentiable.

Let D be a region in $\mathbb{R}^2$. A function $f : D \to \mathbb{R}$ is called differentiable at a point $(a, b) \in D$ if the total increment $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ in f with respect to increments Δx, Δy in x, y, can be written as

$$\Delta f = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

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The following statements give some connections between differentiability, continuity and the partial derivatives.

- Let D be a region in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be such that both f_x and f_y exist on D and at least one of them is continuous at $(a, b) \in D$. Then f is differentiable at (a, b).
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- Let D be a region in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be differentiable at $(a,b) \in D$. Then f is continuous at (a,b).

Differentiable.

Note that the first statement strengthens the increment theorem. Instead of increasing the load on terminology, we will continue with the increment theorem. Note that whenever we assume that fx and fy are continuous, you may replace this with the weaker assumption: "$f(x,y)$ is differentiable".

Other way Definition. Let D be a region in $\mathbb{R}^2$. Let the function $f : D \to \mathbb{R}$. Then we say that f is differentiable at (x_0, y_0) if there is $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0.$$

In this case, the pair $(\alpha, \beta) \in \mathbb{R}^2$ is called the **total derivative** of f at (x_0, y_0). By letting $(h,k) \to (0,0)$ along the x-axis and y-axis, we obtain $\alpha = f_x(x_0, y_0)$ and $\beta = f_y(x_0, y_0)$.

Examples.

- Let $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. Then, check that f is discontinuous at $(0,0)$. Hence f is not differentiable at $(0,0)$.
- Let $f(x,y) := x^2+y^2$ for $(x,y) \in \mathbb{R}^2$. Let $(x_0,y_0) \in \mathbb{R}^2$. Since f_x and f_y exist on $\mathbb{R}^2$ and f_x is continuous at (x_0,y_0), f is differentiable at (x_0,y_0).

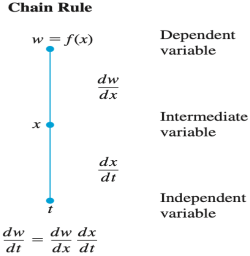
Remember that we formulate and discuss our results for a function $f(x,y)$ of two variables. Analogously, all the notions and the results can be formulated for a function $f(x_1,\cdots,x_n)$ of n variables for $n \geq 2$.

Chain Rules.

Recall that The Chain Rule for functions of a single variable says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, then w is a differential function of t and $\frac{dw}{dt}$ can be calculate by the following formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

For this composite function $w(t) = f(g(t))$, we can think of t as the independent variable and $x = g(t)$ as the "intermediate variable", because t determines the value of x which in turn gives the value of w from the function f.



- For functions of several variables the Chain Rule has more than one form, which depends on how many independent and intermediate variables are involved.
- However, once the variables are taken into account, the Chain Rule works in the same way we just discussed.

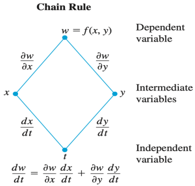


Figure: two intermediate, one independent variable

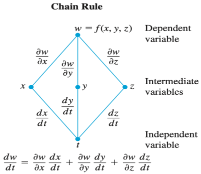


Figure: three intermediate, one independent variable

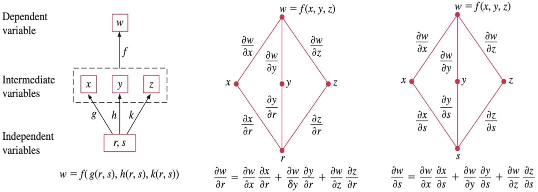


Figure: three intermediate, two independent variables

- Chain rule For Functions of one Independent Variable and Two Intermediate Variables: If $w = f(x, y)$ is differentiable and if $x = x(t), y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot \frac{dx}{dt} + f_y(x(t), y(t)) \cdot \frac{dy}{dt}.$$

or $\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

- Chain rule for functions of one independent Variable and three intermediate Variables: If w = f(x, y, z) is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

- Chain rule for two independent Variables and three intermediate Variables: Suppose that $w = f(x, y, z), x = g(r, s), y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r};$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Examples:

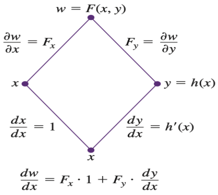
- Using chain rule, find the derivative of $w = xy$ with respect to t along the path $x = \cos t$ and $y = \sin t$. What is the derivative's value at $t = \pi/2$.
  Solution. $\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t$. Value at $t = \pi/2$ is -1.
- Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if $w = x^2 + y^2, x = r - s, y = r + s$.
  Solution. $\partial w/\partial r = (\partial w/\partial x)(\partial x/\partial r) + (\partial w/\partial y)(\partial y/\partial r) = 4r$.
  $\partial w/\partial s = (\partial w/\partial x)(\partial x/\partial s) + (\partial w/\partial y)(\partial y/\partial s) = 4s$.
- Given that $w = x^2 + y^2 + z^2$ and $z(x, y)$ satisfies $z^3 - xy + yz + y^3 = 1$, evaluate $\partial w/\partial x$ at $(2, -1, 1)$.
  Solution. It is now clear that z, w are dependent variables and x, y are independent variables.
  $\partial w/\partial x = 2x + 2z\partial z/\partial x$, $3z^2\partial z/\partial x - y + y\partial z/\partial x = 0$. These two together give $\partial w/\partial x = 2x + \frac{2yz}{y+3z^2}$. Evaluating it at $(2, -1, 1)$ gives $(\partial w/\partial x)(2, -1, 1) = 3$.

The correct procedure to get $\frac{\partial w}{\partial x}$ is:

- w must be dependent variable and x must be independent variable.
- Decide which of the other variables are dependent or independent.
- Eliminate the dependent variables from w using the constraints.
- Then take the partial derivative $\frac{\partial w}{\partial x}$.

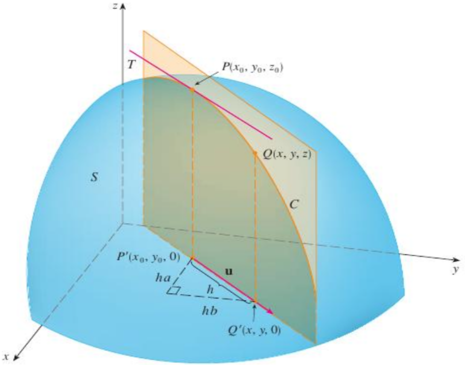
A formula for implicit Differentiation: Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$



Directional Derivatives

- Recall that if $f(x, y)$ is a function, then $f_x(x_0, y_0)$ is the rate of change in f with respect to change in x, at (x_0, y_0), that is, in the direction $\hat{i}$.
- Similarly, $f_y(x_0, y_0)$ is the rate of change at (x_0, y_0) in the direction $\hat{j}$.
- How do we find the rate of change of $f(x, y)$ at (x_0, y_0) in the direction of any unit vector $\hat{u}$?



- Consider the surface S with the equation $z = f(x, y)$.
- Let $z_0 = f(x_0, y_0)$. The point $P(x_0, y_0, z_0)$ lies on S.
- The vertical plane that passes through P in the direction of $\hat{u}$ (containing $\hat{u}$) intersects S in a curve C.
- The slope of the tangent line T to the curve C at the point P is the rate of change of z in the direction of $\hat{u}$.

- Let $f(x,y)$ be a function defined in a region D.
- Let $(x_0,y_0) \in D$.
- The directional derivative of $f(x,y)$ in the direction of a unit vector $\hat{u} = a\hat{u} + b\hat{j}$ at (x_0,y_0) is given by (if following limit exists)

$$(D_u f)(x_0,y_0) = \left(\frac{df}{ds}\right)_u\Big|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+ha,y_0+hb) - f(x_0,y_0)}{h}.$$

Example. Find the derivative of $z = x^2 + y^2$ at $(1,2)$ in the direction of $\hat{u} = (1/\sqrt{2})\hat{i} + (1/\sqrt{2})\hat{j}$.

Solution.
$(D_u z)(1,2) = \lim_{h\to 0}\frac{z(1+h/\sqrt{2},2+h/\sqrt{2})-z(1,2)}{h} = \lim_{h\to 0}\frac{2h/\sqrt{2}+2\cdot 2h/\sqrt{2}}{h} = \frac{6}{\sqrt{2}}$.

Note that: $z_x(1,2)(1/\sqrt{2}) + z_y(1,2)(1/\sqrt{2}) = \frac{6}{\sqrt{2}}$.

Result: Let $f(x,y)$ have continuous first order partial derivatives. Then $f(x,y)$ has a directional derivative at (x,y) in any direction $\hat{u} = a\hat{i} + b\hat{j}$; and it is given by

$$(D_u f)(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f \cdot \hat{u}.$$

That is, at (x_0, y_0), the directional derivative is given by

$$(D_u f)(x_0,y_0) = D_u f|_{(x_0,y_0)} = \nabla f|_{(x_0,y_0)} \cdot \hat{u} = \text{grad} f|_{(x_0,y_0)} \cdot \hat{u}.$$

- Find the directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ in the direction of the line that makes an angle of $\pi/6$ with the x-axis.
  Solution. Here,the direction is given by the unit vector
  $\hat{u} = \cos(\pi/6)\hat{i} + \sin(\pi/6)\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}.$
  Thus $D_u f(x,y) = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y = \frac{1}{2}[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y].$

How much the value of $y\sin x + 2yz$ change if the point (x, y, z) moves 0.1 units from $(0, 1, 0)$ toward $(2, 2, -2)$?

- Let $f(x, y, z) = ysinx + 2yz.$
- If $P := (0, 1, 0), Q := (2, 2, -2).$ Then $\vec{v} = \vec{PQ} = 2\hat{i} + \hat{j} - 2\hat{k}$.
- The unit vector in the direction of $\vec{v}$ is $\hat{u} = \frac{1}{3}\vec{v}$.

Examples of directional derivatives

Examples:

(i) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$.

Let $(x_0, y_0) \in \mathbb{R}^2$ and $\boldsymbol{u} := (u_1, u_2)$ be a unit vector. For $t \neq 0$,

$$\begin{aligned} & \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} \\ = & \frac{(x_0 + tu_1)^2 + (y_0 + tu_2)^2 - x_0^2 - y_0^2}{t} \\ = & 2x_0u_1 + 2y_0u_2 + t. \end{aligned}$$

Letting $t \to 0$, we obtain

$(\boldsymbol{D_u} f)(x_0, y_0) = 2x_0u_1 + 2y_0u_2.$

Note: $(\boldsymbol{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$.

(ii) Let $f(x,y) := \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.
Let $(x_0,y_0) \in \mathbb{R}^2$ and $\boldsymbol{u} := (u_1,u_2)$ be a unit vector. For $t \neq 0$,

$$\begin{aligned} & \frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t} \\ = & \frac{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} - \sqrt{x_0^2+y_0^2}}{t} \\ = & \frac{2x_0u_1+2y_0u_2+t}{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} + \sqrt{x_0^2+y_0^2}}. \end{aligned}$$

Hence if $(x_0,y_0) \neq (0,0)$, then

$$(\boldsymbol{D_u} f)(x_0,y_0) = \frac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}} = (\nabla f)(x_0,y_0) \cdot \boldsymbol{u}.$$

But $(\boldsymbol{D_u} f)(0,0)$ does not exist since $\lim_{t\to 0} t/|t|$ does not exist.

(iii) Let $f(x,y) := xy/(x^2+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$.
For a unit vector $\boldsymbol{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{u_1 u_2}{t}.$$

Hence $(\boldsymbol{D_u} f)(0,0)$ exists $\iff u_1 = 0$ or $u_2 = 0$.

(iv) Let $f(x,y) := x^2y/(x^4+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$.
For a unit vector $\boldsymbol{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{u_1^2 u_2}{t^2 u_1^4 + u_2^2}.$$

Hence $(\boldsymbol{D_u} f)(0,0) = 0$ if $u_2 = 0$, and $(\boldsymbol{D_u} f)(0,0) = u_1^2/u_2$ if $u_2 \neq 0$. In particular, $(\nabla f)(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$.

Thus $(\boldsymbol{D_u} f)(0,0) \neq (\nabla f)(0,0) \cdot \boldsymbol{u}$ unless $u_1 = 0$ or $u_2 = 0$.

(v) Let $f(x,y) := x^3y/(x^4+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$.
For a unit vector $\boldsymbol{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{tu_1^3 u_2}{t^2 u_1^4 + u_2^2}.$$

Hence $(\boldsymbol{D_u} f)(0,0) = 0$ if $u_2 = 0$ and also if $u_2 \neq 0$. In particular, $(\nabla f)(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$

Thus $(\boldsymbol{D_u} f)(0,0) = (\nabla f)(0,0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$.

Tangent Line

Recall the one variable situation:
Let $a < b$ and $x_0 \in (a,b)$. Let a function $f : (a,b) \to \mathbb{R}$ be differentiable at x_0. Then the equation of the **tangent line** to the curve $y = f(x)$ in $\mathbb{R}^2$ at $(x_0, f(x_0))$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0).$$

More generally, suppose $D \subset \mathbb{R}^2$ and $P_0 := (x_0, y_0)$ is an interior point of D. Let a function $F : D \to \mathbb{R}$ have partial derivatives at P_0, and let $(\nabla F)(P_0) \neq (0,0)$. Suppose F defines a curve C in $\mathbb{R}^2$ (implicitly) by the equation $F(x,y) = 0$ for $(x,y) \in D$, and P_0 lies on C. Then the equation of the **tangent line** to C at P_0 is given by

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) = 0.$$

Note: If $D := (a,b) \times \mathbb{R}$, $f : (a,b) \to \mathbb{R}$, and for $(x,y) \in D$, $F(x,y) := y - f(x)$, then $y_0 = f(x_0)$, $F_x(P_0) = -f'(x_0)$ and $F_y(P_0) = 1$. We recover the earlier equation of the tangent line.

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Tangent Plane

Let us turn to the two variable situation:
Let $D \subset \mathbb{R}^2$, (x_0, y_0) be an interior point of D, and let a function $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0). Then the equation of the **tangent plane** to the surface $z = f(x,y)$ in $\mathbb{R}^3$ at $(x_0, y_0, f(x_0, y_0))$ is given by

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More generally, suppose $E \subset \mathbb{R}^3$ and $P_0 := (x_0, y_0, z_0)$ is an interior point of E. Let a function $F : E \to \mathbb{R}$ have partial derivatives at P_0, and let $(\nabla F)(P_0) \neq (0,0,0)$. Suppose F (implicitly) defines a surface S in $\mathbb{R}^3$ by the equation $F(x,y,z) = 0$ for $(x,y,z) \in E$, and P_0 lies on S. Then the equation of the **tangent plane** to S at P_0 is given by

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Normal Line

Now $(\nabla F)(P_0)\cdot(x-x_0,y-y_0,z-z_0)=0$ for all (x,y,z) on the tangent plane to the surface S, that is, $(\nabla F)(P_0)$ is perpendicular to the tangent plane. The line passing through $P_0=(x_0,y_0,z_0)$ and parallel to the nonzero vector $(\nabla F)(P_0)=(F_x(P_0),F_y(P_0),F_z(P_0))$ is called the **normal line** to the surface defined by $F(x,y,z)=0$ at P_0. The parametric equations of this normal line are

$$x=x_0+F_x(P_0)t,\ \ y=y_0+F_y(P_0)t,\ \ z=z_0+F_z(P_0)t,\ \ t\in\mathbb{R}.$$

If all $F_x(P_0),F_y(P_0),F_z(P_0)$ are nonzero, then the equations are

$$\frac{x-x_0}{F_x(P_0)}=\frac{y-y_0}{F_y(P_0)}=\frac{z-z_0}{F_z(P_0)}.$$

Also, the parametric equations of the normal line to the surface defined by $z-f(x,y)=0$ at (x_0,y_0,z_0) are

$$x=x_0-f_x(x_0,y_0)t,\ y=y_0-f_y(x_0,y_0)t,\ z=f(x_0,y_0)+t,\ t\in\mathbb{R}.$$

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$$x = x_0 - f_x(x_0, y_0)t, \; y = y_0 - f_y(x_0, y_0)t, \; z = f(x_0, y_0) + t, \; t \in \mathbb{R}.$$

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$$\frac{x - x_0}{F_x(P_0)} = \frac{y - y_0}{F_y(P_0)} = \frac{z - z_0}{F_z(P_0)}.$$

Also, the parametric equations of the normal line to the surface defined by $z - f(x, y) = 0$ at (x_0, y_0, z_0) are

$$x = x_0 - f_x(x_0, y_0)t, \ y = y_0 - f_y(x_0, y_0)t, \ z = f(x_0, y_0) + t, \ t \in \mathbb{R}.$$

Normal Line

Now $(\nabla F)(P_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$ for all (x, y, z) on the tangent plane to the surface S, that is, $(\nabla F)(P_0)$ is perpendicular to the tangent plane. The line passing through $P_0 = (x_0, y_0, z_0)$ and parallel to the nonzero vector $(\nabla F)(P_0) = (F_x(P_0), F_y(P_0), F_z(P_0))$ is called the **normal line** to the surface defined by $F(x, y, z) = 0$ at P_0. The parametric equations of this normal line are

$$x = x_0 + F_x(P_0)t, \;\; y = y_0 + F_y(P_0)t, \;\; z = z_0 + F_z(P_0)t, \;\; t \in \mathbb{R}.$$

If all $F_x(P_0), F_y(P_0), F_z(P_0)$ are nonzero, then the equations are

$$\frac{x - x_0}{F_x(P_0)} = \frac{y - y_0}{F_y(P_0)} = \frac{z - z_0}{F_z(P_0)}.$$

Also, the parametric equations of the normal line to the surface defined by $z - f(x, y) = 0$ at (x_0, y_0, z_0) are

$$x = x_0 - f_x(x_0, y_0)t, \; y = y_0 - f_y(x_0, y_0)t, \; z = f(x_0, y_0) + t, \; t \in \mathbb{R}.$$