MA-1102

Series and Matrices

Department of Mathematics

IIT Madras

Notation

$\varnothing$ = the empty set.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of integers.

$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N}\}$, the set of rational numbers.

$\mathbb{R}$ = the set of real numbers.

$\mathbb{R} \setminus \mathbb{Q}$ = the set of irrational numbers.

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$.

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This allows the existence of the **greatest integer function**.

That is, corresponding to each $x \in \mathbb{R}$, there exists a unique integer n such that $n \leq x < n + 1$. We write this n as $[x]$; it is the greatest integer less than or equal to x.

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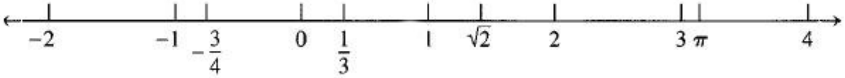
We visualize $\mathbb{R}$ as a straight line made of expansible rubber of no thickness!



Notation Cont.

Let $a, b \in \mathbb{R},\ a < b$.

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, the closed interval $[a, b]$.

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$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$, the closed infinite interval $[a, \infty)$.

$(a, \infty) = \{x \in \mathbb{R} : x \leq b\}$, the open infinite interval (a, ∞).

$(-\infty, \infty) = \mathbb{R}$, both open and closed infinite interval.

$(0, \infty) = \mathbb{R}_+ =$ the set of all positive real numbers.

$(-\infty, 0) = \mathbb{R}_- =$ the set of all negative real numbers.

Absolute Value function

The **absolute value** of $x \in \mathbb{R}$ is defined as

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The distance between x and y is $|x - y|$.

A **neighbourhood** of a point c is an open interval $(c - \delta, c + \delta)$ for some $\delta > 0$.

It is the set of all points having distance from c less than δ.

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We often prove an equality through an inequality:

Let $a, b \in \mathbb{R}$.

1. If for each $\epsilon > 0$, $|a - b| < \epsilon$, then $a = b$.
2. If for each $\epsilon > 0$, $0 \leq a < \epsilon$, then $a = 0$.
3. If for each $\epsilon > 0$, $a < b + \epsilon$, then $a \leq b$.

Infinite sums?

Question: We accept $1111.111\cdots$ as a real number.

But this is same as the infinite sum

$$1000 + 100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots$$

Is this infinite sum a number?

We rather take the partial sums

$$1000,\ 1000 + 100,\ 1000 + 100 + 10,\ 1000 + 100 + 10 + 1,$$

$$1000 + 100 + 10 + 1 + \tfrac{1}{10},\ 1000 + 100 + 10 + 1 + \tfrac{1}{10} + \tfrac{1}{100},\ \ldots$$

which are numbers; and ask whether the sequence of these numbers approximates certain real number?

We may approximate $\sqrt{2}$ by the usual division procedure, and get the sequence

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ 1.414213,\ 1.4142135,\ 1.41421356,\ \ldots$$

Does it approximate $\sqrt{2}$?

Sequences

A **sequence of real numbers** is a function $f : \mathbb{N} \to \mathbb{R}$.

The values of the function are $f(1),\ f(2),\ f(3),\ \ldots$

These are called the **terms of the sequence**.

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With $f(n) = x_n$, the nth term of the sequence, we write the sequence in many ways such as

$$(x_n) = (x_n)_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty} = \{x_n\} = (x_1, x_2, x_3, \ldots)$$

For example,
(n) is the sequence $1, 2, 3, 4, \ldots$
$(1/n)$ is the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$
$(1/n^2)$ is the sequence $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \ldots$
(5) is the sequence $5, 5, 5, 5, \ldots$ It is a constant sequence.

Convergence

A sequence (x_n) **converges to a real number** a iff
corresponding to each $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that if $n > m$ is any natural number, then $|x_n - a| < \epsilon$.

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Example 1 The sequence $(1/n)$ converges to 0.

Reason: Let $\epsilon > 0$. Take $m = [1/\epsilon]$.
That is, m is the natural number such that $m \leq \frac{1}{\epsilon} < m + 1$.

Then $\dfrac{1}{m} \leq \epsilon$.

If $n > m$, then $\dfrac{1}{n} < \dfrac{1}{m} \leq \epsilon$.
That is, corresponding to each $\epsilon > 0$, there exists an m, (we have defined it here) such that for every $n > m$, we see that $|1/n - 0| < \epsilon$.

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Let (x_n) be a sequence. We say that (x_n) **diverges to** ∞ iff corresponding to each $r > 0$, there exists an $m \in \mathbb{N}$ such that if $n > m$ is any natural number, then $x_n > r$.

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We say that (x_n) **diverges to** $-\infty$ iff corresponding to each $r > 0$, there exists an $m \in \mathbb{N}$ such that if $n > m$ is any natural number, then $x_n < -r$.

Divergence

Call an open interval (r, ∞) a neighborhood of ∞.
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When (x_n) converges to a, or when it diverges to ∞, or when it diverges to $-\infty$, we write

$$\lim_{n\to\infty} x_n = \ell \quad \text{for} \quad \ell = a, \ \infty, \ -\infty.$$

We also write it as "$x_n \to \ell$ as $n \to \infty$" or as $x_n \to \ell$.

Example 2

1. $\lim \sqrt{n} = \infty$

Reason: Let $r > 0$. Choose an $m > r^2$. Let $n > m$.

Then $\sqrt{n} > \sqrt{m} > r$.

Therefore, $\lim \sqrt{n} = \infty$.

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2. $\lim \log(1/n) = -\infty$.

Reason: Let $r > 0$. Choose a natural number $m > e^r$. Let $n > m$.

Then $1/n < 1/m < e^{-r}$.

Consequently, $\log(1/n) < \log e^{-r} = -r$.

Therefore, $\log(1/n) \to -\infty$.

Bounded, Monotonic

We say that a sequence (x_n) is **bounded** iff
there exists a positive real number k such that for each $n \in \mathbb{N}$, $|x_n| \leq k$;
that is,
when the whole sequence is contained in an interval of finite length.

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We also say that (x_n) is **bounded below** iff
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A sequence (x_n) is called **increasing** iff $x_n \leq x_{n+1}$ for each n.
Similarly, (x_n) is called **decreasing** iff $x_n \geq x_{n+1}$ for each n.

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A sequence which is either increasing or decreasing is called a **monotonic** sequence.

Cauchy, Subsequence

A sequence (x_n) is called a **Cauchy sequence** iff
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Let (x_n) be a sequence. Choose an increasing sequence of indices $n_1 < n_2 < n_3 < \cdots$. The sequence $\left(x_{n_k}\right)$ for $k = 1, 2, 3, \ldots$, is a **subsequence** of the sequence (x_n).

For example, $(1, 4, 9, 16, \ldots)$ is a subsequence of the sequence $1, 2, 3, 4, \ldots$.

Helpful results

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3. *Algebra of Limits*: Suppose $\lim x_n = a$ and $\lim y_n = b$. Then the following are true:
   3.1 *Sum*: $\lim (x_n + y_n) = a + b$
   3.2 *Difference*: $\lim (x_n - y_n) = a - b$.
   3.3 *Constant Multiple*: $\lim (cx_n) = ca$ for any real number c.
   3.4 *Product*: $\lim (x_n y_n) = ab$.
   3.5 *Division*: $\lim (x_n / y_n) = a/b$, provided no y_n is 0 and $b \neq 0$.
   3.6 *Domination*: If for each n, $x_n \leq y_n$, then $a \leq b$.

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4. *Sandwich Theorem*: Let (x_n), (y_n), and (z_n) be sequences such that $x_n \leq y_n \leq z_n$ holds for all n greater than some m. If $x_n \to \ell$ and $z_n \to \ell$, then $y_n \to \ell$.

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5. *Weirstrass Criterion*: A bounded monotonic sequence converges.
   5.1 an increasing sequence bounded above converges to its *lub*;
   5.2 a decreasing sequence bounded below converges to its *glb*.

Helpful results Cont.

6. *Cauchy Criterion*: A sequence (x_n) converges iff it is a Cauchy sequence.

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7. *Limits of functions to Limits of sequences*: Let $m \in \mathbb{N}$. Let $f(x)$ be a function whose domain includes $[m, \infty)$. Let (x_n) be a sequence such that $x_n = f(n)$ for all $n \geq m$. If $\lim_{n \to \infty} f(n) = \ell$, then $\lim_{n \to \infty} x_n = \ell$.

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8. *Limits of sequences to Limits of functions*: Let $a < c < b$. Let $f : D \to \mathbb{R}$ be a function where D contains $(a, c) \cup (c, b)$. Let $\ell \in \mathbb{R}$. Then $\lim_{x\to c} f(x) = \ell$ iff for each non-constant sequence (x_n) converging to c, the sequence of functional values $\big(f(x_n)\big)$ converges to ℓ.

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10. *Continuity*: Let $f(x)$ be a continuous real valued function whose domain contains each term of a convergent sequence x_n and also its limit. Then $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$.

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In general, even if $(|x_n|)$ converges, (x_n) may not converge.

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In general, even if $(|x_n|)$ converges, (x_n) may not converge.

For example, take $x_n = (-1)^n$.

Example 3

Let (x_n) be a sequence. Show that $x_n \longrightarrow 0$ iff $|x_n| \longrightarrow 0$.

Solution: Suppose $x_n \longrightarrow 0$. Let $\epsilon > 0$. There is $m \in \mathbb{N}$ such that for each $n > m$, $|x_n| < \epsilon$. Since $||x_n| - 0| = |x_n|$, we get: $|x_n| \longrightarrow 0$.

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In general, even if $(|x_n|)$ converges, (x_n) may not converge.

For example, take $x_n = (-1)^n$.

However, if (x_n) converges, say, to ℓ, then $(|x_n|)$ converges to $|\ell|$. It follows from the inequality $||x_n| - |\ell|| \leq |x_n - \ell|$.

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Choose $n_0 \in \mathbb{N}$ such that $n_0 > 5/3\epsilon$. For example, let $n_0 := [5/3\epsilon] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

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Suppose (a_n) is convergent. Then there is a real number a such that $a_n \to a$. Let $\epsilon := 1/2$. Find $n_0 \in \mathbb{N}$ such that

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Since $(-1)^{2n_0} = 1$ and $(-1)^{2n_0+1} = -1$,

$$\begin{aligned} 2 &= |(-1)^{2n_0} - (-1)^{2n_0+1}| \\ &\leq |(-1)^{2n_0} - a| + |a - (-1)^{2n_0+1}| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which is a contradiction.

Recall: Uniqueness of limit

Theorem

A convergent sequence has a unique limit.

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Let $n_0 \in \mathbb{N}$ be such that $n \geq n_0 \implies |a_n - a| < \epsilon$, and let $m_0 \in \mathbb{N}$ be such that $n \geq m_0 \implies |a_n - b| < \epsilon$. Consider $n := \max\{n_0, m_0\}$. Then

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$$|a-b| \leq |a-a_n| + |a_n - b| < \epsilon + \epsilon = |a-b|,$$

which is a contradiction. Hence $a = b$. □

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Define $\alpha := \max\left\{|a_1|, \ldots, |a_{n_0}|, |a| + 1\right\}$. Then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. $\square$

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  Example: $a_n := (-1)^n$ for $n \in \mathbb{N}$.

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Hence (a_n) is bounded. □

- A bounded sequence need not be convergent.
  Example: $a_n := (-1)^n$ for $n \in \mathbb{N}$.
- If $a_n := (-1)^n n$ for $n \in \mathbb{N}$, then (a_n) is divergent since it is not bounded.

Example: Monotonic Bounded

Let $a_n := 1 + (1/2^2) + \cdots + (1/n^2)$ for $n \in \mathbb{N}$. The sequence (a_n) is clearly increasing.

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Let $a_n := 1 + (1/2^2) + \cdots + (1/n^2)$ for $n \in \mathbb{N}$. The sequence (a_n) is clearly increasing. Further, since

$$\begin{aligned} a_n &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \end{aligned}$$

we see that $a_n \leq 2 - (1/n) < 2$. Thus (a_n) is bounded. Hence by the result discussed in a previous slide, (a_n) is convergent.

Examples

Example 4. Does $\lim_{n\to\infty} \frac{5n^2+2n+7}{n^2-11n+5}$ exist?

$$\lim_{n\to\infty} \frac{5n^2+2n+7}{n^2-11n+5} = \lim_{n\to\infty} \frac{5+\frac{2}{n}+\frac{7}{n^2}}{1-\frac{11}{n}+\frac{5}{n^2}} = \frac{5}{1} = 5.$$

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Example 6. As $n<\sqrt{n^2+1}+n$,

$$0<\sqrt{n^2+1}-n=\frac{1}{\sqrt{n^2+1}+n}<\frac{1}{n}.$$

By Sandwich theorem, $\displaystyle\lim_{n\to\infty}\sqrt{n^2+1}-n=0$.

Examples Cont.

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Solution: Let $\epsilon > 0$. Using Archimedean Property, take $m \in \mathbb{N}$ so that $(1/\epsilon)^{1/p} < m$. Now, $1/\epsilon < m^p$; so $1/m^p < \epsilon$. If $n > m$, then

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Example 8. Show that if $|x| < 1$, then $\lim x^n = 0$.

Solution: Write $|x| = \dfrac{1}{1+r}$ for some $r > 0$. By the Binomial theorem, $(1+r)^n \geq 1 + nr > nr$. So, $0 < |x|^n = \frac{1}{(1+r)^n} < \frac{1}{nr}$.
By Sandwich theorem, $\lim |x|^n = 0$. Now, $-|x|^n \leq x^n \leq |x|^n$.
Again, by Sandwich theorem, $\lim x^n = 0$.

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Solution: Let $x_n = n^{1/n} - 1$. We see that $x_n \geq 0$. Using Binomial theorem for $n \geq 2$,

$$x_n + 1 = n^{1/n} \Rightarrow n = (x_n + 1)^n \geq 1 + nx_n + \frac{n(n-1)}{2}x_n^2.$$

Hence $n \geq \dfrac{n(n-1)}{2}x_n^2 \Rightarrow 0 \leq x_n \leq \dfrac{\sqrt{2}}{\sqrt{n-1}}$.

Apply Sandwich theorem to conclude that $\lim x_n = 0$.

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Hence $n \geq \dfrac{n(n-1)}{2} x_n^2 \Rightarrow 0 \leq x_n \leq \dfrac{\sqrt{2}}{\sqrt{n-1}}$.

Apply Sandwich theorem to conclude that $\lim x_n = 0$.

Example 10. Show that if $x > 0$, then $\lim(x^{1/n}) = 1$.

Solution: The function $f(t) = x^t$ is continuous for $t \in [0, 1]$. All terms of the sequence $(1/n)$ and its limit belong to the interval $[0, 1]$. Therefore,

$$\lim x^{1/n} = x^{\lim(1/n)} = x^0 = 1.$$

Examples Cont.

Example 11. Show that $\lim_{n\to\infty} \frac{\log n}{n} = 0$.
Solution: $\log x$ is defined on $[1, \infty)$. Using L' Hospital's rule,

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Therefore, $\lim_{n\to\infty} \frac{\log n}{n} = \lim_{x\to\infty} \frac{\log x}{x} = 0$.

Example 12. Show that $\lim_{n\to\infty} n^{1/n} = 1$.

Solution: $\lim n^{1/n} = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{1}{x}\log x} = e^{\lim_{x\to\infty} \frac{\log x}{x}} = e^0 = 1$.

Exercise 13. Show that $\lim_{n\to\infty} \frac{n^p}{x^n} = 0$ for $x > 1$.

Series

A **series** is an infinite sum.

Consider the series $\sum_{n=1}^{\infty} x_n$, which abbreviates $x_1 + x_2 + x_3 + \cdots$.
Its partial sum is the sequence $s_n = x_1 + x_2 + \cdots + x_n$.
We say that the series $\sum x_n$ **converges to** ℓ iff the sequence (s_n) converges to ℓ.

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That is, when for each $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that for each $n > m$, $|\sum_{k=1}^{n} x_k - \ell| < \epsilon$. In this case, we write $\sum x_n = \ell$.

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The series $\sum x_n$ **diverges to** ∞ iff for each $r > 0$, there exists $m \in \mathbb{N}$ such that for each $n > m$, $\sum_{k=1}^{n} x_k > r$. We write it as $\sum x_n = \infty$.

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A **series** is an infinite sum.

Consider the series $\sum_{n=1}^{\infty} x_n$, which abbreviates $x_1 + x_2 + x_3 + \cdots$. Its partial sum is the sequence $s_n = x_1 + x_2 + \cdots + x_n$. We say that the series $\sum x_n$ **converges to** ℓ iff the sequence (s_n) converges to ℓ.

That is, when for each $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that for each $n > m$, $|\sum_{k=1}^{n} x_k - \ell| < \epsilon$. In this case, we write $\sum x_n = \ell$.

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The series $\sum x_n$ **diverges to** ∞ iff for each $r > 0$, there exists $m \in \mathbb{N}$ such that for each $n > m$, $\sum_{k=1}^{n} x_k > r$. We write it as $\sum x_n = \infty$.

Similarly, the series $\sum x_n$ **diverges to** $-\infty$ iff for each $r > 0$, there exists $m \in \mathbb{N}$ such that for each $n > m$, $\sum_{k=1}^{n} x_k < -r$. We write it as $\sum x_n = -\infty$.

Example 14

1. The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1. Reason?

$$s_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1-(1/2)^n}{1-1/2} = 1 - \frac{1}{2^n} \ \to 1.$$

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$$s_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1 - (1/2)^n}{1 - 1/2} = 1 - \frac{1}{2^n} \rightarrow 1.$$

2. The **Harmonic** series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges to ∞. Reason?

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2. The **Harmonic** series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges to ∞. Reason?

Write $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Let $r > 0$. Choose $m = 2^k$, where $k \in \mathbb{N}$, $k > 2r$. Then

$$\begin{aligned}
s_m &= \sum_{j=1}^{m} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\sum_{j=2^{k-1}+1}^{2^k} \frac{1}{j}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\sum_{j=2^{k-1}+1}^{2^k} \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{k}{2} > r.
\end{aligned}$$

If $n > m$, then $s_n > s_m > r$.

Example 15

1. The series $-1-2-3-4-\cdots-n-\cdots$ diverges to $-\infty$.
2. The **Alternating** series $1-1+1-1+\cdots$ diverges. Reason?
   The sequence of partial sums $1, 0, 1, 0, 1, 0, 1, \ldots$.
   It neither diverges to ∞ nor to $-\infty$.

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   The sequence of partial sums $1, 0, 1, 0, 1, 0, 1, \ldots$.
   It neither diverges to ∞ nor to $-\infty$.
3. Let $a \neq 0$. Consider the **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots .$$

$$s_n = a + ar + ar^2 + ar^3 + \cdots ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

(a) If $|r| < 1$, then $r^n \to 0$. The geometric series converges to

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

(b) If $|r| > 1$, then r^n diverges; so the geometric series diverges.
If $r = \pm 1$, the geometric series clearly diverges.

Helpful results

1. If a series sums to ℓ, then ℓ is unique.
   It follows from the uniqueness of limit of a sequence.
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   Proof: Let $s_n = \sum_{k=1}^{n} a_k$. Then $a_n = s_n - s_{n-1}$.
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Examples: The series $\sum \frac{-n}{3n+1}$ diverges because $\lim \frac{-n}{3n+1} = -\frac{1}{3} \neq 0$.
The series $\sum (-1)^n$ diverges because $\lim (-1)^n$ does not exist.
The harmonic series diverges even though $\lim \frac{1}{n} = 0$.

Algebra with sums

1. If $\sum a_n$ converges to a and $\sum b_n$ sums to b, then the series $\sum(a_n + b_n)$ sums to $a + b$;
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Here, if $\ell \in \mathbb{R}$, then $\ell + \infty = \infty$, $\ell - \infty = -\infty$.
If $\ell > 0$, then $\ell \cdot (\pm\infty) = \pm\infty$.
If $\ell < 0$, then $\ell \cdot (\pm\infty) = \mp\infty$.
Also, $\infty + \infty = \infty$, $-\infty - \infty = -\infty$, $\infty \cdot \pm\infty = \pm\infty$, $-\infty \cdot \pm\infty = \mp\infty$.
Whereas $0 \cdot \pm\infty$, $\infty - \infty$, $\pm\infty/\infty$ etc. are indeterminate.

Improper Integral

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In the definite integral $\int_a^b f(x)dx$, if

at least one of a or b is $\pm\infty$, or
the value of $f(x)$ is $\pm\infty$ for some $x \in [a, b]$, or $f(x)$ is not continuous at some points in (a, b)
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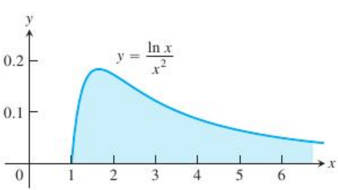

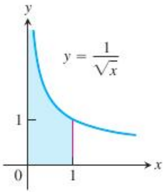

But the area under the curve may still remain bounded.

Convergence

The improper integral $\int_0^\infty f(x)dx$ **converges** provided the limit

$$\lim_{b\to\infty} \int_0^b f(x)dx$$

exists. It means the limit must be a real number, NOT $\pm\infty$.

Then we say that the **value** of the improper integral is this limit. That is,

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The improper integral **diverges** iff the limit above does not exist.

Possible types of improper integrals

1. If $f(x)$ is continuous on $[a, \infty)$, then $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.
2. If $f(x)$ is continuous on $(-\infty, b]$, then $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

   $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \text{ for any } c \in \mathbb{R}.$$

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   $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx, \text{ for any } c \in \mathbb{R}.$$
4. If $f(x)$ is continuous on $(a, b]$ and discontinuous at $x = a$, then

   $$\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx.$$
5. If $f(x)$ is continuous on $[a, b)$ and discontinuous at $x = b$, then

   $$\int_a^b f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx.$$
6. If $f(x)$ is continuous on $[a, c) \cup (c, b]$ and discontinuous at $x = c$, then

   $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Also, when improper integral diverges to $\pm\infty$, we take its value to be that.

Example 16

For what values of $p \in \mathbb{R}$, the improper integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges? What is its value, when it converges?

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Case 1: $p = 1$. $\int_1^b \frac{dx}{x^p} = \int_1^b \frac{dx}{x} = \log b - \log 1 = \log b$.
Since $\displaystyle\lim_{b\to\infty} \log b = \infty$, the improper integral diverges to ∞.

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Case 2: $p < 1$. $\int_1^b \frac{dx}{x^p} = \frac{-x^{-p+1}}{-p+1}\Big|_1^b = \frac{1}{1-p}(b^{1-p} - 1)$.
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Case 3: $p > 1$. $\int_1^b \frac{dx}{x^p} = \frac{1}{1-p}(b^{1-p} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)$.
Since $\displaystyle\lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$, we have

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{p-1}.$$

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Hence, the improper integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges to $\dfrac{1}{p-1}$ for $p > 1$ and diverges to ∞ for $p \leq 1$.

Example 17

For what values of $p \in \mathbb{R}$, the improper integral $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges?

Case 1: $p = 1$. $\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0+}[\log 1 - \log a] = \infty$.
Therefore, the improper integral diverges to ∞.

Case 2: $p < 1$. $\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0+} \int_a^1 \frac{dx}{x^p} = \lim_{a\to 0+} \frac{1-a^{1-p}}{1-p} = \frac{1}{1-p}$.
Therefore, the improper integral converges to $1/(1-p)$.

Case 3: $p > 1$. $\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0+} \frac{1-a^{1-p}}{1-p} = \lim_{a\to 0+} \frac{1}{p-1}\left(\frac{1}{a^{p-1}} - 1\right) = \infty$.
Hence the improper integral diverges to ∞.

The improper integral $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges to $\displaystyle\frac{1}{1-p}$ for $p < 1$ and diverges to ∞ for $p \geq 1$.

Convergence tests

Theorem: **(Comparison Test)** Let $f(x)$ and $g(x)$ be continuous functions on $[a, \infty)$. Suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
2. If $\int_a^\infty f(x)\,dx$ diverges to ∞, then $\int_a^\infty g(x)\,dx$ diverges to ∞.

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Theorem: **(Limit Comparison)** Let $f(x)$ and $g(x)$ be positive continuous functions on $[a, \infty)$. If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = L$, where $0 < L < \infty$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converge, or both diverge.

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Theorem: **(Absolute Convergence)** Let $f(x)$ be a continuous function on $[a, b)$, for $b \in \mathbb{R}$ or $b = \infty$. If the improper integral $\int_a^b |f(x)|\,dx$ converges, then the improper integral $\int_a^b f(x)\,dx$ also converges.

Example 18

1. $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges. Reason?

$\displaystyle\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for all $x \geq 1$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges.

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$$\frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \text{ for all } x \ge 1 \text{ and } \int_1^\infty \frac{dx}{x^2} \text{ converges.}$$

2. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges to ∞. Reason?

$$\frac{1}{\sqrt{x^2-1}} \ge \frac{1}{x} \text{ for all } x \ge 2 \text{ and } \int_2^\infty \frac{dx}{x} \text{ diverges to } \infty.$$

Example 19

$$\int_1^\infty \frac{dx}{1+x^2} \text{ converges or diverges?}$$

Since $\lim_{x\to\infty} \left[\frac{1}{1+x^2} \Big/ \frac{1}{x^2}\right] = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$, the limit comparison test says that the given improper integral and $\int_1^\infty \frac{dx}{x^2}$ both converge or diverge together. The latter converges, so does the former.

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However, they may converge to different values.

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \left(\frac{-1}{b} - \frac{-1}{1} \right) = 1.$$

Example 20

Does the improper integral $\displaystyle\int_1^\infty \frac{10^{10}dx}{e^x+1}$ converge?

$$\lim_{x\to\infty} \frac{10^{10}}{e^x+1} \Big/ \frac{1}{e^x} = \lim_{x\to\infty} \frac{10^{10}e^x}{e^x+1} = 10^{10}.$$

Also, $e \geq 2$ implies that for all $x \geq 1$, $e^x \geq x^2$. So, $e^{-x} \leq x^{-2}$.

Since $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges,

$\displaystyle\int_1^\infty \frac{dx}{e^x}$ also converges.

By limit comparison test, the given improper integral converges.

Gamma function

Example 21: Show that $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ converges for $x > 0$.

Fix $x > 0$. Since $\lim_{t\to\infty} e^{-t}t^{x+1} = 0$, there exists $t_0 \geq 1$ such that $0 < e^{-t}t^{x+1} < 1$ for $t > t_0$. Then

$$0 < e^{-t}t^{x-1} < t^{-2} \quad \text{for} \quad t > t_0.$$

Since $\int_1^\infty t^{-2}\,dt$ is convergent, $\int_{t_0}^\infty t^{-2}\,dt$ is also convergent. By the comparison test,

$$\int_{t_0}^\infty e^{-t}t^{x-1}\,dt \text{ is convergent.}$$

The integral $\int_1^{t_0} e^{-t}t^{x-1}\,dt$ exists and is not an improper integral.

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The integral $\int_1^{t_0} e^{-t} t^{x-1}\, dt$ exists and is not an improper integral.

Next, we consider the improper integral $\int_0^1 e^{-t} t^{x-1}\, dt$.

Gamma function Cont.

Let $0 < a < 1$. For $a \le t \le 1$, we have $0 < e^{-t}t^{x-1} < t^{x-1}$. Since $x > 0$,

$$\int_a^1 e^{-t}t^{x-1}\,dt < \int_a^1 t^{x-1}\,dt = \frac{1-a^x}{x} < \frac{1}{x}.$$

Taking the limit as $a \to 0+$, we see that the

$$\int_0^1 e^{-t}t^{x-1}\,dt \text{ is convergent,}$$

and its value is less than or equal to $1/x$. Therefore,

$$\int_0^\infty e^{-t}t^{x-1}\,dt = \int_0^1 e^{-t}t^{x-1}\,dt + \int_1^{t_0} e^{-t}t^{x-1}\,dt + \int_{t_0}^\infty e^{-t}t^{x-1}\,dt$$

is convergent.

Gamma function Cont.

Let $0 < a < 1$. For $a \le t \le 1$, we have $0 < e^{-t}t^{x-1} < t^{x-1}$. Since $x > 0$,

$$\int_a^1 e^{-t}t^{x-1}\,dt < \int_a^1 t^{x-1}\,dt = \frac{1-a^x}{x} < \frac{1}{x}.$$

Taking the limit as $a \longrightarrow 0+$, we see that the

$$\int_0^1 e^{-t}t^{x-1}\,dt \ \text{ is convergent,}$$

and its value is less than or equal to $1/x$. Therefore,

$$\int_0^\infty e^{-t}t^{x-1}\,dt = \int_0^1 e^{-t}t^{x-1}\,dt + \int_1^{t_0} e^{-t}t^{x-1}\,dt + \int_{t_0}^\infty e^{-t}t^{x-1}\,dt$$

is convergent.

The function $\Gamma(x)$ is defined on $(0, \infty)$.

For $x > 0$, using integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \Big[-t^x e^{-t}\Big]_0^\infty - \int_0^\infty x t^{x-1}(-e^{-t})\,dt = x\Gamma(x).$$

It thus follows that $\Gamma(n+1) = n!$ for any non-negative integer n.

Example 22

Test the convergence of $\int_{-\infty}^{\infty} e^{-t^2}\,dt$.

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For $t > 1$, we have $t < t^2$. So, $0 < e^{-t^2} < e^{-t}$.

Since $\int_{1}^{\infty} e^{-t}\, dt$ is convergent, by the comparison test, $\int_{1}^{\infty} e^{-t^2}\, dt$ is convergent.

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For $t > 1$, we have $t < t^2$. So, $0 < e^{-t^2} < e^{-t}$.

Since $\int_{1}^{\infty} e^{-t}\,dt$ is convergent, by the comparison test, $\int_{1}^{\infty} e^{-t^2}\,dt$ is convergent.

Now, $\int_{-a}^{-1} e^{-t^2}\,dt = \int_{a}^{1} e^{-t^2} d(-t) = \int_{1}^{a} e^{-t^2} dt$.

Taking limit as $a \to \infty$, we see that $\int_{-\infty}^{-1} e^{-t^2}\,dt$ is convergent and its value is equal to $\int_{1}^{\infty} e^{-t^2}\,dt$.

Example 22

Test the convergence of $\int_{-\infty}^{\infty} e^{-t^2}\,dt$.

Since e^{-t^2} is continuous on $[-1, 1]$, $\int_{-1}^{1} e^{-t^2}\,dt$ exists.

For $t > 1$, we have $t < t^2$. So, $0 < e^{-t^2} < e^{-t}$.

Since $\int_{1}^{\infty} e^{-t}\,dt$ is convergent, by the comparison test, $\int_{1}^{\infty} e^{-t^2}\,dt$ is convergent.

Now, $\int_{-a}^{-1} e^{-t^2}\,dt = \int_{a}^{1} e^{-t^2} d(-t) = \int_{1}^{a} e^{-t^2} dt$.

Taking limit as $a \to \infty$, we see that $\int_{-\infty}^{-1} e^{-t^2}\,dt$ is convergent and its value is equal to $\int_{1}^{\infty} e^{-t^2}\,dt$.

Combining the three integrals above, we conclude that $\int_{-\infty}^{\infty} e^{-t^2}\,dt$ converges.

Other forms of $\Gamma(x)$

The Gamma function takes other forms by substitution of the variable of integration. Substituting t by rt we have

$$\Gamma(x) = r^x \int_0^\infty e^{-rt} t^{x-1}\, dt \quad \text{for } \ 0 < r, \ 0 < x.$$

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$$\Gamma(x) = r^x \int_0^\infty e^{-rt} t^{x-1}\, dt \quad \text{for } 0 < r,\ 0 < x.$$

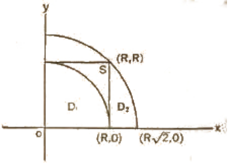
Substituting t by t^2, we have

$$\Gamma(x) = 2 \int_0^\infty e^{-t^2} t^{2x-1}\, dt \quad \text{for } 0 < x.$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}\,dx = 2\int_0^\infty e^{-t^2}\,dt \qquad (x = t^2)$$

To evaluate this integral, consider the double integral of $e^{-x^2-y^2}$ over two circular sectors D_1 and D_2, and the square S as indicated. Since the integrand is positive, we have $\iint_{D_1} < \iint_S < \iint_{D_2}$.



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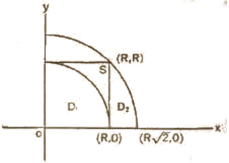

Now, evaluate these integrals by converting them to iterated integrals as follows:

$$\int_0^R e^{-r^2} r\,dr \int_0^{\pi/2} d\theta < \int_0^R e^{-x^2}\,dx \int_0^R e^{-y^2}\,dy < \int_0^{R\sqrt{2}} e^{-r^2} r\,dr \int_0^{\pi/2} d\theta$$

$$\text{So, } \frac{\pi}{4}(1 - e^{-R^2}) < \left(\int_0^R e^{-x^2}\,dx\right)^2 < \frac{\pi}{4}(1 - e^{-2R^2})$$

Take the limit as $R \to \infty$ to obtain

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

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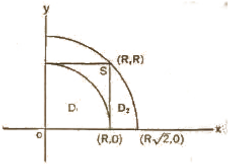

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Take the limit as $R \longrightarrow \infty$ to obtain

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

That is, $[\Gamma(\frac{1}{2})]^2 = 4\frac{\pi}{4} = \pi$. As $\Gamma(\frac{1}{2}) > 0$, we get the required result.

Example 23

Show that $B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt$ converges for $x > 0, y > 0$.

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Write the integral as a sum of two integrals:

$$B(x,y) = \int_0^{1/2} t^{x-1}(1-t)^{y-1}\,dt + \int_{1/2}^{1} t^{x-1}(1-t)^{y-1}\,dt$$

Setting $u = 1 - t$, the second integral looks like

$$\int_{1/2}^{1} t^{x-1}(1-t)^{y-1}\,dt = \int_0^{1/2} u^{y-1}(1-u)^{x-1}\,du$$

Therefore, it is enough to show that the first integral converges.

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Case 1: $x \geq 1$.

For $0 \leq t \leq 1/2$, $1 - t > 0$. For all $y > 0$, the function $(1-t)^{y-1}$ is well defined, continuous, and bounded on $[0, 1/2]$. So is the function t^{x-1}. Therefore, the integral $\int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt$ exists and is not an improper integral.

Beta function Cont.

Case 2: $0 < x < 1$.
The function $(1-t)^{y-1}$ is well defined and continuous on $0 \le t \le 1/2$.
So, let c be an upper bound of it.
Then for $0 < t \le 1/2$, $t^{x-1}(1-t)^{y-1} \le c\, t^{x-1}$.
The function $c\, t^{x-1}$ is well defined and continuous on $(0, 1/2]$.
As done earlier, the integral $\int_0^{1/2} c\, t^{x-1}\, dt$ converges.
Therefore, $\int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt$ converges.

Beta function Cont.

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Other forms of the beta function:

By setting t as $1-t$, we see that $B(x,y) = B(y,x)$.

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By setting t as $1-t$, we see that $B(x, y) = B(y, x)$.

Substituting t with $\sin^2 t$, Beta function can be written as

$$B(x, y) = 2 \int_0^{\pi/2} (\sin t)^{2x-1} (\cos t)^{2y-1}\, dt, \quad \text{for } x > 0,\ y > 0.$$

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$$B(x, y) = \int_0^{\infty} \frac{t^{x+1}}{(1+t)^{x+y}}\, dt \quad \text{for } x > 0,\ y > 0.$$

Beta function Cont.

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$$B(x, y) = \int_0^{\infty} \frac{t^{x+1}}{(1+t)^{x+y}}\, dt \quad \text{for } x > 0,\ y > 0.$$

Using multiple integrals it can be shown that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for } x > 0,\ y > 0.$$

Integral test for series

Let $\sum a_n$ be a series of positive terms. Let $f : [1, \infty) \to \mathbb{R}$ be a continuous, positive and non-increasing function such that $a_n = f(n)$ for each $n \in \mathbb{N}$.

1. If $\int_1^\infty f(t)dt$ is convergent, then $\sum a_n$ is convergent.
2. If $\int_1^\infty f(t)dt$ diverges to ∞, then $\sum a_n$ diverges to ∞.

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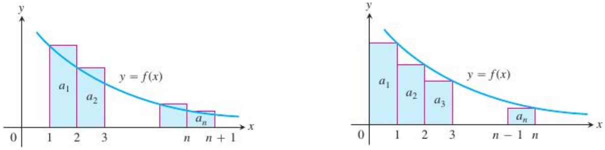
2. If $\int_1^\infty f(t)dt$ diverges to ∞, then $\sum a_n$ diverges to ∞.

Proof: Since f is a positive and non-increasing, the integrals and the partial sums have a certain relation.



$$\int_1^{n+1} f(t)\,dt \;\le\; a_1 + a_2 + \cdots + a_n \;\le\; a_1 + \int_1^n f(t)\,dt.$$

If $\int_1^\infty f(t)\,dt$ is finite, then $\sum a_n$ is convergent.
If $\int_1^\infty f(t)\,dt = \infty$, then $\sum a_n$ diverges to ∞. □

Example 24

Show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

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Example 24

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For $p = 1$, the series is the harmonic series; and it diverges to ∞.
Suppose $p \neq 1$. Consider the function $f(t) = 1/t^p$ from $[1, \infty)$ to $\mathbb{R}$.
This is a continuous, positive and decreasing function.

$$\int_1^{\infty} \frac{1}{t^p}\,dt = \lim_{b\to\infty} \frac{t^{-p+1}}{-p+1}\Big|_1^b = \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1. \end{cases}$$

Then the Integral test proves the statement.

For $p > 1$, the sum of the series $\sum n^{-p}$ need not be equal to $(p-1)^{-1}$.

Comparison Test

Theorem: Let $\sum a_n$ and $\sum b_n$ be series of non-negative terms. Suppose there exists $k > 0$ such that $0 \leq a_n \leq kb_n$ for each n greater than some natural number m.

1. If $\sum b_n$ converges, then $\sum a_n$ converges.
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2. If $\sum a_n$ diverges to ∞, then $\sum b_n$ diverges to ∞.

Proof: (1) Consider all partial sums of the series having more than m terms.

$$a_1 + \cdots + a_m + a_{m+1} + \cdots + a_n \le a_1 + \cdots + a_m + k \sum_{j=m+1}^{n} b_j.$$

Since $\sum b_n$ converges, so does $\sum_{j=m+1}^{n} b_j$.
By Weirstrass criterion, $\sum a_n$ converges.

(2) Similar to (1). $\square$

Ratio Comparison Test

Theorem: Let $\sum a_n$ and $\sum b_n$ be series of non-negative terms.
Suppose there exists $m \in \mathbb{N}$ such that for each $n > m$, $a_n > 0$, $b_n > 0$, and $\dfrac{a_{n+1}}{a_n} \leq \dfrac{b_{n+1}}{b_n}$.

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1. If $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $\sum a_n$ diverges to ∞, then $\sum b_n$ diverges to ∞.

Proof: For $n > m$,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{m+2}}{a_{m+1}} a_{m+1} \leq \frac{b_n}{b_{n-1}} \frac{b_{n-1}}{b_{n-2}} \cdots \frac{b_{m+2}}{b_{m+1}} a_{m+1} = \frac{a_{m+1}}{b_{m+1}} b_n.$$

By the Comparison test, if $\sum b_n$ converges, then $\sum a_n$ converges. This proves (1).

And, (2) follows from (1) by contradiction. $\square$

Limit Comparison Test

Theorem: Let $\sum a_n$ and $\sum b_n$ be series of non-negative terms. Suppose that there exists $m \in \mathbb{N}$ such that for each $n > m$, $a_n > 0,\ b_n > 0$, and that $\lim_{n\to\infty} \frac{a_n}{b_n} = k$.

1. If $k > 0$ then $\sum b_n$ and $\sum a_n$ converge or diverge to ∞, together.
2. If $k = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $k = \infty$ and $\sum b_n$ diverges to ∞ then $\sum a_n$ diverges to ∞.

Limit Comparison Test

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1. If $k > 0$ then $\sum b_n$ and $\sum a_n$ converge or diverge to ∞, together.
2. If $k = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $k = \infty$ and $\sum b_n$ diverges to ∞ then $\sum a_n$ diverges to ∞.

Proof: (1) $k > 0$. Let $\epsilon = k/2$. The limit condition implies that there exists $m \in \mathbb{N}$ such that

$$\frac{k}{2} < \frac{a_n}{b_n} < \frac{3k}{2} \quad \text{for each } n > m.$$

By the Comparison test, the conclusion is obtained.

(2) $k = 0$. Let $\epsilon = 1$. The limit condition implies that there exists $m \in \mathbb{N}$ such that

$$-1 < \frac{a_n}{b_n} < 1 \quad \text{for each } n > m.$$

Using the right hand inequality and the Comparison test we conclude that convergence of $\sum b_n$ implies the convergence of $\sum a_n$.

(3) $k = \infty$. Then $\lim(b_n/a_n) = 0$. Use (2).

Example 25

For each $n \in \mathbb{N}$, $n! \geq 2^{n-1}$. That is, $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$.

Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent, $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent. Therefore, adding 1 to it, the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

is convergent.

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$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

is convergent.

In fact, this series converges to e. Consider

$$s_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}, \quad t_n = \left(1 + \frac{1}{n}\right)^n.$$

By the Binomial theorem,

$$t_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)\right] \leq s_n.$$

Thus taking limit as $n \to \infty$, we have

$$e = \lim_{n \to \infty} t_n \leq \lim_{n \to \infty} s_n.$$

Example 25 Cntd.

Also, for $n > m$, where m is any fixed natural number,

$$t_n > \left(1+\frac{1}{n}\right)^m = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{m!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\right]$$

Taking limit as $n \longrightarrow \infty$ we have

$$e = \lim_{n\to\infty} t_n \geq s_m.$$

Since m is arbitrary, taking the limit as $m \longrightarrow \infty$, we have

$$e \geq \lim_{m\to\infty} s_m.$$

Therefore, $\lim_{m\to\infty} s_m = e$. That is,

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Example 26

Determine when the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+7}{n(n+3)\sqrt{n+5}}$ converges.

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Let $a_n = \frac{n+7}{n(n+3)\sqrt{n+5}}$ and $b_n = \frac{1}{n^{3/2}}$. Then

$$\frac{a_n}{b_n} = \frac{\sqrt{n}(n+7)}{(n+3)\sqrt{n+5}} \to 1 \text{ as } n \to \infty.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent,
Limit comparison test says that the given series is convergent.

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Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent,
Limit comparison test says that the given series is convergent.

We will next use some properties of integrals with infinite limits.

Cauchy's Root test

Let $\sum a_n$ be a series of positive terms. Suppose $\lim_{n\to\infty} (a_n)^{1/n} = \ell$.

1. If $\ell < 1$, then $\sum a_n$ converges.
2. If $\ell > 1$ or $\ell = \infty$, then $\sum a_n$ diverges to ∞.
3. If $\ell = 1$, then no conclusion is obtained.

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2. If $\ell > 1$ or $\ell = \infty$, then $\sum a_n$ diverges to ∞.
3. If $\ell = 1$, then no conclusion is obtained.

Proof: (1) Suppose $\ell < 1$. Choose δ such that $\ell < \delta < 1$.
Due to the limit condition, there exists an $m \in \mathbb{N}$ such that for each $n > m$, $(a_n)^{1/n} < \delta$. That is, $a_n < \delta^n$. Since $0 < \delta < 1$, $\sum \delta^n$ converges. By Comparison test, $\sum a_n$ converges.

(2) Given that $\ell > 1$ or $\ell = \infty$, we see that $(a_n)^{1/n} > 1$ for infinitely many values of n. That is, the sequence $\left((a_n)^{1/n}\right)$ does not converge to 0. Therefore, $\sum a_n$ is divergent. It diverges to ∞ since it is a series of positive terms.

(3) Once again, for both the series $\sum(1/n)$ and $\sum(1/n^2)$, we see that $(a_n)^{1/n}$ has the limit 1. But one is divergent, the other is convergent. $\square$

Root to Ratio

Theorem Let (a_n) be a sequence of positive terms.
If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \in \mathbb{R}$, then $\ell = \lim_{n\to\infty} a_n^{1/n}$.

Root to Ratio

Theorem Let (a_n) be a sequence of positive terms.
If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \in \mathbb{R}$, then $\ell = \lim_{n\to\infty} a_n^{1/n}$.

Proof: Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$. Suppose $\ell \in \mathbb{R}$. Let $\epsilon > 0$. Then we have an $m \in \mathbb{N}$ such that for all $n > m$, $\ell - \epsilon < \frac{a_{n+1}}{a_n} < \ell + \epsilon$. Use the right side inequality first. Let $n > m$. Then

$$\begin{aligned} a_n &= \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{m+2}}{a_{m+1}} \cdot a_{m+1} \leq (\ell+\epsilon)^{n-m-1} a_{m+1}. \\ (a_n)^{1/n} &\leq (\ell+\epsilon)((\ell+\epsilon)^{-m-1} a_{m+1})^{1/n} \to \ell + \epsilon \text{ as } n \to \infty. \end{aligned}$$

By Weirstrass criterion, the sequence $(a_n)^{1/n}$ converges, and $\lim(a_n)^{1/n} \leq \ell + \epsilon$ for every $\epsilon > 0$. That is, $\lim(a_n)^{1/n} \leq \ell$.
Similarly, the left side inequality gives $\lim(a_n)^{1/n} \geq \ell$.
This shows that $\lim(a_n)^{1/n} = \ell$. □

D' Alembert's Ratio test

Let $\sum a_n$ be a series of positive terms. Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$.

1. If $\ell < 1$, then $\sum a_n$ converges.
2. If $\ell > 1$ or $\ell = \infty$, then $\sum a_n$ diverges to ∞.
3. If $\ell = 1$, then no conclusion is obtained.

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3. If $\ell = 1$, then no conclusion is obtained.

Proof: When $\ell \in \mathbb{R}$, the conclusions follow from Cauchy's root test and the last theorem.

If $\ell = \infty$, then there exists $m \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} > 1$.

Then $\sum_{n=m+1}^{\infty} a_n$ is a series of positive and increasing terms.

Therefore, it diverges to ∞.

The series $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

Example 27

Does the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

Write $a_n = n!/(n^n)$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{(n+1)^{n+1}(n!} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1 \text{ as } n \to \infty.$$

By D' Alembert's ratio test, the series converges.

Then it follows that the sequence $\left(\frac{n!}{n^n}\right)$ converges to 0.

Example 28

Does the series $\sum_{n=1}^{\infty} 2^{(-1)^{n+1}-n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \cdots$ converge?

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Let $a_n = 2^{(-1)^{n+1}-n}$. Then

$$\frac{a_{n+1}}{a_n} = \begin{cases} 1/8 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd.} \end{cases}$$

Clearly, its limit does not exist. But

$$(a_n)^{1/n} = \begin{cases} 2^{1/n-1} & \text{if } n \text{ even} \\ 2^{-1/n-1} & \text{if } n \text{ odd} \end{cases}$$

This has limit $1/2 < 1$.
Therefore, by Root test, the series converges. □

Alternating Series

(Leibniz) Let (a_n) be a sequence of positive terms decreasing to 0; that is, for each n, $a_n \geq a_{n+1} > 0$, and $\lim_{n \to \infty} a_n = 0$. Then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges, and its sum lies between $a_1 - a_2$ and a_1.

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Proof: Let s_n be the partial sum upto n terms. Then

$$\begin{aligned} s_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \\ &= a_1 - \left[(a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1})\right] - a_{2n}. \end{aligned}$$

For each $n \in \mathbb{N}$, s_{2n} is a sum of n positive terms bounded above by a_1 and below by $a_1 - a_2$.
By Weirstrass criterion, $\lim_{n\to\infty} s_{2n} = s$, where $a_1 - a_2 \leq s \leq a_1$.
Now, $s_{2n+1} = s_{2n} + a_{2n+1}$. It follows that $\lim_{n\to\infty} s_{2n+1} = s$.
Therefore, $\lim_{n\to\infty} s_n = s$.
That is, the series sums to some s with $a_1 - a_2 \leq s \leq a_1$. □

Absolute Convergence

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$ is convergent to some s with $1/2 \leq s \leq 1$.

But $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is divergent to ∞.

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An alternating series $\sum a_n$ is said to be **conditionally convergent** iff it is convergent but it is not absolutely convergent.

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We say that the series $\sum a_n$ is **absolutely convergent** iff the series $\sum |a_n|$ is convergent.

An alternating series $\sum a_n$ is said to be **conditionally convergent** iff it is convergent but it is not absolutely convergent.

Thus, the alternating harmonic series is conditionally convergent.

The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent.

It is also convergent. Why?

Abs. Conv. ⇒ Conv.

Theorem: An absolutely convergent series is convergent.

Abs. Conv. $\Rightarrow$ Conv.

Theorem: An absolutely convergent series is convergent.

Proof: Let $\sum a_n$ be an absolutely convergent series.
Then $\sum |a_n|$ is convergent.

Let $\epsilon > 0$. By Cauchy criterion, there exists an $n_0 \in \mathbb{N}$ such that for all $n > m > n_0$, we have

$$|a_m| + |a_{m+1}| + \cdots + |a_n| < \epsilon.$$

Now,

$$|a_m + a_{m+1} + \cdots + a_n| \leq |a_m| + |a_{m+1}| + \cdots + |a_n| < \epsilon.$$

Again, by Cauchy criterion, the series $\sum a_n$ is convergent. $\square$

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   Therefore, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n}$ converges absolutely; hence it converges.

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2. $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.
   By comparison test, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely; and hence it converges.

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   By comparison test, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely; and hence it converges.

3. For $p > 1$, the series $\sum n^{-p}$ converges. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges absolutely for $p > 1$.

   For $0 < p \leq 1$, by Leibniz test, the series converges.
   But $\sum n^{-p}$ does not converge.
   Therefore, the given series converges conditionally for $0 < p \leq 1$.

   For $p \leq 0$, $\lim \frac{(-1)^{n+1}}{n^p} \neq 0$. Therefore, the given series diverges.

Example 30

Does the series $1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} - \cdots$ converge?

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Solution: Here, the series has been made up from the terms $1/n^2$ by taking first one term, next two negative terms of squares of next even numbers, then three positive terms which are squares of next three odd numbers, and so on. This is a rearrangement of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$$

which is absolutely convergent (since $\sum(1/n^2)$ is convergent).

Therefore, the given series is convergent and its sum is the same as that of the alternating series $\sum(-1)^{n+1}(1/n^2)$.

Power series

Let $a \in \mathbb{R}$. A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

The point a is called the **center** of the power series and the real numbers $a_0, a_1, \cdots, a_n, \cdots$ are its **co-efficients**.

If the power series converges to $f(x)$ for all $x \in D$, for some subset D of $\mathbb{R}$, then we say that the power series **sums to** the function $f(x)$, whose domain is D.

In such a case, we also say that the power series **represents** the function $f(x)$.

Example 1

Show that the following power series converges for $0 < x < 4$.

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \frac{(-1)^n}{2^n}(x-2)^n + \cdots$$

It is a geometric series with the ratio as $r = (-1/2)(x-2)$.
Thus it converges for $|(-1/2)(x-2)| < 1$.
Simplifying we get the constraint as $0 < x < 4$.

The power series sums to

$$\frac{1}{1-r} = \frac{1}{1+\frac{1}{2}(x-2)} = \frac{2}{x}.$$

Thus, the power series gives a series expansion of the function $\frac{2}{x}$ for $0 < x < 4$.
Truncating the series to n terms give us polynomial approximations of the function $\frac{2}{x}$.

Convergence of Power series

Theorem: (1) If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c$ for some $c > 0$, then it converges absolutely for all x with $|x| < c$.
(2) If the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = d$ for some $d > 0$, then the it diverges for all x with $|x| > d$.

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Proof: (1) Suppose the power series $\sum a_n x^n$ converges for $x = c$ for some $c > 0$. Then the series $\sum a_n c^n$ converges. Thus $\lim_{n\to\infty} a_n c^n = 0$. Then we have an $m \in \mathbb{N}$ such that for all $n > m$, $|a_n c^n| < 1$.
Let $x \in \mathbb{R}$ be such that $|x| < c$. Write $t = |\frac{x}{c}|$. For each $n > m$, we have

$$|a_n x^n| = |a_n c^n| \, |\tfrac{x}{c}|^n < |\tfrac{x}{c}|^n = t^n.$$

As $0 \leq t < 1$, the geometric series $\sum_{n=0}^{\infty} t^n$ converges. By comparison test, for any x with $|x| < c$, the series $\sum_{n=0}^{\infty} |a_n x|^n$ converges. That is, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x with $|x| < c$.

(2) If the power series converges for some α with $|\alpha| > d$, then by (1), it must converge for $x = d$, a contradiction. $\square$

Radius & Interval of Convergence

For the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the real number

$R = \text{lub}\{c \geq 0 : \text{ the power series converges for all } x \text{ with } |x-a| < c\}$

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That is, R is such non-negative number that the power series converges for all x with $|x-a| < R$ and it diverges for all x with $|x-a| > R$.

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is called the **radius of convergence** of the power series.

That is, R is such non-negative number that the power series converges for all x with $|x-a| < R$ and it diverges for all x with $|x-a| > R$.

If the radius of convergence of the power series $\sum a_n(x-a)^n$ is R, then the **interval of convergence** of the power series is

$[a-R, a+R]$ if it converges at $x = a-R$ and converges at $x = a+R$.

$[a-R, a+R)$ if it converges at $x = a-R$ and diverges at $x = a+R$.

$(a-R, a+R]$ if it diverges at $x = a-R$ and converges at $x = a+R$.

$(a-R, a+R)$ if it diverges at both $x = a-R$ and $x = a+R$.

Determining R

Theorem: Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$. Then

(1) $R = \lim_{n\to\infty} |a_n|^{-1/n}$ if this limit is in $\mathbb{R} \cup \{\infty\}$.

(2) $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if this limit is in $\mathbb{R} \cup \{\infty\}$.

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Proof: (1A) Let $r = \lim_{n\to\infty} |a_n|^{1/n}$. We consider two cases.

(a) Let $|x| < \frac{1}{r}$. Then $\lim\limits_{n\to\infty} |a_n x^n|^{1/n} = |x| \lim\limits_{n\to\infty} |a_n|^{1/n} = |x| \cdot \frac{1}{r} < 1$.

By the root test, the series $\sum a_n x^n$ is absolutely convergent, and hence, convergent.

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By the root test, the series $\sum a_n x^n$ is absolutely convergent, and hence, convergent.

(b) Let $|x| > \frac{1}{r}$. Choose α such that $\frac{1}{r} < \alpha < |x|$. If the power series converges for x, then by the convergence theorem for power series, the series $\sum |a_n \alpha^n|$ converges. However,

$$\lim_{n\to\infty} |a_n \alpha^n|^{1/n} = |\alpha| \lim_{n\to\infty} |a_n|^{1/n} = |\alpha| \cdot \frac{1}{r} > 1.$$

By the root test, $\sum |a_n \alpha^n|$ diverges. This is a contradiction.
Therefore, The power series $\sum a_n x^n$ diverges when $|x| > \frac{1}{r}$.
From (a) and (b), we conclude that $R = \frac{1}{r}$.

Determining R Cont.

(1B) Let $r = \infty$. Then for any $x \neq a$,

$$\lim |a_n(x-a)^n| = \lim |x-a||a_n|^{1/n} = \infty.$$

By Root test, $\sum a_n(x-a)^n$ diverges for each $x \neq a$. Thus, $R = 0$.

Determining R Cont.

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(1C) Let $r = 0$. Then for any $x \in \mathbb{R}$,

$$\lim |a_n(x-a)^n|^{1/n} = |x-a| \lim |a_n|^{1/n} = 0.$$

By Root test, the series converges for each $x \in \mathbb{R}$. So, $R = \infty$.

This proves (1).

For (2), use Ratio test instead of Root test. $\square$

Special Property

Theorem: $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} |a_n| x^n$ have equal radii of convergence.

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Proof: Let R be the radius of convergence of $\sum |a_n| x^n$, and let r be the radius of convergence of $\sum a_n x^n$.
Suppose $R \in \mathbb{R}$. If $\sum |a_n| x^n$ converges for $x = c$, then $\sum a_n c^n$ is absolutely convergent; therefore, $\sum a_n x^n$ converges for $x = c$.
So, $R \leq r$.

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Proof: Let R be the radius of convergence of $\sum |a_n| x^n$, and let r be the radius of convergence of $\sum a_n x^n$.
Suppose $R \in \mathbb{R}$. If $\sum |a_n| x^n$ converges for $x = c$, then $\sum a_n c^n$ is absolutely convergent; therefore, $\sum a_n x^n$ converges for $x = c$.
So, $R \leq r$.
Now, if $R < r$, then choose β such that $R < \beta < r$. Now, $\sum |a_n| \beta^n$ diverges. By the convergence theorem for power series, $\sum a_n \beta^n$ converges absolutely. This is a contradiction. Therefore, $R = r$.
When $R = \infty$, then the power series $\sum a_n x^n$ is absolutely convergent for every $x \in \mathbb{R}$. Therefore, $\sum a_n x^n$ converges for every $x \in \mathbb{R}$.
That is, $r = \infty$. $\square$

Example 2

For what values of x, do the following power series converge?

$$\text{(a)} \quad \sum_{n=0}^{\infty} n! x^n \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$a_n = n!$. Thus $\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{1}{n+1} = 0$.

Hence $R = 0$.

That is, the series is only convergent for $x = 0$.

(b) $a_n = 1/n!$. Thus $\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim(n+1) = \infty$.

Hence $R = \infty$. That is, the series is convergent for all $x \in \mathbb{R}$.

Example 3

Consider the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

Think of it as

$$x\left(1 - \frac{x^2}{3} + \frac{x^4}{5} + \cdots\right) = x\sum_{n=0}^{\infty}(-1)^n \frac{t^n}{2n+1} \quad \text{for } \; t = x^2$$

Then $a_n = (-1)^n/(2n+1)$. $\lim |a_n/a_{n+1}| = \lim \frac{2n+3}{2n+1} = 1$.

Hence $R = 1$. That is, for $|t| = x^2 < 1$, the series converges. For $|t| = x^2 > 1$, the series diverges.

Notice that $|t| = x^2 < 1$ means $-1 < x < 1$.

For $x = -1$, the (original) power series is an alternating series; it converges due to Leibniz. Similarly, for $x = 1$, the alternating series also converges.
Hence the interval of convergence for the original power series (in x) is $[-1, 1]$.

Operations with power series

Theorem: Let the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ have radius of convergence $R > 0$. Then the power series defines a function $f : (a-R, a+R) \to \mathbb{R}$. Further, $f'(x)$ and $\int f(x)dx$ exist as functions from $(a-R, a+R)$ to $\mathbb{R}$ and these are given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \quad f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}, \quad \int f(x)dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C$$

where all the three power series converge for all $x \in (a-R, a+R)$.

Theorem: Let the power series $\sum a_n x^n$ and $\sum b_n x^n$ have the same radius of convergence $R > 0$. Then their multiplication has the same radius of convergence R. Moreover, the functions they define satisfy the following: If $f(x) = \sum a_n x^n, \ g(x) = \sum b_n x^n$, then

$$f(x)g(x) = \sum c_n x^n \quad \text{for } \ a-R < x < a+R$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0$.

Example 4

(a) Determine power series expansions of (a) $\dfrac{2}{(x-1)^3}$ (b) $\tan^{-1} x$.

(a) For $-1 < x < 1$, $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.

Differentiating term by term, we have

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Differentiating once more, we get

$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \quad \text{for } -1 < x < 1.$$

(b) $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad$ for $|x^2| < 1$.

Integrating term by term and evaluating at $x = 0$, we have

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } -1 < x < 1.$$

Taylor's Formula

For a function $f(x)$, the **Taylor's polynomial** is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem: Let $n \in \mathbb{N}$. Suppose that $f^{(n)}(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(x) = p(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Integral Form: Let $f(x)$ be an $(n+1)$-times continuously differentiable function on an open interval I containing a. Let $x \in I$. Then

$$f(x) = p(x) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

An estimate for the error $R_n(x) = f(x) - p(x)$ is given by

$$\frac{m\, x^{n+1}}{(n+1)!} \le R_n(x) \le \frac{M\, x^{n+1}}{(n+1)!},$$

with $m \le f^{n+1}(x) \le M$ for $x \in I$.

Taylor Series

You have seen the proof of the differential form in MA1010. The integral form is proved by using induction on n. The basis case is fundamental theorem of calculus. The induction step is proved using integration by parts.

When $f(x)$ is infinitely differentiable and the error $R_n(x) \to 0$ as $n \to \infty$, the function $f(x)$ is represented by a power series, called **Taylor series** at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Conversely, if a function $f(x)$ has a power series expansion about $x = a$, then the power series is same as the Taylor series. Reason?

Take the power series. Differentiate repeatedly and evaluate at $x = a$. You get the co-efficients of the power series as $\frac{f^{(n)}(a)}{n!}$. When $x = 0$, we get the **Maclaurin Series**:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Example 5

Find the Taylor series expansion of the function $f(x) = 1/x$ at $x = 2$.
In which interval around $x = 2$, the series converges?

$$f(x) = x^{-1}, f(2) = \frac{1}{2}; \cdots ; f^{(n)}(x) = (-1)^n n! x^{-(n+1)}, f^{(n)}(2) = (-1)^n n! 2^{-(n+1)}.$$

Hence the Taylor series for $f(x) = 1/x$ is

$$\frac{1}{2} - \frac{x-2}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

It is a geometric series with ratio $r = -(x-2)/2$. It converges absolutely for

$$|r| < 1, \quad \text{i.e.,} \quad |x-2| < 2 \quad \text{i.e.,} \quad 0 < x < 4.$$

Does this convergent series converge to the given function?

For any c, x in an interval around $x = 2$,

$$|R_n| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-2)^{n+1} \right| = \left| \frac{(x-2)^{n+1}}{c^{n+2}} \right|$$

Here, c lies between x and 2. If x is near 2, then $|R_n| \to 0$.
Hence the Taylor series represents the function near $x = 2$.

Example 6

Consider the function $f(x) = e^x$. We find that

$$f(0) = 1,\ f'(0) = 1, \cdots, f^{(n)}(0) = 1, \cdots$$

Hence its Taylor series is

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots .$$

By the ratio test, this power series has the radius of convergence

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty.$$

Therefore, for every $x \in \mathbb{R}$ the above series converges. Using the integral form of the remainder,

$$|R_n(x)| = \left| \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt \right| = \left| \int_0^x \frac{(x-t)^n}{n!} e^t\, dt \right| \to 0 \ \text{ as } \ n \to \infty.$$

Hence, for each $x \in \mathbb{R}$,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots .$$

Example 7

Similarly, Taylor series for $\cos x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

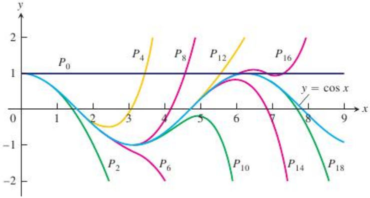
The absolute value of the remainder in the differential form is

$$|R_{2n}(x)| = \frac{|x|^{2n+1}}{(2n+1)!} \to 0 \text{ as } n \to \infty$$

for any $x \in \mathbb{R}$. Hence the series represents $\cos x$ for each $x \in \mathbb{R}$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for each } x \in \mathbb{R}.$$

Taylor polynomials approximating $\cos x$ for $0 \le n \le 9$ are



Example 8

Let $m \in \mathbb{R}$. Consider the function $f(x) = (1+x)^m$. The derivatives are

$$f(x) = (1+x)^m, \quad f^{(n)}(x) = m(m-1)\cdots(m-n+1)x^{m-n}.$$

Show that the series converges for $-1 < x < 1$. Also, the remainder term in the Maclaurin series expansion goes to 0 as $n \to \infty$. So, for $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n, \quad \text{where } \binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

The series so obtained is called a **binomial series** expansion of $(1+x)^m$.

For example, with $m = 1/2$, we have

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots \quad \text{for } -1 < x < 1.$$

When $m \in \mathbb{N}$, the binomial series terminates to give a polynomial and it represents $(1+x)^m$ for each $x \in \mathbb{R}$.

Trigonometric Series

In the power series $\sin x = x - x^3/3! + \cdots$, the periodicity of $\sin x$ is not obvious. Recall: A function $f(x)$ is 2ℓ-**periodic** for $\ell > 0$ iff $f(x + 2\ell) = f(x)$ for all $x \in \mathbb{R}$.

For the time being, we consider 2π-periodic functions.

A **trigonometric series** is of the form $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

If it converges to $f(x)$, then $f(x)$ is also 2π-periodic.

How to get the coefficients a_n and b_n?

multiply $f(t)$ by $\cos mt$ and integrate to obtain:

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)\cos mt\,dt &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos mt\,dt + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nt \cos mt\,dt \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nt \cos mt\,dt.\end{aligned}$$

Coefficients in a Trigonometric series

For $m, n = 0, 1, 2, 3, \ldots,$

$$\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m > 0 \\ 2\pi & \text{if } n = m = 0 \end{cases} \quad \text{and} \quad \int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0.$$

Thus, we obtain

$$\int_{-\pi}^{\pi} f(t) \cos mt \, dt = \pi a_m, \quad \text{for all } m = 0, 1, 2, 3, \cdots$$

Similarly, by multiplying $f(t)$ by $\sin mt$ and integrating, and using the fact that

$$\int_{-\pi}^{\pi} \sin nt \sin mt \, dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m > 0 \\ 0 & \text{if } n = m = 0 \end{cases}$$

we obtain

$$\int_{-\pi}^{\pi} f(t) \sin mt \, dt = \pi b_m, \quad \text{for all } m = 1, 2, 3, \cdots$$

Fourier Series

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function integrable on $[-\pi, \pi]$.

Let $a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(t) \cos nt \, dt$ for $n = 0, 1, 2, 3, , \ldots,$

and $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(t) \sin nt \, dt$ for $n = 1, 2, 3, \ldots.$

Then the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the **Fourier series** of $f(x)$.

Notation and Terminology

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

- At any point $c \in \mathbb{R}$,
  $f(c+) = \lim_{h\to 0+} f(c+h), \quad f(c-) = \lim_{h\to 0+} f(c-h).$
- $f(x)$ has a **finite jump** at $x = c$ iff
  $f(c+)$ exists, $f(c-)$ exists, $f(c+) \neq f(c-)$.
- $f(x)$ is **piecewise continuous** iff on any finite interval $f(x)$ is continuous except for at most a finite number of finite jumps.
- At any point $c \in \mathbb{R}$, the **right hand slope** of $f(x)$ is equal to

$$\lim_{h\to 0+} \frac{f(c+h) - f(c+)}{h}.$$

Convergence of Fourier series

- At any point $c \in \mathbb{R}$, the **left hand slope** of $f(x)$ is equal to

$$\lim_{h \to 0+} \frac{f(c-h) - f(c-)}{h}.$$

- $f(x)$ is **piecewise smooth** iff $f(x)$ is piecewise continuous and $f(x)$ has both left hand slope and right hand slope at every point.

Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic piecewise smooth function. Then the Fourier series of $f(x)$ converges at each $x \in \mathbb{R}$. Further, the sum of the Fourier series $s(c)$ at any point $c \in \mathbb{R}$ is as follows:

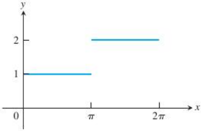
1. if $f(x)$ is continuous at c, then $s(c) = f(c)$; and
2. if $f(x)$ is not continuous at c, then $s(c) = \frac{1}{2}[f(c+) + f(c-)]$.

Fourier series can represent functions which cannot be represented by a Taylor series, or a conventional power series; for example, a step function.

Example 9

Consider the function $f(x)$ given by the following which is extended to $\mathbb{R}$ with the periodicity 2π:

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \pi \\ 2 & \text{if } \pi \le x < 2\pi \end{cases}$$



Due to periodic extension, rewrite the function $f(x)$ on $[-\pi, \pi)$ as

$$f(x) = \begin{cases} 2 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x < \pi. \end{cases}$$

The coefficients of the Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\,dt = 3.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 2\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \cos nt\,dt = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 2\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin nt\,dt = \frac{(-1)^n - 1}{n\pi}.$$

Example 9 Cntd.

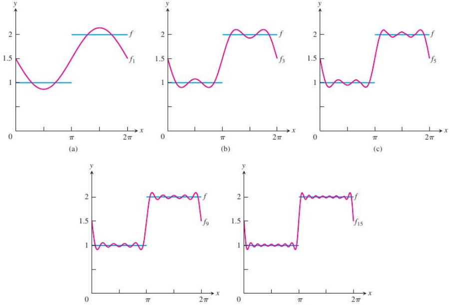
Here, $b_1 = -\frac{2}{\pi}$, $b_2 = 0$, $b_3 = -\frac{2}{3\pi}$, $b_4 = 0, \ldots$. Therefore, for all $x \in [-\pi, \pi)$ except at $x = 0$,

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

And the Fourier series sums to $3/2$ at $x = 0$.
Write the mth partial sum of the series as

$$f_m(x) = \frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx).$$

The approximations $f_1(x)$, $f_3(x)$, $f_5(x)$, $f_9(x)$ and $f_{15}(x)$ to $f(x)$ are



Example 10

Consider the function $f(x) = x^2$ defined on $[0, 2\pi)$ and extended to $\mathbb{R}$ with periodicity 2π. Here, we take $f(2\pi) = f(0) = 0$. Then

$$f(-\pi) = f(-\pi + 2\pi) = f(\pi) = \pi^2,$$

$$f(-\pi/2) = f(-\pi/2 + 2\pi) = f(3\pi/2) = (3\pi/2)^2.$$

Thus the function $f(x)$ on $[-\pi, \pi)$ is defined by

$$f(x) = \begin{cases} (x + 2\pi)^2 & \text{if } \; -\pi \leq x < 0 \\ x^2 & \text{if } \; 0 \leq x < \pi. \end{cases}$$

Notice that $f(x)$ is neither odd nor even. The coefficients of the Fourier series for $f(x)$ are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\pi} \int_0^{2\pi} t^2\, dt = \frac{8\pi^2}{3}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos nt\, dt = \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \sin nt\, dt = -\frac{4\pi}{n}.$$

Example 10 Cntd.

Hence

$$f(x) = \frac{4\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{4}{\pi^2} \cos nx - \frac{4\pi}{n} \sin nx \right).$$

Due to the periodic extension, in the interval $(2k\pi, 2(k+1)\pi)$, $f(x) = (x - 2k\pi)^2$.

It has discontinuities at the points $x = 0, \ \pm 2\pi, \ \pm 4\pi, \ \ldots$

At such a point $x = 2k\pi$, the series converges to the average value of the left and right side limits, i.e., the series when evaluated at $2k\pi$ yields the value

$$\begin{aligned} & \frac{1}{2}\left[\lim_{x \to 2k\pi-} f(x) + \lim_{x \to 2k\pi+} f(x) \right] \\ = \ & \frac{1}{2}\left[\lim_{x \to 2k\pi-} (x - 2k\pi)^2 + \lim_{x \to 2k\pi+} (x - 2(k+1)\pi)^2 \right] = 2\pi^2. \end{aligned}$$

Notice that since $f(x)$ is extended by periodicity, whether we take the basic interval as $[-\pi, \pi]$ or as $[0, 2\pi]$ does not matter in the calculation of coefficients.

Odd and even functions

If $f(x)$ and $f'(x)$ are continuous on $[-\pi, \pi)$ with period 2π and if $f(x)$ is an odd function on $(-\pi, \pi)$, i.e., $f(-x) = -f(x)$, then for $n = 0, 1, 2, 3, \ldots,$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{for all } \ x \in \mathbb{R}.$$

$$\text{with } \ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt.$$

If $f(x)$ and $f'(x)$ are continuous on $[-\pi, \pi)$ with period 2π and if $f(x)$ is an even function, i.e., $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{for all } \ x \in \mathbb{R}$$

$$\text{with } \ a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt \quad \text{for } \ n = 0, 1, 2, 3, \ldots$$

Example 11

Consider $f(x) = x^2$ for $-\pi \leq x \leq \pi$. Its periodic extension to $\mathbb{R}$ is not the function $f(x) = x^2$. For example, in the interval $[\pi, 3\pi]$, its extension looks like $f(x) = (x - 2\pi)^2$. Notice that $f(x)$ is an even function. The coefficients of the cosine series are as follows:

$$a_0 = \frac{2}{\pi} \int_0^\pi t^2 \, dt = \frac{2}{3}\pi^2.$$

$$a_n = \frac{2}{\pi} \int_0^\pi t^2 \cos nt \, dt = \frac{4}{n^2}(-1)^n.$$

Therefore,

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2} \quad \text{for all } \ x \in [-\pi, \pi].$$

In particular, by taking $x = 0$ and $x = \pi$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 12

Show that for $0 < x < 2\pi$, $\quad \dfrac{1}{2}(\pi - x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

Let $f(x) = x$ for $0 \leq x < 2\pi$. Extend $f(x)$ to $\mathbb{R}$ by taking the periodicity as 2π and with the condition that $f(2\pi) = f(0)$.

Here, $f(x)$ is not an odd function;
$f(-\pi/2) = f(3\pi/2) = 3\pi/2 \neq f(\pi/2) = \pi/2$.

The coefficients of the Fourier series for $f(x)$ are as follows:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = 0.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{1}{\pi}\left[\frac{-n\cos nt}{n}\right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi} \cos nt\,dt = -\frac{2}{n}.$$

By the convergence theorem, $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ for $0 < x < 2\pi$, which yields the required result.

Functions on $[0, \pi]$

Suppose a function $f : [0, \pi] \to \mathbb{R}$ is given. To find its Fourier series, we need to extend it to $\mathbb{R}$ so that the extended function is 2π-periodic Such an extension can be done in many ways.

1. Odd Extension:

First, extend $f(x)$ from $[0, \pi)$ to $[-\pi, \pi]$ by requiring that $f(x)$ is an odd function. This requirement forces $f(-x) = -f(x)$ for each $x \in [-\pi, \pi]$. In particular, the extended function $f(x)$ will satisfy $f(-\pi) = -f(\pi) = f(\pi)$ leading to $f(-\pi) = f(\pi) = 0$. Next, we extend this $f(x)$ which has now been defined on $[-\pi, \pi]$ to $\mathbb{R}$ with periodicity 2π. Then the Fourier series will represent the function on $[0, \pi)$. Notice that if $f(\pi)$ is already 0, then the Fourier series will represent $f(x)$ on $[0, \pi]$.

The Fourier series expansion of this extended $f(x)$ is a sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt, \quad n = 1, 2, 3, \ldots, \quad x \in \mathbb{R}.$$

We say that this Fourier series is a **sine series expansion** of $f(x)$.

Functions on $[0, \pi]$ Cont.

2. Even Extension:

First, extend $f(x)$ to $[-\pi, \pi]$ by requiring that $f(x)$ is an even function. This requirement forces $f(-x) = f(x)$ for each $x \in [-\pi, \pi]$.

Next, we extend this $f(x)$ which has now been defined on $[-\pi, \pi]$ to $\mathbb{R}$ with periodicity 2π.

The Fourier series expansion of this extended $f(x)$ is a cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt, \quad n = 0, 1, 2, 3, \ldots,$$

In this case, we say that the Fourier series is a **cosine series expansion** of $f(x)$.

Example 13

Find the Fourier series for $f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 \le x \le \pi. \end{cases}$

1. With an odd extension, the Fourier coefficients are given by

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - t)\sin nt\,dt \\
&= \frac{2}{\pi}\Big[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\Big]_0^{\pi/2} + \frac{2}{\pi}\Big[\frac{t-\pi}{n}\cos nt - \frac{1}{n^2}\sin nt\Big]_{\pi/2}^{\pi} \\
&= \frac{2}{\pi}\Big(-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big) + \frac{2}{\pi}\Big(\frac{\pi}{2n}\cos\frac{2\pi}{n} + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big) \\
&= \frac{4}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} \frac{4}{\pi n^2}(-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}$$

Thus $f(x) = \frac{4}{\pi}\Big(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\Big), \quad \text{for } x \in [0, \pi].$

Example 13 Cntd.

2. With an even extension, the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt \\ &= \frac{2}{\pi}\int_0^{\pi/2} t\cos nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - t)\cos nt\,dt \\ &= \begin{cases} \pi/4 & n = 0 \\ 0 & n = 4k,\ k \geq 1 \\ -\frac{2}{n^2\pi} & n \neq 4k,\ k \geq 1. \end{cases} \end{aligned}$$

Thus

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right), \quad \text{for } x \in [0, \pi].$$

Sine expansion of cosine in $[0, \pi]$

Find the Fourier sine expansion of $\cos x$ in $[0, \pi]$.

We work with the odd extension of $\cos x$ with period 2π to $\mathbb{R}$.

The Fourier coefficients a_n are 0, and b_n are given by

$$b_1 = \frac{2}{\pi} \int_0^{\pi} \cos t \sin t \, dt = 0.$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos t \sin nt \, dt = \begin{cases} 0 & \text{for } n \text{ odd} \\ \frac{4n}{\pi(n^2-1)} & \text{for } n \text{ even.} \end{cases}$$

Therefore, $\cos x = \dfrac{8}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{n \sin(2nx)}{4n^2 - 1}$ for $x \in [0, \pi]$.

Similarly, you can find out that $\sin x = \dfrac{2}{\pi} - \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}$ for $x \in [0, \pi]$. This is cosine expansion of $\sin x$ in $[0, \pi]$.

Scaling Approach

There is a third approach in extending the function from $[0, \pi]$ to $\mathbb{R}$ keeping periodicity as 2π.

3. Scaling to length 2π:

We define a bijection $g : [-\pi, \pi] \to [0, \pi]$. Then consider the composition $h = (f \circ g) : [-\pi, \pi] \to \mathbb{R}$. We find the Fourier series for $h(y)$ and substitute $y = g^{-1}(x)$ for obtaining Fourier series for $f(x)$. Notice that in computing the Fourier series for $h(y)$, we must extend $h(y)$ to $\mathbb{R}$ using 2π-periodicity.

We consider Example 13 once again to illustrate this method of scaling. There, we had

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 \le x \le \pi. \end{cases}$$

To scale the interval $[0, \pi]$ to $[-\pi, \pi]$, we define a bijection $g : [-\pi, \pi] \to [0, \pi]$ given by $g(y) = \frac{1}{2}(y + \pi)$, $\quad g^{-1}(x) = 2x - \pi$.

Example 13 with scaling

$$x = g(y) = \frac{1}{2}(y+\pi), \quad h(y) = f\left(\frac{y+\pi}{2}\right) = \begin{cases} \frac{1}{2}(y+\pi) & \text{if } -\pi \le y \le 0 \\ \frac{1}{2}(\pi - y) & \text{if } 0 \le y \le \pi. \end{cases}$$

Notice that the function $h(y)$ happens to be an even function here. Thus $b_n = 0$ and other Fourier coefficients are as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} \frac{t+\pi}{2}\,dt + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi - t}{2}\,dt = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \frac{t+\pi}{2}\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi - t}{2}\cos nt\,dt = \begin{cases} \frac{2}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Then the Fourier series for $h(y)$ is given by $\displaystyle \frac{\pi}{4} + \sum_{n \text{ odd}} \frac{2}{\pi n^2} \cos ny.$

Using $y = g^{-1}(x) = 2x - \pi$, we have

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos\left((2n+1)(2x-\pi)\right) \quad \text{for } x \in [-\pi, \pi].$$

In general, such a series obtained by scaling need neither be a sine series nor a cosine series.

Half-Range expansion

Suppose a real valued function $f(x)$ is only defined on an interval $(0, \ell)$. Then all the three approaches are applicable.

- We take an odd extension of $f(x)$, with the domain as $(-\ell, \ell)$. Then we scale $(-\ell, \ell)$ to $(-\pi, \pi)$; extend it to $\mathbb{R}$ using 2π-periodicity. Finally construct the Fourier series of this extended function. This is called the *half range sine expansion*.
- We take an even extension of $f(x)$, with the domain as $(-\ell, \ell)$. Then we scale $(-\ell, \ell)$ to $(-\pi, \pi)$; extend it to $\mathbb{R}$ using 2π-periodicity. Finally construct the Fourier series of this extended function. This is called the *half range cosine expansion*.
- We scale $(0, \ell)$ to $(-\pi, \pi)$; extend it to $\mathbb{R}$ using 2π-periodicity. Finally construct the Fourier series of this extended function. This is the *scaling approach*.

We may also use the interval $[-\ell, \ell]$ directly in the integrals while evaluating the Fourier coefficients instead of first scaling to $[-\pi, \pi]$ and then constructing the Fourier series.

In $[-\ell, \ell]$

In case, our function is defined on $(-\ell, \ell)$, having period as 2ℓ, we consider $g(x) = f(\ell x/\pi)$. Now, $g : [-\pi, \pi] \to \mathbb{R}$ is 2π-periodic. It has the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big),$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\Big(\frac{\ell}{\pi} s\Big) \cos ns \, ds, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\Big(\frac{\ell}{\pi} s\Big) \cos ns \, ds.$$

Substituting $t = \frac{\ell}{\pi} s, \quad ds = \frac{\pi}{\ell} dt$, we have

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(t) \cos\left(\frac{n\pi t}{\ell}\right) dt, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(t) \sin\left(\frac{n\pi t}{\ell}\right) dt.$$

And the Fourier series for $f(x)$ is then obtained by substituting x with $\pi x/\ell$ in the above Fourier series. It is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi}{\ell} x + b_n \sin \frac{n\pi}{\ell} x \Big).$$

Example 14

Construct the Fourier series for $f(x) = |x|$ for $x \in [-\ell, \ell]$ for some $\ell > 0$.

We extend the given function to $f : \mathbb{R} \to \mathbb{R}$ with period 2ℓ. Here, $f(x)$ is not $|x|$ on $\mathbb{R}$; it is $|x|$ on $[-\ell, \ell]$. Due to its period as 2ℓ, it is $|x - 2\ell|$ on $[\ell, 3\ell]$ etc.

It is an even function. Thus all b_n are 0. The Fourier coefficients a_n are

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} |s|\, ds = \frac{2}{\ell}\int_0^{\ell} s\, ds = \ell,$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} s\cos\left(\frac{n\pi s}{\ell}\right) ds = \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{4\ell}{n^2\pi^2} & \text{if } n \text{ odd} \end{cases}$$

Fourier series for $f(x)$ in $[-\ell, \ell]$ is

$$|x| = \frac{\ell}{2} - \frac{4\ell}{\pi^2}\left[\frac{\cos(\pi/\ell)x}{1} + \frac{\cos(3\pi/\ell)x}{3^2} + \cdots + \frac{\cos((2n+1)\pi/\ell)x}{(2n+1)^2} + \cdots\right]$$