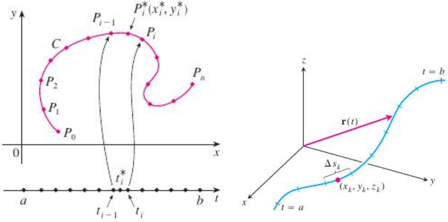
MA 1101

Functions of Several Variables

Vector Integrals

Line Integral

Line integrals are single integrals which are obtained by integrating a function over a curve instead of integrating over an interval.



Let $f(x, y, z)$ be a real valued function with region D. Let C be a curve that lies in D given in parametric form as

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \quad a \le t \le b.$$

The values of f on the curve C are given by the composite function $f(x(t), y(t), z(t))$. We want to integrate this composite function on the curve C.

How to proceed?

Partition C into n subarcs. Choose a point (x_k, y_k, z_k) on the kth subarc. Suppose the kth subarc has length Δs_k. Form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k.$$

Suppose the partition is such that when n approaches ∞, the length s_k approaches 0.
If $\lim_{n\to\infty} S_n$ exists, then this limit is called the line integral of f over the curve C. We write

$$\int_C f(x, y, z)ds = \lim_{n\to\infty} S_n.$$

Notice that the line integral is computed by parameterizing the curve C.

How to Compute it?

Theorem 1: Let $C: \quad \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \quad a \le t \le b$ be a parametrization of the curve C lying in a region $D \subseteq \mathbb{R}^3$.
Let $f: D \to \mathbb{R}$ have continuous partial derivatives and let the component functions $x(t), y(t), z(t)$ have continuous derivatives. Then the line integral of f over C exists, and it is given by

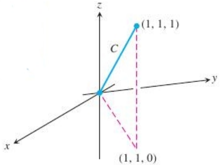
$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

We also write

$$ds = |\vec{r}\,'(t)|dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

Examples

Example 1: Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment from the origin to the point $(1, 1, 1)$.



Parametrize the curve C : $\vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath} + t\,\hat{k}, \quad 0 \le t \le 1.$

Then $x(t) = y(t) = z(t) = t$. So, $|\vec{r}\,'(t)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

$$\int_C f ds = \int_0^1 \left(x(t) - 3y^2(t) + z(t)\right)\sqrt{3}dt = \int_0^1 (t - 3t^2 + t)\sqrt{3}dt = 0.$$

Example 2: Evaluate $\int_C (2 + x^2 y) ds$, where C is the upper half of the unit circle in the xy-plane.

Here, $f = f(x, y) = 2 + x^2 y$ is a function of two variables.
Parametrize the curve. C : $x(t) = \cos t,\ y(t) = \sin t, \quad 0 \le t \le \pi.$

Then

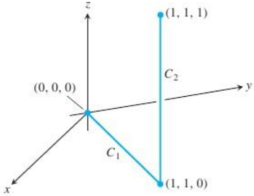
$$\int_C (2 + x^2 y) ds = \int_0^{\pi} (2 + \cos^2 t \ \sin t)\sqrt{(x'(t))^2 + (y'(t))^2} dt = 2\pi + 2/3.$$

Piecewise Smooth Curve

If C is a piecewise smooth curve, i.e., it is a join of finite number of smooth curves, written as $C = C_1 \cup \cdots \cup C_m$, then we define

$$\int_C f(x,y,z)ds = \int_{C_1} f(x,y,z)ds + \cdots + \int_{C_m} f(x,y,z)ds.$$

Example 3: Let C be the curve consisting of line segments joining $(0,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$. Evaluate $\int_C (x - 3y^2 + z)ds$.



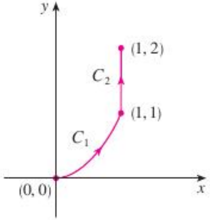
Parametrize. The curve C is the join of C_1 and C_2, where
$C_1 : \vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath},\ 0 \le t \le 1;$
$C_2 : \vec{r}(t) = \hat{\imath} + \hat{\jmath} + t\,\hat{k},\ 0 \le t \le 1.$
Then On C_1, $|\vec{r}\,'(t)| = \sqrt{2}$ and on C_2, $|\vec{r}\,'(t)| = 1$. Now,

$$\int_C (x - 3y^2 + z)ds = \int_{C_1} (x - 3y^2 + z)ds + \int_{C_2} (x - 3y^2 + z)ds$$

$$= \int_0^1 (t - 3t^2 + 0)\sqrt{2}dt + \int_0^1 (1 - 3 + t)dt = \frac{-\sqrt{2} - 3}{2}.$$

Examples Contd.

Example 4: Evaluate $\int_C 2x\,ds$, where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment joining $(1,1)$ to $(1,2)$.



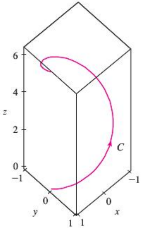
Here, $C = C_1 \cup C_2$ with

$$C_1 : x = t,\ y = t^2,\ \ 0 \le t \le 1; \quad C_2 : x = 1,\ y = t,\ \ 1 \le t \le 2.$$

Then $\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$

$$= \int_0^1 2t\sqrt{1^2 + (2t)^2}dx + \int_1^2 2\sqrt{0^2 + 1^2}dt = \frac{5\sqrt{5}-1}{6} + 2.$$

Example 5: Evaluate $\int_C y \sin z\,ds$, where C is the circular helix given by
$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = t,\ \ 0 \le t \le 2\pi.$



$$\int_C y \sin z\,ds = \int_0^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \sqrt{2}\pi.$$

Line Integral on x-axis

If the curve C happens to be a line segment on the x-axis, then $ds = dx$.
In that case, the line integral over the curve becomes

$$\int_C f(x, y, z)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta x_k.$$

In this case, we see that

$$\int_C f(x, y, z)\, ds = \int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t)) x'(t)\, dt.$$

Other Line Integrals

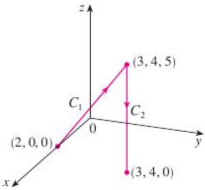
We generalize this observation and define the Line integrals of f over C with respect to $x, y, z,$ as follows:

$$\begin{aligned}
\int_C f(x,y,z)\,dx &= \int_a^b f(x(t),y(t),z(t))x'(t)\,dt,\\
\int_C f(x,y,z)\,dy &= \int_a^b f(x(t),y(t),z(t))y'(t)\,dt,\\
\int_C f(x,y,z)\,dz &= \int_a^b f(x(t),y(t),z(t))z'(t)\,dt.
\end{aligned}$$

Notice that in order to be meaningful, we assume here that $f(x,y,z)$ has continuous first order partial derivatives and $\vec{r}(t)$ is smooth, that is, if C has parmeterization as $x = x(t),\ y = y(t),\ z = z(t),\ \ a \le t \le b$, then $dx/dt,\ dy/dt$ and dz/dt are continuous.

Example 6

Evaluate $\int_C ydx + zdy + xdz$, where C is the curve joining the line segments from $(2,0,0)$ to $(3,4,5)$ to $(3,4,0)$.



Parameterize: $C = C_1 \cup C_2$, where
$C_1 : \; x = 2 + t, \; y = 4t, \; z = 5t, \quad 0 \le t \le 1;$
$C_2 : \; x = 3, \; y = 4, \; z = 5 - 5t, \quad 0 \le t \le 1.$

Then

$$\int_C ydx + zdy + xdz = \int_{C_1} ydx + zdy + xdz + \int_{C_2} ydx + xdz + zdx$$

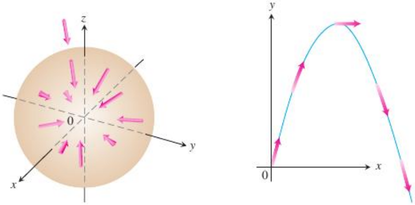
$$= \int_0^1 (4t)\,dt + (5t)4\,dt + (2+t)5\,dt \; + \int_0^1 3(-5)\,dt = 49/2 - 15 = 9.5.$$

Vector Fields

We want to generalize line integrals to vector fields.

A vector field is a function defined on a region D in the plane or space that assigns a vector to each point in D. If D is a region in space, a vector field on D may be written as

$$F(x, y, z) = M(x, y, z)\,\hat{\imath} + N(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k}.$$



Vectors in a gravitational field point toward the center of mass that gives the source of the field.

The velocity vectors on a projectile's motion make a vector field along the trajectory.

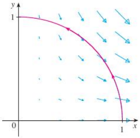
Line Integral of Vector Fields

Let $F(x,y,z)$ be a continuous vector field defined over a curve C given by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ for $a \le t \le b$. The line integral of F along C, also called the work done by moving a particle on C under the force field F is

$$\int_C F \cdot d\vec{r} = \int_C F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C F \cdot \hat{T}\,ds,$$

where $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the unit tangent vector at points on C.

Example 7: Evaluate the line integral of the vector field $F(x,y,z) = x^2\,\hat{\imath} - xy\,\hat{\jmath}$ along the first quarter unit circle in the first quadrant.



The curve C is given by $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$, $0 \le t \le \pi/2$. Then

$$F(\vec{r}(t)) = \cos^2 t\,\hat{\imath} - \cos t \sin t\,\hat{\jmath} \quad \text{and} \quad \vec{r}\,' = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}.$$

The work done is $\displaystyle\int_C F \cdot d\vec{r} = \int_0^{\pi/2} F(\vec{r}) \cdot \vec{r}\,'\,dt = \frac{-2}{3}.$

Line Integrals - Vector fields and Scalar fields

Let the vector filed F be given by

$$F(x,y,z) = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}.$$

Suppose the curve C is given by

$$C: \quad \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \qquad a \le t \le b.$$

Then

$$\begin{aligned} \int_C F \cdot d\vec{r} &= \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \\ &= \int_a^b [M(x(t), y(t), z(t))\,x'(t) + N\,y'(t) + P\,z'(t)]\,dt \\ &= \int_C Mdx + Ndy + Pdz. \end{aligned}$$

This formula connects the line integral of a vector field to the line integrals of the component scalar fields.

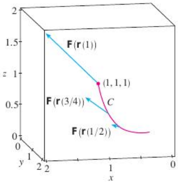
Example 8

Evaluate $\int_C F \cdot d\vec{r}$, where

$$F = xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$$

and C is the twisted cube given by

$$x = t,\ y = t^2,\ z = t^3,\quad 0 \le t \le 1.$$



$$\int_C M dx = \int_0^1 t\, t^2\, 1\, dt = 1/4,$$

$$\int_C N dy = \int_0^1 t^2\, t^3\, 2t\, dt = 2/7,$$

$$\int_C P dz = \int_0^1 t^3\, t\, 3t^2\, dt = 3/7.$$

So, $\int_C F \cdot d\vec{r} = 1/4 + 2/7 + 3/7 = 27/28.$

Also, a direct computation shows

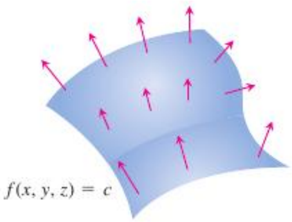
$$\int_C F \cdot d\vec{r} = \int_0^1 [xyx' + yzy' + zxz']dt = \int_0^1 [t^3 + 2t^6 + 3t^6]dt = 27/28.$$

Gradient Field

The gradient field of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}.$$

The gradient field of the surface $f(x, y, z) = c$ may be drawn (typical) as:



At each point on the surface, we have a vector, the gradient vector, which is normal to the surface. And we draw it there itself to show it.

For example, the gradient field of $f(x, y, z) = xyz$ is

$$\text{grad} f = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}.$$

Conservative Fields

A vector field F is called conservative if there exists a scalar function f such that $F = \text{grad}\, f$. In such a case, the scalar function f is called the potential of the vector field F.

For example, the gravitational force field $F = -\frac{mMG}{|r|^3}\vec{r}$ or written in the $F(x, y, z)$ form:

$$F(x, y, z) = -\frac{mMG}{(x^2 + y^2 + z^2)^{3/2}}[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}]$$

is a conservative field. Reason?

Define $f(x, y, z) = \dfrac{mMG}{(x^2 + y^2 + z^2)^{1/2}}$. Then

$$\text{grad}\, f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k} = F.$$

Physically, the law of conservation of energy holds in every conservative field.

Fundamental Theorem for Line Integrals

Recall: $\int_a^b f'(t)dt = f(b) - f(a)$ for a function $f(t)$. Gradient acts as a sort of derivative.

Theorem 2: Let C be a smooth curve given by

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k} \text{ for } a \le t \le b.$$

Suppose C joins points (x_1, y_1, z_1) to (x_2, y_2, z_2). That is,

$$\vec{r}(a) = x_1\,\hat{\imath} + y_1\,\hat{\jmath} + z_1\,\hat{k} \quad \text{and} \quad \vec{r}(b) = x_2\,\hat{\imath} + y_2\,\hat{\jmath} + z_2\,\hat{k}.$$

Let $f(x, y, z)$ be a function whose gradient vector is continuous on a region containing C. Then

$$\int_C \nabla f \cdot d\,\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

Proof: $\int_C \nabla f \cdot d\,\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$= \int_a^b [\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}]\,dt$

$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(t))\Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a)).$ $\square$

Path Independence

Theorem 2 says that if F is a conservative vector field with potential f, then the line integral over any smooth curve joining points A to B can be evaluated from the potential: $\int_C F \cdot d\vec{r} = f(B) - f(A)$.
Observe that the line integral in such a case, is independent of path of C; it only depends on the initial and end points of C.

We say that a line integral $\int_C F \cdot d\vec{r}$ is independent of path if for any curve C' that is lying in the domain of F, and having the same initial and end points as that of C, we have $\int_C F \cdot d\vec{r} = \int_{C'} F \cdot d\vec{r}$.

The line integral $\int_C F \cdot d\vec{r}$ is path independent if F is conservative.

Example 9: Find the line integral of the field $F = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}$ along any smooth curve joining the points $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Notice that F is conservative since $F = \text{grad}\,(xyz)$. That is, with $f = xyz$, we have $F = \nabla f$. Let C be any smooth curve. Then

$$\int_C F \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A) = 3.$$

Path independence implies Conservative

As a corollary to Theorem 2, we have the following result:

Theorem 3: Let F be a continuous vector field defined on a region D. Let C be any smooth curve lying in D. The line integral $\int_C F \cdot d\vec{r}$ is path independent iff $\int_{C'} F \cdot d\vec{r} = 0$ for each closed curve C' lying in D.

Note: A closed curve has same initial and end points.
In Theorems 2-3, "Smooth curve" may be replaced by "Piecewise smooth curve."
When C is a closed curve, we write

$$\int_C F \cdot d\vec{r} \quad \text{as} \quad \oint_C F \cdot d\vec{r}.$$

Theorem 4: Let F be a continuous vector field defined on an open connected region D. If $\int_C F \cdot d\vec{r}$ is path independent for each smooth curve C lying in D, then F is conservative.

We will not prove this theorem.

Checking Conservative

Consider a conservative vector field $F = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ in the plane. We have a scalar function $f(x,y)$ such that $f_x = M$, $f_y = N$. Suppose M_y and N_x are continuous.
Using Clairaut's theorem, we have $f_{xy} = M_y = f_{yx} = N_x$. That is,

$$\text{If } F = M\,\hat{\imath} + N\,\hat{\jmath} \text{ is conservative, then } M_y = N_x.$$

Similarly, in 3d, if F is a conservative vector field, then $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k} = \nabla f$ for some scalar function f. We have

$$M_y = N_x,\ N_z = P_y,\ P_x = M_z.$$

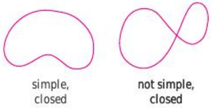
Theorem 5: Let $F(x,y,z) = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}$ be a vector field, where the component functions have continuous partial derivatives on a region D. If F is conservative, then on D, we have $M_y = N_x,\ N_z = P_y,\ P_x = M_z$.

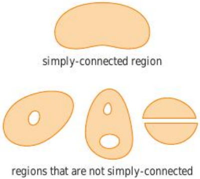
Converse of Theorem 5

The converse of Theorem 5 also holds provided the domain of F is a simply connected region.

A simple curve is a curve which does not intersect itself.

A connected region D is said to be a simply connected region if every simple closed curve lying in D encloses only points from D.





Theorem 6: Let $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ be a vector field on a simply connected region D, where M, N, P have continuous partial derivatives. If $M_y = N_x,\ N_z = P_y,\ P_x = M_z$ hold on D, then F is conservative.

Again, we omit the proof.

These equations help in determining the potential function of a conservative field.

Example 10

Are the following vector fields conservative?
(a) $F(x, y) = (x - y)\,\hat{\imath} + (x - 2)\,\hat{\jmath}$.
(b) $F(x, y) = (3 + 2xy)\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$.
(c) $F(x, y, z) = (2x - 3)\,\hat{\imath} + z\,\hat{\jmath} + \cos z\,\hat{k}$.

(a) $F = M\,\hat{\imath} + N\,\hat{\jmath}$, where $M = x - y,\ N = x - 2$.
$M_y = -1,\ N_x = 1$. Since $M_y \neq N_x$, it is not a conservative field.

(b) Here, $M = 3 + 2xy,\ N = x^2 - 3y^2$. $M_y = 2x = N_x$.

The vector filed is defined on $\mathbb{R}^2$, which is a simply connected region. The partial derivatives of M and N are continuous. Therefore, F is a conservative field.

(c) $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, where $M = 2x - 3,\ N = z,\ P = \cos z$.
$M_y = 0,\ N_x = 0,\ N_z = 1, P_y = 0,\ P_x = 0,\ M_z = 0$.
Since $N_z \neq P_y$, the field F is not conservative.

Example 11

Find a potential for the vector field $F = (3 + 2xy)\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$. Then evaluate $\int_C F \cdot d\,\vec{r}$, where C is given by
$\vec{r}(t) = e^t \sin t\,\hat{\imath} + e^t \cos t\,\hat{\jmath},\ 0 \le t \le \pi.$

To determine the scalar function $f(x, y, z)$ such that $F = \operatorname{grad} f$, we take

$$f_x = 3 + 2xy, \quad f_y = x^2 - 3y^2.$$

Integrate the first one with respect to x and integrate the second with respect to y to obtain:

$$f(x, y) = 3x + x^2y + g(y), \quad f(x, y) = x^2y - y^3 + h(x).$$

Taking $g(y) = -y^3 + \text{const.}$ and $h(x) = 3x + \text{const.}$, we have

$$f(x, y) = 3x + x^2y - y^3 + k \quad \text{for any constant } k.$$

Next, $\int_C F \cdot d\,\vec{r} = f(x(\pi), y(\pi)) - f(x(0), y(0)) = e^{3\pi} + 1.$

Example 12

Find a potential for the vector field $F = y^2\,\hat{\imath} + (2xy + e^{3z})\,\hat{\jmath} + 3ye^{3z}\,\hat{k}$.

Denote the potential by $f(x, y, z)$. Then

$$f_x = y^2,\ f_y = 2xy + e^{3z},\ f_z = 3ye^{3z}.$$

Integrate with respect to suitable variables:

$$f = xy^2 + g(y, z),\ f = xy^2 + ye^{3z} + h(x, z),\ f = ye^{3z} + \phi(x, y).$$

Sometimes matching may not be obvious. So, differentiate the first:

$$f_y = 2xy + g_y(y, z) = 2xy + e^{3z}.$$

Thus, $g_y(y, z) = e^{3z}$. Integrate: $g(y, z) = ye^{3z} + \psi(z)$. Then

$$f = xy^2 + ye^{3z} + \psi(z).$$

This gives $f_z = 3ye^{3z} + \psi'(z) = 3ye^{3z}$. Thus, $\psi(z) = k$, a const. Therefore,

$$f(x, y, z) = xy^2 + ye^{3z} + k.$$

Example 13

Show that the following vector field is conservative by finding a potential for it:
$F = (e^x \cos y + yz)\,\hat{\imath} + (xz - e^x \sin y)\,\hat{\jmath} + (xy + z)\,\hat{k}$

Let the potential be $f(x, y, z)$. Then

$$f_x = e^x \cos y + yz,\ f_y = xz - e^x \sin y,\ f_z = xy + z.$$

Integrate the first w.r.t. x to get

$$f = e^x \cos y + xyz + g(y, z).$$

Differentiate w.r.t. y to get

$$f_y = -e^x \sin y + xz + g_y(y, z) = xz - e^x \sin y \Rightarrow g_y(y, z) = 0.$$

Thus $g(y, z) = h(z)$. And then $f = e^x \cos y + xyz + h(z)$. Differentiate w.r.t. z to obtain

$$f_z = xy + h'(z) = xy + z \Rightarrow h'(z) = z \Rightarrow h(z) = z^2/2 + k.$$

Then $f(x, y, z) = e^x \cos y + xyz + z^2/2 + k$.

Exact Differential Forms

If M, N, P are functions of x, y, z, on a region D in space, then the expression

$$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$$

Is called a differential form. The differential form is called exact if there exists a function $f(x,y,z)$ such that

$$M(x,y,z) = \frac{\partial f}{\partial x},\ N(x,y,z) = \frac{\partial f}{\partial y},\ P(x,y,z) = \frac{\partial f}{\partial z}.$$

Notice that if the differential form is exact, then

$$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz = df,$$

which is an exact differential. In that case, if C is any curve joining points A to B in the region D, then

$$\int_C [Mdx + Ndy + Pdz] = \int_C \nabla f \cdot d\vec{r} = \int_A^B df = f(B) - f(A).$$

Therefore, the differential form is exact iff $F = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ is conservative. Then the f is the potential of the field F.

Example 14

Show that the differential form $ydx + xdy + 4dz$ is exact. Evaluate the integral $\int_C (ydx + xdy + 4dz)$ over the line segment C joining the points $(1, 1, 1)$ to $(2, 3, -1)$.

$M = y,\ N = x,\ P = 4$. Then

$$M_y = 1 = N_x,\ N_z = 0 = P_y,\ P_x = 0 = M_z.$$

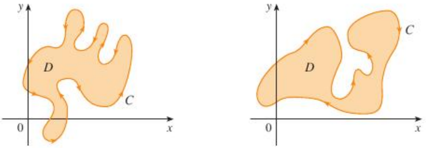
Therefore, the differential form is exact. Notice that $ydx + xdy + 4dz = d(xy + 4z + k)$. Hence it is exact. In case, f is not obvious, we can determine it as earlier by differentiating and integrating etc. Next,

$$\int_C (ydx + xdy + 4dz) = \int_{(1,1,1)}^{(2,3,-1)} d(xy + 4z + k)$$

$$= (xy + 4z + k)\Big|_{(1,1,1)}^{(2,3,-1)} = -3.$$

Green's Theorem

Let C be a simple closed curve in the plane. The positive orientation of C refers to a single counter-clockwise traversal of C. If C is given by $\vec{r}(t)$, $a \leq t \leq b$, then its positive orientation refers to a traversal of C keeping the region D bounded by the curve to the left.



(a) Positive orientation

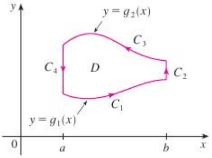
(b) Negative orientation

Theorem 7: (Green's) Let C be a positively oriented simple piecewise smooth curve in the plane. Let D be the region with boundary as C. (That is, $C = \partial D$.) If $M(x, y)$ and $N(x, y)$ have continuous partial derivatives on an open region containing D, then

1. $\oint_C (Mdx + Ndy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$
2. $\oint_C (Mdy - Ndx) = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA.$

Proof

Green's theorem helps in evaluating an integral of the type $\int_a^b F \cdot \vec{r}$ in a non-conservative vector field F. It gives a relationship between a line integral around a simple closed curve C and the double integral over the plane region D bounded by this closed curve.



We only prove for a special kind of regions to give an idea of how it is proved.
Consider the region $D = \{(x, y) : a \le x \le b,\ f(x) \le y \le g(x)\}$.
Assume that f, g are continuous functions. Then

$$\iint_D \frac{\partial M}{\partial y} dA = \int_a^b \int_{f(x)}^{g(x)} M_y dy dx = \int_a^b [M(x, g(x)) - M(x, f(x))] dx.$$

Now we compute $\int_C M dx$ by breaking C into four parts C_1, C_2, C_3 and C_4.

Proof Contd.

The curve C_1 is given by $x = x,\ y = f(x),\ a \leq x \leq b$. Thus

$$\int_{C_1} Mdx = \int_a^b M(x, f(x))dx.$$

On C_2 and also on C_4, the variable x is a single point. So,

$$\int_{C_2} Mdx = \int_{C_4} Mdx = 0.$$

As x increases, C_3 is traversed backward. That is, $-C_3$ is given by $x = x,\ y = g(x),\ a \leq x \leq b$. So,

$$\int_{C_3} Mdx = -\int_{-C_3} Mdx = -\int_a^b M(x, g(x))dx.$$

Therefore, $\iint_D \frac{\partial M}{\partial y} dA = -\int_C Mdx$. Similarly, express D using the variable of integration as y. Then we have $\iint_D \frac{\partial N}{\partial x} dA = \int_C Ndy$. Next, add the two results obtained to get

$$\int_C (Mdx + Ndy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

The second form follows similarly.

Example 15

Verify Green's theorem for the field $F = (x - y)\,\hat{\imath} + x\,\hat{\jmath}$, where C is the unit circle oriented positively.

Here, we have $C:\ \vec{r}(t) = x\hat{\imath} + y\,\hat{\jmath} = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath},\ 0 \le t \le 2\pi.$
The region D is the unit disk.

$M = \cos t - \sin t,\ N = \cos t.$

$dx = -\sin t\,dt,\ dy = \cos t\,dt.$

$M_x = 1,\ M_y = -1,\ N_x = 1,\ N_y = 0.$ Now,

$\oint_C (Mdy - Ndx) = \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)]dt = \pi.$
$\iint_D (M_x + N_y)dA = \iint_D (1+0)dA = \text{Area of } D = \pi.$
Similarly,
$\oint_C (Mdx + Ndy) = \int_0^{2\pi} [(\cos t - \sin t)(-\sin t) + \cos^2 t]dt = 2\pi.$
$\iint_D (N_x - M_y)dA = \iint_D (1 - (-1))dA = 2 \times \text{Area of } D = 2\pi.$

Examples

Example 16: Evaluate the integral $I = \oint_C xy\,dy + y^2\,dx$, where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$, with positive orientation.

Take $M = y^2,\ N = xy,\ D$ as the region bounded by C. Then

$$I = \oint_C (Mdx + Ndy) = \iint_D (N_x - M_y)dA = \int_0^1 \int_0^1 (y - 2y)dxdy = -1/2.$$

Also, taking $M(x, y) = xy,\ N(x, y) = -y^2$ we have

$$I = \oint_C (Mdy - Ndx) = \iint_D (M_x + N_y)dA = \int_0^1 \int_0^1 (y - 2y)dxdy = -1/2.$$

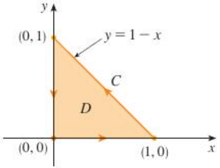
Example 17: Evaluate the integral
$I = \oint_C (3y - e^{\sin x})dx + (7x + \sqrt{1 + y^4})dy$, where C is the positively oriented circle $x^2 + y^2 = 9$.

Take D as the disk $x^2 + y^2 \leq 9$. Then by Green's theorem,

$$I = \iint_D [(7x + \sqrt{1 + y^4})_x - (3y - e^{\sin x})_y]dA = \iint_D (7 - 3)dA = 36\pi.$$

Examples Contd.

Example 18: Evaluate $I = \oint_C x^4\,dx + xy\,dy$, where C is the triangle with vertices at $(0,0)$, $(0,1)$ and $(1,0)$; its orientation being from $(0,0)$ to $(1,0)$ to $(0,1)$ to $(0,0)$.



The triangle is positively oriented. Let D be the region bounded by the triangle. Take $M = x^4$, $N = xy$. Then

$$I = \iint_D [(xy)_x - (x^4)_y]\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{2}\int_0^1 (1 - x^2)dx = \frac{1}{6}.$$

Example 19: Evaluate $\int_C (xdy - y^2dx)$, where C is the positively oriented square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Here, $M = x$, $N = y^2$, and D as the region bounded by C.

By Green's theorem,

$$\oint_C (Mdy - Ndx) = \iint_D (M_x + N_y)dA = \int_{-1}^1 \int_{-1}^1 (1 + 2y)dxdy = 4.$$

Area and Green's Theorem

Consider the formula $\iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy)$.

1. $M = 0,\ N = x \Rightarrow N_x - M_y = 1$. So,
$$\text{Area of } D = \iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy) = \oint_C xdy.$$
2. $M = -y,\ N = 0 \Rightarrow N_x - M_y = 1$. Then
$$\text{Area of } D = \iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy) = -\oint_C ydx.$$
3. Combine both to get Area of $D = \frac{1}{2}\oint_C (x\,dy - y\,dx)$.

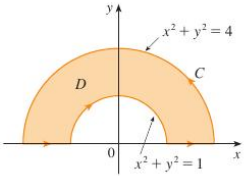
For example, area enclosed by the ellipse $C : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is (Note: C is $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$.)

$$\frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} [(a\cos t\, b\cos t) - (b\sin t\,(-b\sin t))]dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi\,ab.$$

A Different Region

Example 20: Evaluate $\oint_C (y^2\,dx + xy\,dy)$, where C is the boundary of the semi-annular region between the semicircles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the upper half plane.



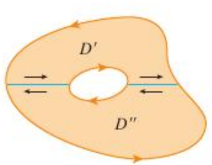
Write, in polar co-ordinates, $D = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$. Then

$$\oint_C (y^2\,dx + xy\,dy) = \iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(y^2)\right] dA = -\iint_D y\,dA$$

$$= -\int_1^2 \int_0^{\pi} r\sin\theta\, r\,dr\,d\theta = -\int_1^2 r^2 dr \int_0^{\pi} \sin\theta\,d\theta = -\frac{14}{3}.$$

In fact, Green's theorem can be applied to regions having holes, provided the region can be divided into simply connected regions.

Regions with holes



The boundary C of the region D consists of two simple closed curves C_1 (Outer) and C_2 (inner). Assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.

Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. Divide D into two regions D' and D'' as shown in the figure. Green's theorem on D' and D'' gives

$$\iint_D (N_x - M_y) dA = \iint_{D'} (N_x - M_y) dA + \iint_{D''} (N_x - M_y) dA$$

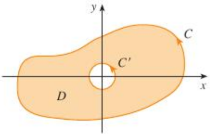
$$= \int_{\partial D'} (Mdx + Ndy) + \int_{\partial D''} (Mdx + Ndy) = \int_C (Mdx + Ndy).$$

This is the general version of Green's Theorem.

Example 21

Let C be any positively oriented simple closed curve that encloses the origin. Show that $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 2\pi$.

Take a positively oriented circle C', of radius a, around origin that lies entirely in the region bounded by C. Let D be the annular region bounded by C and C'.



Take $F(x,y) = (-y\,\hat{\imath} + x\,\hat{\jmath})/(x^2+y^2)$. Then the positively oriented boundary of D is $\partial D = C \cup (-C')$.

Here, $F = M\,\hat{\imath} + N\,\hat{\jmath}$ gives $N_x = M_y = (y^2 - x^2)/(x^2+y^2)^2$.

Green's theorem on D gives

$$\oint_C (Mdx + Ndy) + \oint_{-C'} (Mdx + Ndy) = \iint_D (N_x - M_y)dA = 0$$

Then $\oint_C (Mdx + Ndy) = \oint_{C'} (Mdx + Ndy)$.

But C' is parameterized by $x(t) = a\cos t,\ y(t) = a\sin t,\ 0 \le t \le 2\pi$.

So, $\int_{C'} (Mdx + Ndy) = \int_0^{2\pi} (a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}) \cdot (a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath})'dt = 2\pi$.

Curl of a vector field

If $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ is a vector field in $\mathbb{R}^3$, where the partial derivatives of the component functions exist, then curl F is a vector field given by

$$\text{curl } F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\imath} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

Writing in operator notation, recall that

$$\text{grad } = \nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

Then curl $F = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}.$

For example, if $F = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$, then

$$\text{curl } F = -y(2+x)\,\hat{\imath} + x\,\hat{\jmath} + yz\,\hat{k}.$$

Curl of a Conservative Field

Theorem 8: Let F be a vector field defined over a simply connected region D whose component functions have continuous partial derivatives. Then F is conservative iff curl $F = 0$.

Proof of $\Longrightarrow$: Let f be any scalar function defined on D. Now,

$$\text{curl } \nabla f = \nabla \times (\nabla f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= (f_{yz} - f_{zy})\,\hat{\imath} + (f_{zx} - f_{xz})\,\hat{\jmath} + (f_{xy} - f_{yx})\,\hat{k} = 0.$$

if F is conservative, then $F = \nabla f$ for some f. Thus, curl $F = 0$.

The converse follows from Stokes' theorem, which we will discuss later. □

Remember: The curl of gradient of any scalar function is zero:

$$\text{curl grad } f = 0.$$

Examples

Example 22: Is the vector field $F = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$ conservative?

Here, curl $F = -y(2+x)\,\hat{\imath} + x\,\hat{\jmath} + yz\,\hat{k} \neq 0$. So, F is not conservative.

Example 23: Is the vector field $F = y^2z^3\,\hat{\imath} + 2xyz^3\,\hat{\jmath} + 3xy^2z^2\,\hat{k}$ conservative?

Here, F is defined on $\mathbb{R}^2$ and

$$\text{curl } F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \begin{matrix} (6xy^2z^2 - 6xy^2z^2)\,\hat{\imath} \\ -(3y^2z^2 - 3y^2z^2)\,\hat{\jmath} \\ +(2yz^3 - 2yz^3)\,\hat{k} \end{matrix} = 0.$$

Hence F is conservative.

In fact, $F = \text{grad}\, f$, where $f(x, y, z) = xy^2z^3$.

The name game: curl F measures how quickly a tiny peddle (at a point) in some fluid in a vector field moves around itself. If curl $F = 0$, then there is no rotation of such a tiny peddle.

Divergence

If $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ is a vector field defined on a region, where its component functions have first order partial derivatives, then

$$\text{div } F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The divergence is also called flux density.

For example, if $F = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$, then div $F = z + xz$.

The divergence of the vector field $F = (x^2 - y)\,\hat{\imath} + (xy - y^2)\,\hat{\jmath}$ is

$\frac{\partial(x^2-y)}{\partial x} + \frac{\partial(xy-y^2)}{\partial y} = 3x - 2y.$

Intuitively, div F measures the tendency of the fluid to diverge from the point (a, b). When the gas (fluid) is expanding, divergence is positive; and when it is compressing, the divergence is negative.
If div $F = 0$, then the fluid is said to be incompressible.

Divergence of Curl

Theorem 9: Let $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ be a vector field defined on a simply connected region $D \subseteq \mathbb{R}^3$, where M, N, P have continuous second order partial derivatives. Then div curl $F = 0$.

Proof: div curl $F = \nabla \cdot (\nabla \times F)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$= 0$ due to Clairaut's Theorem. □

Example 24: Does there exist a vector field G such that $F = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k} = \text{curl } G$?

div $F = z + xz \neq 0$. Hence there is no such G.

Divergence of grad f is the Laplacian of a scalar function f since

$$\text{div grad } f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} := \nabla^2 f.$$

Green's Theorem - Tangent form

Let D be a simply connected region whose boundary is the simple closed curve C. Let $F = M\,\hat{\imath} + N\,\hat{\jmath}$ be a vector field defined on D.
Let C be parameterized by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$.
Let $\hat{T}$ be the unit tangent vector to C at the point $(x(t), y(t))$. Then

$$F \cdot \hat{T}(t)ds = F \cdot d\,\vec{r} = Mdx + Ndy.$$

The line integral of F over C is

$$\oint F \cdot \hat{T}(t)ds = \oint_C F \cdot d\,\vec{r} = \oint_C (M\,dx + N\,dy).$$

Consider F as a vector field on $\mathbb{R}^3$ with $P = 0$. Then

$$\text{curl } F = (N_x - M_y)\,\hat{k} \Longrightarrow \text{curl } F \cdot \hat{k} = N_x - M_y.$$

Thus Green's theorem takes the form

$$\oint_C F \cdot \hat{T}(t)\,ds = \oint_C F \cdot d\,\vec{r} = \iint_D (\text{curl } F \cdot \hat{k})\,dA.$$

Green's Theorem - Normal form

Let C be given by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$.
Let $\hat{n}$ be the unit normal to C at the point $(x(t), y(t))$. Then

$$\hat{T} = \frac{x'(t)}{|\vec{r}\,'(t)|}\,\hat{\imath} + \frac{y'(t)}{|\vec{r}\,'(t)|}\,\hat{\jmath}, \qquad \hat{n}(t) = \frac{y'(t)}{|\vec{r}\,'(t)|}\,\hat{\imath} - \frac{x'(t)}{|\vec{r}\,'(t)|}\,\hat{\jmath}.$$

Then $\vec{F}\cdot\hat{n} = [M\,y'(t) - N\,x'(t)]/|\vec{r}\,'(t)|$.

Now, $\oint_C F \cdot \hat{n}\,ds = \int_a^b \vec{F}\cdot\hat{n}\,|\vec{r}\,'(t)|\,dt = \oint_C (Mdy - Ndx)$.

Also, $\iint_D \text{div}\,F\,dA = \iint_D (M_x + N_y)dA$.

Hence the second form of Green's theorem takes the form

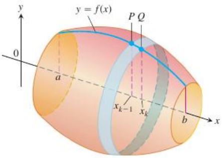
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \text{div}\,F\,dA.$$

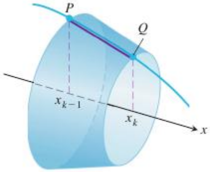
Both the tangent-form and the normal-form are called vector forms of Green's theorem.

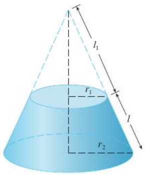
Surface area of Revolution

Consider a smooth curve $y = f(x),\ f(x) \geq 0$. Its arc when $a \leq x \leq b$ is revolved about the x-axis to generate a solid. How do we compute the area of the surface of this solid?

It is similar to computing the volume of revolution.
Partition $[a, b]$ into n subintervals $[x_{k-1}, x_k]$. When each Δx_k is small, the surface area corresponding to this subinterval is approximately same as the area on the frustum of a right circular cone.







Area of Frustum

Suppose a circular cone has base radius R and slant height ℓ. Its surface area is $\pi R\ell$. So, the area of the frustum is

$$A = \pi r_2(\ell_1 + \ell) - \pi r_1 \ell_1 = \pi[(r_2 - r_1)\ell_1 + r_2\ell].$$

Using similarity of triangles, $\dfrac{\ell_1}{r_1} = \dfrac{\ell_1 + \ell}{r_2}$.

So, $r_2\ell_1 = r_1\ell_1 + r_1\ell \Rightarrow (r_2 - r_1)\ell_1 = r_1\ell$. Therefore,

$$A = \pi(r_1\ell + r_2\ell) = 2\pi\, r\, \ell, \quad \text{where } r = \frac{r_1 + r_2}{2}.$$

The slant height ℓ is approximated by $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, where

$$\Delta x_k = x_k - x_{k-1} \text{ and } \Delta y_k = f(x_k) - f(x_{k-1}).$$

the average radius $r = \dfrac{r_1 + r_2}{2} = \dfrac{f(x_{k-1}) + f(x_k)}{2}$.

Thus the area of the frustum is

$$A_k = 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Approximating the area

Due to MVT, we have $c_k \in [x_{k-1}, x_k]$ such that

$$\Delta y_k = f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1}) = f'(c_k)\Delta x_k.$$

So, $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + (f'(c_k))^2}\,\Delta x_k$.

The surface of revolution is approximated by

$$\sum_{k=1}^{n} A_k = 2\pi \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{1 + (f'(c_k))^2}\,\Delta x_k.$$

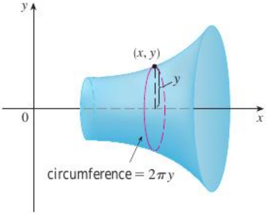
Its limit as $n \to \infty$ is the Riemann sum of an integral, which is the required area:

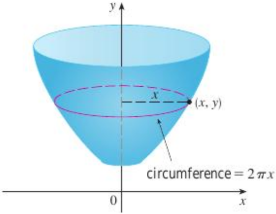
$$S = \int_a^b 2\pi\, y \sqrt{1 + (f'(x))^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

In summary

1. If the arc of the curve $y = f(x)$ for $a \leq x \leq b$ is revolved about the x-axis, then write $ds = \sqrt{1 + (f'(x))^2}\, dx$. The area of the surface of the solid of revolution is given by

$$S = \int 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1 + (f'(x))^2}\, dx.$$





2. If the arc of the curve $x = g(y)$ for $c \leq y \leq d$ is revolved about the y-axis, then write $ds = \sqrt{1 + (g'(y))^2}\, dy$. The area of the surface of the solid of revolution is given by

$$S = \int 2\pi x\, ds = \int_c^d 2\pi x \sqrt{1 + (g'(y))^2}\, dy.$$

Particular cases

Suppose the curve is parameterized by $x = x(t),\ y = y(t)$ for $a \le t \le b$; it is traversed exactly once while t increases from a to b.

Then $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

The surface area S of the solid generated by revolving the curve about the coordinate axes are as follows:

1. If the revolution is about the x-axis, then

$$S = \int_a^b 2\pi\, y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

2. If the revolution is about the y-axis, then

$$S = \int_a^b 2\pi\, x(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Examples

Example 25: Find the surface area of the solid obtained by revolving about x-axis, the arc of the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$.

Since $y = 2\sqrt{x}$, $y' = 1/\sqrt{x}$, $\sqrt{1 + (y')^2} = \sqrt{1 + 1/x}$. Then

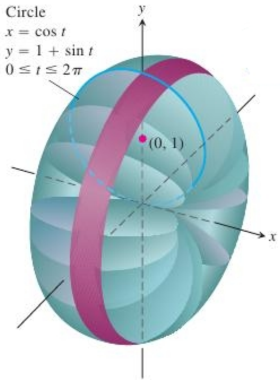
$$S = \int_1^2 2\pi y\left(1+[y']^2\right)^{1/2} dx = \int_1^2 2\pi\, 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx = \frac{8\pi}{3}(3\sqrt{3}-2\sqrt{2}).$$

Example 26: The arc of the parabola $y = x^2$, $1 \leq x \leq 2$ is revolved about the y-axis. Find the surface area of revolution.

Since $x = \sqrt{y}$, $1 \leq y \leq 4$, the surface area is

$$\begin{aligned} S &= \int_1^2 2\pi\, x\sqrt{1 + (x')^2}\, dy = 2\pi\int_1^4 \sqrt{y}\sqrt{1 + 1/(4y)}\, dy \\ &= \pi\int_1^4 \sqrt{1 + 4y}\, dy = \frac{\pi}{4}\int_1^4 \sqrt{1 + 4y}\, d(1 + 4y) = \frac{\pi}{4}\cdot\frac{2}{3}\left[(1 + 4y)^{3/2}\right]_1^4 \\ &= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right) \end{aligned}$$

Example 27



The circle of radius 1 centered at $(0, 1)$ is revolved about the x-axis. Find the surface area of the solid so generated.

The circle can be parameterized as

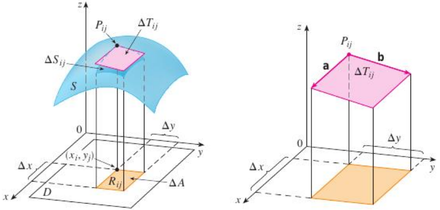
$$x = \cos t,\ y = 1 + \sin t,\ 0 \le t \le 2\pi.$$

Then $(x'(t))^2 + (y'(t))^2 = 1$. Thus the area is

$$S = \int_0^{2\pi} 2\pi\,(1 + \sin t)\,dt = 4\pi^2.$$

Surface area in General

Let S be a surface given by $z = f(x, y)$. For simplicity, assume that $f(x, y) \geq 0$ over the region D, which is rectangular.



Divide D into smaller rectangles R_{ij} with area $\Delta(R_{ij}) = \Delta x \Delta y$. For the corner (x_i, y_j) in R_{ij}, closest to the origin, let P_{ij} be the point $(x_i, y_j, f(x_i, y_j))$ on the surface. The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. The area T_{ij} of the portion of the tangent plane that lies above R_{ij} approximates the area of S_{ij}, the portion of S that is directly above R_{ij}. Therefore, we define the area of the surface S as

$$\Delta(S) = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} T_{ij}.$$

Surface Area - Formula

Let $\vec{a}$ and $\vec{b}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram whose area is T_{ij}. Then $T_{ij} = \vec{a} \times \vec{b}$. However, $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of $\vec{a}$ and $\vec{b}$, respectively. Therefore,

$$\vec{a} = \Delta x\, \hat{\imath} + f_x(x_i, y_j)\Delta x\, \hat{k}, \quad \vec{b} = \Delta y\, \hat{\jmath} + f_y(x_i, y_j)\Delta y\, \hat{k}.$$

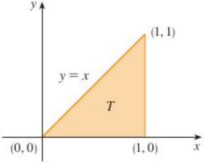
$$T_{ij} = |\vec{a} \times \vec{b}| = |-f_x(x_i, y_j)\,\hat{\imath} - f_y(x_i, y_j)\,\hat{\jmath} + k|\,\Delta(R_{ij})$$
$$= \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1}\,\Delta A.$$

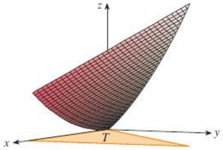
Summing over these T_{ij} and taking the limit, we obtain:

$$\Delta(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

Example 28

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region in the xy-plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.





$$T = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}, \quad f(x, y) = x^2 + 2y.$$

The required surface area is

$$\iint_T \sqrt{(2x)^2 + 2^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy dx = \frac{1}{12}(27 - 5\sqrt{5}).$$

Surface Area - A generalized form

Recall that for a surface S given by $f(x, y) = z$, the surface area is $\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$. Here, D is the rectangle on the xy-plane obtained by projecting S onto the plane.

Look at this surface as $f(x, y) - z = 0$. Then $\nabla f = f_x\,\hat{\imath} + f_y\,\hat{\jmath} - 1\,\hat{k}$.

If $\vec{p}$ is the unit normal to the projected rectangle, then $\vec{p} = \hat{k}$. Then

$\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|} = \frac{\sqrt{f_x^2+f_y^2+1}}{1^2}$, which is the integrand in the surface area formula.

A derivation similar to the surface area formula gives the following:

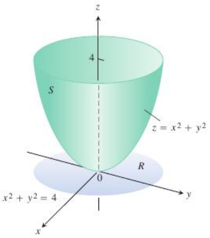
Let the surface S be given by $f(x, y, z) = c$. Let R be a closed bounded region which is obtained by projecting the surface to a plane whose unit normal is $\vec{p}$. Suppose that ∇f is continuous on R and $\nabla f \cdot \vec{p} \neq 0$ on R. Then

$$\text{The surface area of } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\, dA.$$

Of course, whenever possible, we project onto the co-ordinate planes.

Example 29

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 = z$ by the plane $z = 4$.



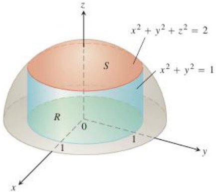
Surface S is given by $f(x, y, z) = x^2 + y^2 - z = 0$. Project it onto xy-plane to get the region R as $x^2 + y^2 \leq 4$. Then $\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - \hat{k}$.
$|\nabla f| = \sqrt{1 + 4x^2 + 4y^2}$.
$\vec{p} = \hat{k}$. $|\nabla f \cdot \vec{p}\,| = 1$.

R is given by $x = r\cos\theta,\ y = r\sin\theta,\ 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$.
So, the surface area is

$$\iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr d\theta = \frac{\pi}{6}(17\sqrt{17} - 1).$$

Example 30

Find the surface area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2,\ z \geq 0$ by the cylinder $x^2 + y^2 = 1$.



The surface projected on xy-plane gives R as the disk $x^2 + y^2 \leq 1$. The surface is $f(x, y, z) = 2$, where $f(x, y, z) = x^2 + y^2 + z^2$.
$\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}$. $|\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2}$. $\vec{p} = k$.
$|\nabla f \cdot \vec{p}\,| = |2z| = 2z$. Thus the surface area is

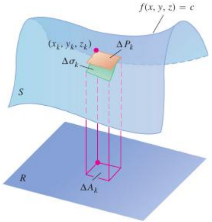
$$S = \iint_R \frac{2\sqrt{2}}{2z}\, dA = \sqrt{2} \iint_R z^{-1}\, dA = \sqrt{2} \iint_R (2 - x^2 - y^2)^{-1/2}\, dA.$$

R is given by $x = r\cos\theta, y = r\sin\theta,\ 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1$. So,

$$S = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r\,dr\,d\theta}{\sqrt{2 - r^2}} = 2\pi(2 - \sqrt{2}).$$

Integrating over a surface

Suppose a function $g(x, y, z)$ is defined over a surface S given by $f(x, y, z) = c$. To compute the integral of g, as area elements are taken over the surface, we look at the region R on which this surface is defined as a function.



Divide the region R into smaller rectangles ΔA_k. Consider the corresponding surface areas $\Delta\sigma_k$. Then

$$\Delta\sigma_k \approx \left(\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\right)_k.$$

Assuming that g is nearly constant on the smaller surface fragment σ_k, we form the sum

$$\sum_k g(x_k, y_k, z_k)\Delta\sigma_k \approx \sum_k g(x_k, y_k, z_k)\left(\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\right)_k.$$

If this sum converges, then we define that limit as the integral of g over the surface S.

In Summary

We summarize:

Let S be a surface given by $f(x, y, z) = c$.
Let the projection of S onto a plane with unit normal $\vec{p}$ be the region R.
Let $g(x, y, z)$ be defined over S.
Then the surface integral of g over S is

$$\iint_S g\, d\sigma = \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\, dA.$$

Also, we say $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\, dA.$

Linearity

If the surface S can be represented as a union of non-overlapping smooth surfaces $S_1, \ldots, S_n$, then

$$\iint_S g\, d\sigma = \iint_{S_1} g\, d\sigma + \cdots + \iint_{S_n} g\, d\sigma.$$

If $g(x, y, z) = g_1(x, y, z) + \cdots + g_m(x, y, z)$ over the surface S, then

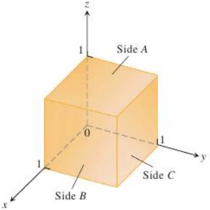
$$\iint_S g\, d\sigma = \iint_S g_1\, d\sigma + \cdots + \iint_S g_m\, d\sigma.$$

Similarly, if $g(x, y, z) = k\, h(x, y, z)$ holds for a constant k, over S, then

$$\iint_S g(x, y, z)\, d\sigma = k \iint_S h(x, y, z)\, d\sigma.$$

Example 31

Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1$, and $z = 1$.



We integrate g over the six surfaces and add the results. As $g = xyz$ is zero on the co-ordinate planes, we need integrals on sides A, B and C. Side A is the surface defined on the region $R_A : 0 \leq x \leq 1,\ 0 \leq y \leq 1$ on the xy-plane. For this surface and the region,

$$\vec{p} = \hat{k},\ \nabla f = \hat{k},\ |\nabla f| = 1,\ |\nabla f \cdot \vec{p}\,| = |\,\hat{k} \cdot \hat{k}| = 1,\ g(x, y, z) = xyz\big|_{z=1} = xy.$$

Therefore,

$$\iint_A g(x, y, z)\, d\sigma = \iint_{R_1} xy \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|} dA = \int_0^1 \int_0^1 xy dx dy = \int_0^1 \frac{y}{2} = \frac{1}{4}.$$

Similarly, $\iint_B g(x, y, z)\, d\sigma = \frac{1}{4} = \iint_C g(x, y, z)\, d\sigma$.

Thus, $\displaystyle\iint_S g\, d\sigma = \frac{3}{4}$.

Example 32

Evaluate the surface integral of $g(x,y,z) = x^2$ over the unit sphere.

It can be divided into the upper hemisphere and the lower hemisphere. Let S be the upper hemisphere $f(x,y,z) := x^2 + y^2 + z^2 = 1,\ z \geq 0$. Its projection on the xy-plane is the region

$R: \ x = r\cos\theta, y = r\sin\theta,\ 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi.$

Here, $\vec{p} = \hat{k},\ |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2,$

$|\nabla f \cdot \vec{p}\,| = 2|z| = 2\sqrt{1 - (x^2 + y^2)} = 2\sqrt{1 - r^2}.$

Hence $\iint_S x^2\, d\sigma = \iint_R x^2 \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\, dA = \iint_R \frac{x^2}{\sqrt{1-r^2}} dA$

$$= \int_0^{2\pi} \int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^1 \frac{r^3}{\sqrt{1-r^2}} dr = \frac{2\pi}{3}.$$

Since the integral of x^2 on the upper hemisphere is equal to that on the lower hemisphere, the required integral is $2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$.

A simplification

Recall that when $\vec{p} = \hat{k}$, that is, when the region R is obtained by projecting the surface S onto the xy-plane, $\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} = \sqrt{1 + z_x^2 + z_y^2}$. Now, if the surface $f(x, y, z) = c$ can be written explicitly by $z = h(x, y)$, then the surface integral takes the form

$$\iint_S g(x, y, z)\, d\sigma = \iint_R g(x, y, h(x, y))\, \sqrt{1 + h_x^2 + h_y^2}\, dx\, dy.$$

Similarly, if the surface can be written as $y = h(x, z)$ and R is obtained by projecting S onto the xz-plane, then

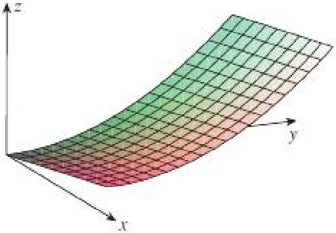
$$\iint_S g(x, y, z)\, d\sigma = \iint_R g(x, h(x, z), z)\, \sqrt{1 + h_x^2 + h_z^2}\, dx\, dz.$$

If the surface can be written as $x = h(y, z)$ and R is obtained by projecting S onto the yz-plane, then

$$\iint_S g(x, y, z)\, d\sigma = \iint_R g(h(y, z), y, z)\, \sqrt{1 + h_y^2 + h_z^2}\, dy\, dz.$$

Example 33

Evaluate $\iint_S y\, d\sigma$, where S is the surface $z = x + y^2$, where $0 \leq x \leq 1$ and $0 \leq y \leq 2$.



Projecting the surface to xy-plane, we obtain the region R as the rectangle $0 \leq x \leq 1,\ 0 \leq y \leq 2$.

Here, the surface is given by $z = h(x, y) = x + y^2$. So,

$$\iint_S y\, d\sigma = \iint_R y\sqrt{1 + 1 + (2y)^2}\, dA = \int_0^1 \int_0^2 \sqrt{2}y\sqrt{(1 + 2y^2)}\, dy\, dx = \frac{13\sqrt{2}}{3}.$$

Another Formulation

Suppose the surface S is given in a parameterized form:

$$\vec{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k},$$

where (u,v) ranges over the region D in the uv-plane. Here, a change of variable happens. Then

$$d\sigma = |\,\vec{r}_u \times \vec{r}_v|\,du\,dv,$$

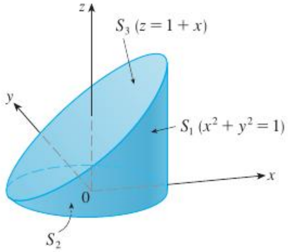
where $\vec{r}_u = x_u\,\hat{\imath} + y_u\,\hat{\jmath} + z_u\,\hat{k}$ and $\vec{r}_v = x_v\,\hat{\imath} + y_v\,\hat{\jmath} + z_v\,\hat{k}$.

Then

$$\iint_S f(x,y,z)\,d\sigma = \iint_D f(\vec{r}(u,v))|\,\vec{r}_u \times \vec{r}_v|\,du\,dv.$$

Example 34

Evaluate $\iint_S z\, d\sigma$, where S is the surface whose sides S_1, S_2, S_3 are:
S_1 is given by the cylinder $x^2 + y^2 = 1$,
bottom S_2 is the disk $x^2 + y^2 \leq 1,\ z = 0$, and
whose top S_3 is part of the plane $z = 1 + x$ that lies above S_2.



S_1 is given by $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ with $x = \cos\theta,\ y = \sin\theta,\ z = z$,
where D is given by $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1 + x = 1 + \cos\theta$.
Then $|\vec{r}_\theta \times \vec{r}_z| = |\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}| = 1.$ So,

Example 34 Contd.

$$\iint_{S_1} z\,d\sigma = \iint_D z\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \frac{3\pi}{2}.$$

S_2 lies in the plane $z = 0$. Hence,

$$\iint_{S_2} z\,d\sigma = 0.$$

S_3 lies above the unit disk and lies in the plane $z = 1 + x$. So,

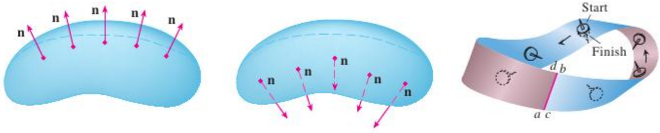
$$\begin{aligned}\iint_{S_3} z\,d\sigma &= \iint_D (1+x)\sqrt{1 + z_x^2 + z_y^2}\,dA \\ &= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{1+1+0}\,r\,dr\,d\theta = \sqrt{2}\pi.\end{aligned}$$

Hence,

$$\iint_S z\,d\sigma = \iint_{S_1} z\,d\sigma + \iint_{S_2} z\,d\sigma + \iint_{S_3} z\,d\sigma = \frac{3\pi}{2} + 0 + \sqrt{2}\pi.$$

Oriented Surface

A smooth surface is called orientable if it is possible to define a vector field of unit normal vectors $\hat{n}$ to the surface which varies continuously with position. Once such normal vectors are chosen, the surface is considered an oriented surface.



If the surface S is given by $z = f(x, y)$, then we take its orientation by considering the unit normal vectors $\hat{n} = \frac{-f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \hat{k}}{\sqrt{1+f_x^2+f_y^2}}$.

If S is a part of a level surface $g(x, y, z) = c$, then we may take $\hat{n} = \frac{\nabla g}{|\nabla g|}$.

If S is given parametrically as $\vec{r}(u, v) = x(u, v)\,\hat{\imath} + y(u, v)\,\hat{\jmath} + z(u, v)\,\hat{k}$, then $\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

Sometimes we may take negative sign if it is preferred.
Conventionally, the outward direction is taken as the positive direction.

Examples of Parametrization

1. The cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$ can be parametrized by $x = r\cos\theta$, $y = r\sin\theta$, $z = r$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Then its vector form is

$$\vec{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\,\hat{k}.$$

2. The sphere $x^2 + y^2 + z^2 = a^2$ can be parametrized by $x = a\cos\theta\sin\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\phi$. Here, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. In vector form the parametrization is

$$\vec{r}(\theta,\phi) = a\cos\theta\sin\phi\,\hat{\imath} + a\sin\theta\sin\phi\,\hat{\jmath} + a\cos\phi\,\hat{k}.$$

3. The cylinder $x^2 + y^2 = a^2$, $0 \le z \le 5$ can be parametrized by

$$\vec{r}(\theta,z) = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath} + z\,\hat{k}.$$

Surface Integral of a Vector Field

Let F be a continuous vector field defined over an oriented surface S with unit normal $\hat{n}$. The surface integral of F over S, also called, the flux of F across S is

$$\iint_S F \cdot \hat{n}\, d\sigma.$$

The flux is the integral of the scalar component of F along the unit normal to the surface. Thus in a flow, the flux is the net rate at which the fluid is crossing the surface S in the chosen positive direction.

If S is part of a level surface $g(x, y, z) = c$, which is defined over the region D, then $d\sigma = \frac{|\nabla g|}{|\nabla g . \vec{p}|}$. So, the flux across S is

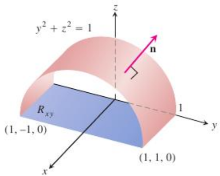
$$\iint_S F \cdot \hat{n}\, d\sigma = \iint_S F \cdot \frac{\pm\nabla g}{|\nabla g|} d\sigma = \iint_D F \cdot \frac{\pm\nabla g}{|\nabla g \cdot \vec{p}\,|} dA.$$

If S is parametrized by $\vec{r}(u, v)$, where D is the region in uv-plane, then $d\sigma = |\,\vec{r}_u \times \vec{r_v}\,|\, dA$. So, flux across S is

$$\iint_S F \cdot \hat{n}\, d\sigma = \iint_S F \cdot \frac{\vec{r}_u \times \vec{r_v}}{|\,\vec{r}_u \times \vec{r_v}\,|} d\sigma = \iint_D F(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r_v})\, dA.$$

Example 35

Find the flux of $F = yz\,\hat{j} + z^2\,\hat{k}$ outward through the surface S which is cut from the cylinder $y^2 + z^2 = 1, z \geq 0$ by the planes $x = 0$ and $x = 1$.



S is given by $g(x, y, z) := y^2 + z^2 - 1 = 0$, defined over the rectangle $R = R_{xy}$ as in the figure.

The outward unit normal is $\hat{n} = +\frac{\nabla g}{|\nabla g|} = y\,\hat{j} + z\,\hat{k}$.

Here, $\vec{p} = \hat{k}$. So, $d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \hat{k}|} dA = \frac{1}{2z}\,dA$.

Therefore, outward flux through S is

$$\iint_S F \cdot \hat{n}\,d\sigma = \iint_S z\,d\sigma = \iint_R z\,\frac{1}{2z}\,dA = \tfrac{1}{2}\text{Area of } R = 1.$$

Example 36

Find the flux of the vector field $F = z\,\hat{\imath} + y\,\hat{\jmath} + x\,\hat{k}$ across the unit sphere.

If no direction of the normal vector is given and the surface is a closed surface, we take $\hat{n}$ in the positive direction, which is directed outward.

Using the spherical co-ordinates, the unit sphere S is parametrized by

$$\vec{r}(\phi, \theta) = \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k},$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ give the region D. Then

$\vec{F}(\vec{r}(\phi, \theta)) = \cos\phi\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \sin\phi\cos\theta\,\hat{k}.$

$\vec{r}_{\phi} \times \vec{r}_{\theta} = \sin^2\phi\cos\theta\,\hat{\imath} + \sin^2\phi\sin\theta\,\hat{\jmath} + \sin\phi\cos\phi\,\hat{k}.$

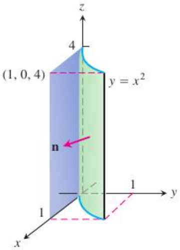
$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_D \vec{F}\cdot(\vec{r}_{\phi} \times \vec{r}_{\theta})\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi}(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\phi\,d\theta$

$= 2\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi\int_0^{2\pi}\cos\theta\,d\theta + \int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta$

$= 0 + \int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{4\pi}{3}.$

Example 37



Find the surface integral of $F = yz\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}$ over the portion of the parabolic cylinder $y = x^2,\ 0 \le x \le 1,\ 0 \le z \le 4.$

We assume the positive direction of the normal $\hat{n}$.

On the surface, we have $x = x,\ y = x^2,\ z = z$ giving the parametrization as $\vec{r}(x, z) = x\,\hat{\imath} + x^2\,\hat{\jmath} + z\,\hat{k}$,

whereas D is given by $0 \le x \le 1,\ 0 \le z \le 4.$

Again, on the surface, $F = x^2 z\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}$. So,

$$\iint_S F \cdot \hat{n}\, d\sigma = \iint_D F \cdot (\vec{r}_x \times \vec{r}_z)\, dxdz$$

$$= \iint_D (x^2 z\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}) \cdot (2x\,\hat{\imath} - \hat{\jmath}) dxdz$$

$$= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz = \int_0^4 (z-1)/2\, dz = 2.$$

A Simplification

If S is given by $z = f(x, y)$, then think of x, y as the parameters u and v. We have

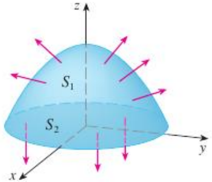
$$F = M(x, y)\,\hat{\imath} + N(x, y)\,\hat{\jmath} + P(x, y)\,\hat{k} \text{ and } \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + f(x, y)\,\hat{k}.$$

Then $\vec{r}_x \times \vec{r}_y = (\,\hat{\imath} + f_x\,\hat{k}) \times (\,\hat{\jmath} + f_y\,\hat{k}) = -f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \,\hat{k}$.

Therefore, the flux is

$$\iint_S F \cdot \,\hat{n}\,d\sigma = \iint_D F \cdot (\vec{r}_x \times \vec{r}_y)\,dxdy = \iint_D (-Mf_x - Nf_y + P)\,dxdy.$$

Example 38: Evaluate $\iint_S F \cdot \,\hat{n}\,d\sigma$, where $F = y\,\hat{\imath} + x\,\hat{\jmath} + z\,\hat{k}$ and S is the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



The surface S has two parts: the top portion S_1 and the base S_2. Since S is a closed surface, we consider its outward normal $\,\hat{n}$. Projections of both S_1 and S_2 on xy-plane are D, the unit disk.

Example 38 Contd.

By the simplified formula for the flux, we have

$\iint_{S_1} F \cdot \hat{n}\, d\sigma = \iint_D (-Mf_x - Nf_y + P) dxdy$

$= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2] dxdy$

$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta$

$= \int_0^{2\pi} (\frac{1}{4} + \cos\theta \sin\theta)\, d\theta = \frac{\pi}{2}.$

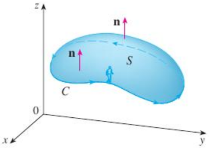
The disk S_2 has positive direction, when $\hat{n} = -\hat{k}$. Thus

$$\iint_{S_2} F \cdot \hat{n}\, d\sigma = \iint_{S_2} (-F \cdot \hat{k})\, d\sigma = \iint_D (-z) dxdy = 0$$

since on $D = S_2$, $z = 0$. Then

$$\iint_S F \cdot \hat{n}\, d\sigma = \iint_{S_1} F \cdot \hat{n}\, d\sigma + \iint_{S_2} F \cdot \hat{n}\, d\sigma = \frac{\pi}{2}.$$

Boundary of an Oriented Surface



Consider an oriented surface with a normal vector $\hat{n}$. Call the boundary curve of S as C. The orientation of S induces a positive orientation of the boundary of S.

If you walk in the positive direction of C keeping your head pointing towards $\hat{n}$, then S will be to your left.

Recall that Green's theorem relates a double integral in the plane to a line integral over its boundary.

We will have a generalization of this to 3 dimensions.

Write the boundary curve of a given smooth surface as ∂S.

The boundary is assumed to be a closed curve, positively oriented unless specified otherwise.

Stokes' Theorem

Theorem 10: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve ∂S with positive orientation. Let $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ be a vector field with M, N, P having continuous partial derivatives on an open region in space that contains S. Then

$$\oint_{\partial S} F \cdot d\vec{r} = \iint_S \text{curl } F \cdot \hat{n}\, d\sigma.$$

In particular, if S is a bounded region D in the xy-plane, $\partial S = C$, the smooth boundary of D, then $\hat{n} = \hat{k}$ and $d\sigma = dA$. We obtain

$$\oint_C F \cdot d\vec{r} = \iint_D \text{curl } F \cdot \hat{k}\, dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} (N_x - M_y)\, dx\, dy.$$

This is Green's theorem. We omit the proof of Stoke's theorem.

Example 39

Consider S as the hemisphere $x^2+y^2+z^2=9,\ z\ge 0$. Let
$F(\vec{r})=y\,\hat{\imath}-x\,\hat{\jmath}$. The bounding curve for S is in the xy-plane;
it is ∂S given by $x^2+y^2=9,\ z=0$.
Its parameterization is $\vec{r}(\theta)=3\cos\theta\,\hat{\imath}+3\sin\theta\,\hat{\jmath}$ for $0\le\theta\le 2\pi$.

$$\oint_{\partial S} F\cdot d\vec{r}=\int_0^{2\pi}[(3\sin\theta)\,\hat{\imath}-(3\cos\theta)\,\hat{\jmath}]\cdot[(-3\sin\theta)\,\hat{\imath}+(3\cos\theta)\,\hat{\jmath}]\,d\theta$$
$$=-\int_0^{2\pi}[9\sin^2\theta+9\cos^2\theta]\,d\theta=-18\pi.$$

This is the line integral in Stokes' theorem.

For the surface integral, we have

curl $F=(P_y-N_z)\,\hat{\imath}+(M_z-P_x)\,\hat{\jmath}+(N_x-M_y)\,\hat{k}=-2\,\hat{k}$.

On the surface $g:=x^2+y^2+z^2=9$, we have

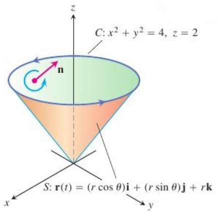
$\hat{n}=(\nabla g)/|\nabla g|=\frac{1}{3}(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k}),\ \vec{p}=\hat{k}$,

$d\sigma=|\nabla g|/|\nabla g\cdot\vec{p}\,|\,dA=\frac{2\times 3}{2z}dA=\frac{3}{z}dA$,

where dA is the differential in the projected area $D:\ x^2+y^2\le 9$.

$\iint_S \text{curl } F\cdot\hat{n}\,d\sigma=\iint_S\frac{-2z}{3}\,d\sigma=\iint_D\frac{-2z}{3}\,\frac{3}{z}\,dA=\iint_D(-2)\,dA=-18\pi.$

Example 40



Evaluate $\oint_C ((x^2 - y)\,\hat{\imath} + 4z\,\hat{\jmath} + x^2\,\hat{k}) \cdot d\vec{r}$, where C is the intersection of the plane $z = 2$ and the cone $z = \sqrt{x^2 + y^2}$.
Parameterize the cone as
$\vec{r}(r, \theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\,\hat{k}$,
for $0 \le r \le 2,\ 0 \le \theta \le 2\pi$.

Then $\hat{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|} = \frac{1}{\sqrt{2}}(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} + \hat{k})$.

curl $F = (P_y - N_z)\,\hat{\imath} + (M_z - P_x)\,\hat{\jmath} + (N_x - M_y)\,\hat{k} = -4\,\hat{\imath} - 2r\cos\theta\,\hat{\jmath} + \hat{k}$.

curl $F \cdot \hat{n} = \frac{1}{\sqrt{2}}(4\cos\theta + r\sin(2\theta) + 1)$ and $d\sigma = r\sqrt{2}\,dr\,d\theta$.

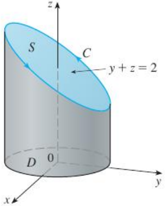
By Stokes' theorem,

$$\oint_C F \cdot d\vec{r} = \iint_S \text{curl } F \cdot \hat{n}\,d\sigma$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{\sqrt{2}}(4\cos\theta + r\sin(2\theta) + 1)r\sqrt{2}\,dr\,d\theta = 4\pi.$$

Example 41

Evaluate $\oint_C (-y^2\,\hat{\imath} + x\,\hat{\jmath} + z^2\,\hat{k}) \cdot d\,\vec{r}$, where C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clock-wise when looked from above.



$F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, where $M = -y^2,\ N = x,\ P = z^2$.

curl $F = (P_y - N_z)\,\hat{\imath} + (M_z - P_x)\,\hat{\jmath} + (N_x - M_y)\,\hat{k} = (1 + 2y)\,\hat{k}$.

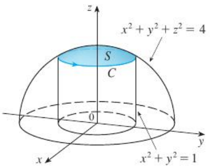
Here, there are many surfaces with boundary C.
We choose a convenient one:
the surface S on the plane $y + z = 2$ with boundary as C.
Its projection on the xy-plane is the disc $D:\ x^2 + y^2 \leq 1$.
Thus, with $z = g(x, y) = 2 - y$, Stokes' theorem gives

$$\oint_C F \cdot d\,\vec{r} = \iint_S \text{curl } F \cdot \hat{n}\, d\sigma = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3}\sin\theta\right) d\theta = \pi.$$

Example 42

Compute $\iint_S \text{curl}\, F \cdot \hat{n}\, d\sigma$, where $F = xz\,\hat{\imath} + yz\,\hat{\jmath} + xy\,\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.



The boundary curve C is obtained by solving the two equations to get $z^2 = 3$. Since $z > 0$, we have the curve C as $x^2 + y^2 = 1,\; z = \sqrt{3}$. In vector parametric form,

$$C: \quad \vec{r}(\theta) = \cos\theta\;\hat{\imath} + \sin\theta\;\hat{\jmath} + \sqrt{3}\;\hat{k} \text{ for } 0 \le \theta \le 2\pi.$$

Then $F(\vec{r}(\theta)) = \sqrt{3}\cos\theta\,\hat{\imath} + \sqrt{3}\sin\theta\;\hat{\jmath} + \cos\theta\sin\theta\,\hat{k}$.

By Stokes' theorem,

$$\iint_S \text{curl}\, F \cdot \hat{n}\, d\sigma = \oint_C F \cdot d\vec{r} = \int_0^{2\pi} F \cdot \vec{r}\,'(\theta)\, d\theta$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\sin\theta\cos\theta)\, d\theta = 0.$$

Conservative Field Again

Stokes' theorem can be generalized to piecewise smooth surfaces like union of sides of a polyhedra.
Here, we take the integral over the sides as the sum of integrals over each individual side.

Similarly, Stokes' theorem can be generalized to surfaces with holes. The line integrals are to be taken over all the curves which form the boundaries of the holes.
The surface integral over S of the normal component of curl F equals the sum of the line integrals around all the boundary curves of the tangential component of F, where the curves are to be traced in the direction induced by the orientation of S.
Recall that in a conservative field curl $F = 0$.
Then by Stokes' theorem, it follows that $\oint_C F \cdot d\vec{r} = 0$.

Theorem 11: If curl $F = 0$ at each point of an open simply connected region D in space, then on any piecewise smooth closed path C lying in D, $\oint_C F \cdot d\vec{r} = 0$.

Unification

We have seen how to relate an integral of a function over a region with the integral of possibly some other related function over the boundary of the region.

For definite integrals on intervals: $\int_a^b f'(t)\,dt = f(b) - f(a)$.

For a path from a point P to a point Q in $\mathbb{R}^3$: $\int_C \nabla f \cdot ds = f(Q) - f(P)$.

For a region D in $\mathbb{R}^2$: $\iint_D (N_x - M_y)\,dA = \int_{\partial D} F \cdot d\vec{r}$.

For a surface S in $\mathbb{R}^3$: $\iint_S \text{curl } F \cdot \hat{n}\,d\sigma = \int_C F \cdot d\vec{r}$.

It suggests a unification; and we use the divergence of a vector field for this purpose.

Gauss' Divergence Theorem

Recall that div $F = \nabla \cdot F$. That is, the divergence of a vector field $F = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}$ is the scalar function div $F = M_x + N_y + P_z$.

Our generalization is $\iiint_D \text{div } F\, dV = \iint_S F \cdot \hat{n}\, d\sigma$.

Theorem 12: Let S be a piecewise smooth simple closed bounded surface that encloses a solid region D in $\mathbb{R}^3$. Suppose S has been oriented positively by its outward normals. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains D. Then

$$\iint_S F \cdot \hat{n}\, d\sigma = \iiint_D \text{div } F\, dV.$$

Again, we omit its proof.

Example 43

Consider the field $F = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ over the sphere $S:\ x^2 + y^2 + z^2 = a^2$.

The outer unit normal to S computed from ∇f, where

$$f = x^2 + y^2 + z^2 - a^2, \quad \hat{n} = \frac{2(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{1}{a}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}).$$

On the given surface, $F \cdot \hat{n}\, d\sigma = \dfrac{1}{a}(x^2 + y^2 + z^2)\, d\sigma = a\, d\sigma.$

Therefore, $\displaystyle\iint_S F \cdot \hat{n}\, d\sigma = \iint_S a\, d\sigma = a \times \text{Area of } S = 4\pi a^3.$

Now, $\operatorname{div} F = M_x + N_y + P_z = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3.$

Therefore, with D as the ball bounded by S,

$$\iiint_D \operatorname{div} F\, dV = \iiint_D 3\, dV = 3 \times \text{Volume of } D = 4\pi a^3.$$

Example 44

Find the outward flux of the vector field $xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$ through the surface cut from the first octant by the planes $x = 1,\; y = 1$ and $z = 1$.

The solid D is a cube having six faces. Call the surface of the cube as S. Instead of computing the surface integral, we use Divergence theorem.

With $F = xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$, we have

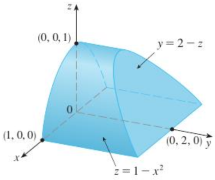
$$\text{div } F = \frac{\partial(xy)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(zx)}{\partial z} = y + z + x.$$

Therefore the required flux is

$$\iint_S F \cdot \hat{n}\, d\sigma = \iiint_D \text{div } F\, dV = \int_0^1 \int_0^1 \int_0^1 (y + z + x)\, dxdydz = \frac{3}{2}.$$

Example 45

Evaluate $\iint_S F \cdot \hat{n}\, d\sigma$, where S is the surface of the solid D bounded by the parabolic cylinder $z = 1 - x^2$, and the planes $y = 0$, $z = 0$, and $y + z = 2$; and $F = xy\,\hat{\imath} + y^2 + e^{xz^2}\,\hat{\jmath} + \sin(xy)\,\hat{k}$.



S has four sides. Instead of computing the surface integrals, we use Divergence theorem. We have

$$\text{div } F = (xy)_x + (y^2 + e^{xz^2})_y + (\sin(xy))_z = 3y.$$

And D is given by $-1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z$.

Therefore,

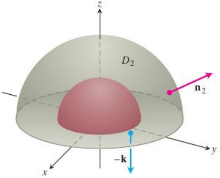
$\iint_S F \cdot \hat{n}\, d\sigma = \iiint_D \text{div } F\, dV = \iiint_D 3y\, dV$

$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx = \int_{-1}^{1} \int_0^{1-x^2} \frac{(2-z)^2}{2}\, dz\, dx$

$= -\frac{1}{2} \int_{-1}^{1} [(x^2+1)^3 - 8]\, dx = \frac{184}{35}.$

Example 46

Find the outward flux of the vector field $F = (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})/(x^2+y^2+z^2)^{3/2}$ across the boundary of the solid $D:\ 0 < a^2 \le x^2+y^2+z^2 \le b^2$.



Write $\rho = \sqrt{x^2+y^2+z^2}$. Then $\dfrac{d\rho}{dx} = \dfrac{x}{\rho}$. With $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, we have $M_x = \dfrac{\partial(x\rho^{-3})}{\partial x} = \rho^{-3} - 3x\rho^{-4}\dfrac{\partial\rho}{\partial x} = \dfrac{1}{\rho^3} - \dfrac{3x^2}{\rho^5}$.

Similarly, $N_y = \dfrac{1}{\rho^3} - \dfrac{3y^2}{\rho^5}$ and $P_z = \dfrac{1}{\rho^3} - \dfrac{3z^2}{\rho^5}$.

Then div $F = \dfrac{3}{\rho^3} - \dfrac{3x^2+3y^2+3z^2}{\rho^5} = 0.$

Thus the required flux is $\iiint_D \text{div } F\,dV = 0$.

In fact, flux through the inner surface and flux through the outer surface are in opposite directions.

Example 47

Consider the same vector field $F = (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})/(x^2 + y^2 + z^2)^{3/2}$ of Example 46. Let S be any sphere centered at the origin. Show that the flux through S is a constant.

We compute the flux directly. Let S be the sphere $x^2 + y^2 + z^2 = a^2$ for any $a > 0$. The gradient computed from $f = x^2 + y^2 + z^2 - a^2$ gives the outward unit normal to S as

$$\hat{n} = \frac{2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{a}.$$

Therefore, on the sphere S,

$$F \cdot \hat{n} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{1}{a^2}.$$

Then

$$\iint_S F \cdot \hat{n}\, d\sigma = \iint_S \frac{1}{a^2}\, d\sigma = \frac{1}{a^2} \times \text{Area of } S = 4\pi.$$

Review Problems

Problem 1: Compute the line integral of the vector function

$$x^3\,\hat{\imath} + 3zy^2\,\hat{\jmath} - x^2 y\,\hat{k}$$

along the straight line segment L from the point $(3, 2, 1)$ to $(0, 0, 0)$.

The parametric equation of the line segment joining these points is

$$x = -3t,\ y = -2t,\ z = -t \text{ for } -1 \leq t \leq 0.$$

The derivatives of these with respect to t are

$$x_t = -3,\ y_t = -2,\ z_t = -1.$$

Then the required line integral is

$$\int_L x^3\,dx + 3zy^2\,dy - x^2 y\,dz$$

$$= \int_{-1}^{0} [(-3t)^3(-3) + 3(-t)(-2t)^2(-2) - (-3t)^2(-2t)(-1)]\,dt = \frac{-87}{4}.$$

Problem 2

Let C be the portion of the curve $y = x^3$ from $(1, 1)$ to $(2, 8)$. Compute

$$\int_C (6x^2 y\, dx + 10xy^2\, dy).$$

C has the parametrization: $x = t,\ y = t^3,\ 1 \le t \le 2.$

Then $x_t = 1,\ y_t = 3t^2.$

The line integral is

$$\int_C (6x^2 y\, dx + 10xy^2\, dy) = \int_1^2 (6t^5 \cdot 1 + 10t^7 \cdot 3t^2)\, dt = 3132.$$

Problem 3

Let a closed smooth surface S be such that any straight line parallel to the z-axis cuts it in no more than two points. Let n_3 denote the z-component of the unit outward normal $\vec{n}$ to the surface S. Then what is $\iint_S z n_3 \, d\sigma$?

In this case, S has an upper part and a lower part.

Suppose they are given, respectively, by the equations

$$z = f_u(x,y), \quad z = f_b(x,y).$$

Let D be the projection of S on the xy-plane. Then

$$\iint_S z\, n_3 \, d\sigma = \iint_D f_u(x,y)\, d\sigma - \iint_D f_b(x,y)\, d\sigma.$$

Problem 4

Prove that the integral of the Laplacian over a planar region is the same as the integral, over the boundary curve, of the directional derivative in the direction of the unit normal to the boundary curve.

We rephrase: Let $f(x, y)$ be a function defined over a simply connected region D in the xy-plane. Let C be the boundary curve of D. Denote by $D_n f(x, y)$ the directional derivative of f in the direction of the unit outer normal $\hat{n}$ to C. Show that $\iint_D (f_{xx} + f_{yy}) dA = \int_C D_n f \, ds$.

Let θ be the angle between $\hat{n}$ and $\hat{\imath}$, the x-axis. Then $\hat{n} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$. If α is the angle between the tangent line to C and the x-axis, then $\cos\alpha = -\sin\theta$ and $\sin\alpha = \cos\theta$. Then

$$dx = \cos\alpha\, ds = -\sin\theta\, ds \text{ and } dy = \sin\alpha\, ds = \cos\theta\, ds.$$

Consequently, the directional derivative $D_n f$ is given by

$$D_n f(x, y) = (f_x\,\hat{\imath} + f_y\,\hat{\jmath}) \cdot \hat{n} = f_x \cos\theta + f_y \sin\theta.$$

For the vector function $F = f_x\,\hat{\imath} + f_y\,\hat{\jmath}$, by Green's theorem, we obtain

$$\iint_D (f_{xx} + f_{yy}) dA = \int_C f_x dy - f_y dx = \int_C (f_x \cos\theta + f_y \sin\theta) ds = \int_C D_n f \, ds.$$

Problem 5

Let f and g be functions with continuous partial derivatives up to second order on a region D in space, which has a smooth boundary ∂D. Denote by Δf and Δg their Laplacians. Prove the Green's formula: $\iiint_D (g\Delta f - f\Delta g) dV = \iint_{\partial D} \left(g\frac{\partial f}{\partial \hat{n}} - f\frac{\partial g}{\partial \hat{n}} \right) d\sigma$.

Let $F = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$. Gauss' divergence theorem says that

$$\iiint_D \text{div}\, F\, dV = \iint_S F \cdot \hat{n}\, d\sigma.$$

Suppose the unit normal $\hat{n}$ has the components a, b, c in the x, y, z-directions. Then

$$\iiint_D (M_x + N_y + P_z)\, dV = \iint_D (aM + bN + cP)\, d\sigma.$$

Substitute $M = gf_x - fg_x,\ N = gf_y - fg_y,\ P = gf_z - fg_z$. Then

$$M_x + N_y + P_z = g(f_{xx} + f_{yy} + f_{zz}) - f(g_{xx} + g_{yy} + g_{zz}) = g\Delta f - f\Delta g.$$

$$aM + bN + cP = g(af_x + bf_y + cf_z) - f(ag_x + bg_y + cg_z) = g\frac{\partial f}{\partial \hat{n}} - f\frac{\partial g}{\partial \hat{n}}.$$

Now Green's formula follows from Gauss' divergence theorem.

Problem 6

Evaluate $\int_C (-y\,\hat{\imath} - xy\,\hat{\jmath}) \cdot d\,\vec{r}$, where C is the circular arc joining $(1, 0)$ to $(0, 1)$ of a circle centered at the origin.

Prameterize C by $\vec{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$, for $0 \leq \theta \leq \pi/2$.

Thus $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$. Then

$$\begin{aligned}
\int_C F \cdot d\,\vec{r} &= \int_0^{\pi/2} F(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\,d\theta \\
&= \int_0^{\pi/2} (-\sin\theta\,\hat{\imath} - \cos\theta\sin\theta\,\hat{\jmath}) \cdot (-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})\,d\theta \\
&= \int_0^{\pi/2} (\sin^2\theta - \cos^2\theta\sin\theta)\,d\theta = \frac{\pi}{4} - \frac{1}{3}.
\end{aligned}$$

Problem 7

Let $F = 5z\,\hat{\imath} + xy\,\hat{\jmath} + x^2z\,\hat{k}$. Is $\int_C F \cdot d\vec{r}$ the same if C is a curve from $(0,0,0)$ to $(1,1,1)$, given by

(a) $\vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath} + t\,\hat{k}$ for $0 \le t \le 1$;
(b) $\vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath} + t^2\,\hat{k}$ for $0 \le t \le 1$?

(a) $F(\vec{r}(t)) = 5t\,\hat{\imath} + t^2\,\hat{\jmath} + t^3\,\hat{k}$. $d\,\vec{r}(t) = \hat{\imath} + \hat{\jmath} + \hat{k}$. Thus

$$\int_C F \cdot d\,\vec{r} = \int_0^1 (5t + t^2 + t^3)dt = \frac{37}{12}.$$

(b) $F(\vec{r}(t)) = 5t\,\hat{\imath} + t^2\,\hat{\jmath} + t^3\,\hat{k}$. $d\,\vec{r}(t) = \hat{\imath} + \hat{\jmath} + 2t\,\hat{k}$. Thus

$$\int_C F \cdot d\,\vec{r} = \int_0^1 (5t + t^2 + 2t^5)dt = \frac{28}{12}.$$

Review Problems Contd.

Problem 8: Let D be a simply connected region containing a smooth curve C from $(0,0,0)$ to $(2,2,2)$. Evaluate $\int_C (2xdx + 2ydy + 4zdz)$.

$F = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 4z\,\hat{k} = \text{grad}\, f$, where $f = x^2 + y^2 + 2z^2$.

Therefore, the line integral is independent of path C.

Hence its value is $f(2,2,2) - f(0,0,0) = 16$.

Problem 9: Evaluate $I = \iint_S (7x\,\hat{\imath} - z\,\hat{k}) \cdot \hat{n}\, d\sigma$ over the surface $S: \quad x^2 + y^2 + z^2 = 4$.

$\text{div}\, F = \text{div}\left(7x\,\hat{\imath} - z\,\hat{k}\right) = 7 - 1 = 6$.

So, $I = 6 \times \text{volume of } S = 64\pi$.

Problem 10

Evaluate $I = \int_C (3x^2\,dx + 2yz\,dy + y^2\,dz)$, where C is a smooth curve from $(0, 1, 2)$ to $(1, -1, 7)$ by showing that F here is a potential.

In order that $F = \text{grad}\, f$, we should have

$$f_x = M = 3x^2, \quad f_y = N = 2yz, \quad f_z = P = y^2.$$

To obtain such a possible f, we use integration and differentiation:

$$f = x^3 + g(y,z), \quad f_y = g_y = 2yz, \quad g = y^2 z + h(z),$$

$$f_z = y^2 + h'(z) = y^2, \quad h'(z) = 0, \quad h(z) = \text{ constant.}$$

Then $f = x^3 + y^2 z$. We verify that $F = \text{grad}\, f$. Therefore,

$I = F(1, -1, 7) - f(0, 1, 2) = 6.$

Problem 11

Determine whether the following integral is independent of Path:

$$I = \int_C (2xyz^2\, dx + (x^2z^2 + z\cos(yz))\, dy + (2x^2yz + y\cos(yz)\, dz)$$

and then evaluate it for a line segment joining $(0,0,1)$ to $(1,\pi/4,2)$.

Here, $M = 2xyz^2,\ N = x^2z^2 + z\cos(yz),\ P = 2x^2yz + y\cos(yz)$. Then

$$M_y = 2xz^2 = N_x,\ N_z = 2x^2z + \cos(yz) - yz\sin(yz) = P_y,\ P_x = 4xyz = M_z.$$

Hence the line integral is independent of path.

We find f such that $F = \text{grad}\, f$. Now,

$$f = \int N dy = x^2z^2y + \sin(yz) + g(x,z),\ f_x = 2xz^2y + g_x = M = 2xyz^2.$$

$$g_x = 0,\ g = h(z),\ f_z = 2x^2yz + y\cos(yz) + h'(z) = P = 2x^2yz + y\cos(yz),\ h'(z) = 0.$$

Taking $h(z) = 0$, we get $f(x,y,z) = x^2yz^2 + \sin(yz)$ as a possible potential. Then $I = f(1,\pi/4,2) - f(0,0,1) = \pi + 1$.

Problem 12

Use Green's theorem to compute the area of the region
(a) bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.
(b) bounded by the cardioid $r = a(1 - \cos\theta)$ for $0 \le \theta \le 2\pi$.

(a) Recall: Green's theorem gave Area of $D = \frac{1}{2}\oint_{\partial D}(x\,dy - y\,dx)$.
The ellipse $x^2/a^2 + y^2/b^2 = 1$ has the parameterization
$x(t) = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$. Then its area is

$$\frac{1}{2}\int_0^{2\pi}(xy' - yx')\,dt = \frac{1}{2}\int_0^{2\pi}(ab\cos^2 t - (-ab\sin^2 t))\,dt = \pi ab.$$

(b) In polar form, $x = r\cos\theta,\ y = r\sin\theta$. Then
$dx = \cos\theta\,dr - r\sin\theta\,d\theta$ and $dy = \sin\theta\,dr + r\cos\theta\,d\theta$.
Consequently the area is equal to $\frac{1}{2}\oint_{\partial D} r^2\,d\theta$. Thus the area of the region bounded by the cardioid $r = a(1 - \cos\theta)$ for $0 \le \theta \le 2\pi$, is

$$\frac{a^2}{2}\int_0^{2\pi}(1 - \cos\theta)^2\,d\theta = \frac{3\pi}{2}a^2.$$

Problem 13

Compute the flux of the water through the parabolic cylinder $S: \; y = x^2, \; 0 \le x \le 2, \; 0 \le z \le 3$ if the velocity vector $v = F = 3z^2\,\hat{\imath} + 6\,\hat{\jmath} + 6zx\,\hat{k}$, speed being measured in m/sec.

Write $x = u, \; z = v$. We have $y = x^2 = u^2$. The surface is

$$S: \quad \vec{r} = u\,\hat{\imath} + u^2\,\hat{\jmath} + v\,\hat{k}, \quad \text{for } 0 \le u \le 2, \; 0 \le v \le 3.$$

Then

$$\vec{n} = \vec{r}_u \times \vec{r}_v = (\hat{\imath} + 2u\,\hat{\jmath}) \times \hat{k} = 2u\,\hat{\imath} - \hat{\jmath}.$$

On S,

$$F(\vec{r}(u,v)) = 3v^2\,\hat{\imath} + 6\,\hat{\jmath} + 6uv\,\hat{k}.$$

Hence $F(\vec{r}(u,v)) \cdot \vec{n} = 6uv^2 - 6$. Consequently the flux is

$$\iint_S F\cdot\vec{n}\; dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv = \int_0^3 (12v^2 - 12)\, dv = 72\,\mathrm{m}^3/\mathrm{sec}.$$

Problem 14

A torus is generated by rotating a circle C about a straight line L in space so that C does not intersect or touch L. If L is the z-axis and C has radius b and its centre has distance a $(> b)$ from L, then compute the surface area of the torus.

The surface S of the torus is represented by

$$\vec{r}(u, v) = (a + b\cos v)\cos u\,\hat{\imath} + (a + b\cos v)\ \sin u\ \hat{\jmath} + b\sin v\ \hat{k}.$$

Here, v is the angle in describing the circle and u is the angle of rotation. Thus $0 \le u, v \le 2\pi$. And

$$\begin{aligned}
\vec{r}_u &= -(a + b\cos v)\sin u\,\hat{\imath} + (a + b\cos v)\cos u\ \hat{\jmath} \\
\vec{r}_v &= -b\sin v\cos u\,\hat{\imath} - b\sin v\sin u\ \hat{\jmath} + b\cos v\ \hat{k} \\
\vec{r}_u \times \vec{r}_v &= b(a + b\cos v)(\cos u\cos v\,\hat{\imath} + \sin u\cos v\ \hat{\jmath} + \sin v\ \hat{k})
\end{aligned}$$

Hence $|\vec{r}_u \times \vec{r}_v| = b(a + b\cos v)$ and the area is

$$\iint_C |\vec{r}_u \times \vec{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)\,du\,dv = 4\pi^2 ab.$$

Problem 15

Let S be the closed surface consisting of the cylinder $x^2+y^2=a^2,\ 0\le z\le b$ and the circular disks $x^2+y^2\le a^2$ one with $z=0$ and the other with $z=b$. By transforming to a triple integral evaluate

$$I=\iint_S (x^3\,dy\,dz+x^2y\,dz\,dx+x^2z\,dx\,dy).$$

$F=M\,\hat{\imath}+N\,\hat{\jmath}+P\,\hat{k}$, where $M=x^3,\ N=x^2y,\ P=x^2z.$

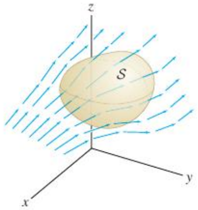
Then div $F=5x^2$.

Let D be the solid bounded by S. In cylindrical co-ordinates, using Gauss' divergence theorem,

$$I=\iiint_D 5x^2 dV=5\int_0^b\int_0^a\int_0^{2\pi} r^2\cos^2\theta\, r\,dr\,d\theta\,dz=\frac{5}{4}\pi a^4 b.$$

Problem 16

Compute the flux of the vector field
$F = (z^2 + xy^2)\,\hat{\imath} + \cos(x+z)\,\hat{\jmath} + (e^{-y} - zy^2)\,\hat{k}$
through the boundary of the surface given in the figure:



$$\text{div }(F) = \frac{\partial}{\partial x}(z^2 + xy^2) + \frac{\partial}{\partial y}\cos(x+z) + \frac{\partial}{\partial z}(e^{-y} - zy^2) = 0.$$

Let D be the region enclosed by S. By the Divergence theorem,

$$\text{Flux through } S = \iiint_D \text{div } F \, dV = 0.$$