Comparing Cardinalities of Sets

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Abstract

Students often have motivational difficulty in accepting the definitions of comparing cardinalities of sets. The key to the application of the concepts lies in the so called Schröder-Bernstein theorem, the proof of which is often avoided due to its difficulty level. In this short article, we discuss the remarkable results around this theorem and finally give a very simple proof. On the way, we also construct a simple proof of uncountability of the set of real numbers, which follows directly from the completeness property.

1 Introduction

To determine the number of elements in a finite set, we count the elements. If the result is a certain natural number n, then by such a counting, we have put the set in one-one correspondence with the set $\{1, 2, 3, \ldots, n\}$. Thus even if we do not know the names of different numbers, we still can determine whether two given finite sets have the same number of elements or not. Well, we simply try to put them in one-one correspondence; if we succeed, then they have the same number of elements, else, they have different number of elements. For arbitrary sets, the *number of elements* is called *cardinality*.

In the sequel, we denote the cardinality of a set A by $|A|$. For sets A and B, we write $A - B$ instead of $A \setminus B$, which is the set of elements of A that are not in B. If $f : A \to B$ is any map, $x \in A$, and $X \subseteq A$, then we write $f(x)$ for that element in B to which f associates x, and $f(X)$ denotes the set of all elements $f(x)$ where $x \in X$. Thus $f(A)$ is the range of f. For any subset C of B, $f^{-1}(C)$ denotes the set $\{x \in A : f(x) \in C\}$. If $C \cap f(A) = \emptyset$, then $f^{-1}(C) = \emptyset$. When f is one-one and onto, $f^{-1} : B \to A$ is a map, and $f^{-1}(b) = a$ if and only if $f(a) = b$ for $a \in A$ and $b \in B$.

We agree that if A and B are sets, then $|A| = |B|$ if and only if there exists a one-one map from one onto the other. But then how to define the notion of $|A| \le |B|$? Looking at the definition of $|A| = |B|$, it is tempting to break it into two parts, that is,

> There exists a one-one map from A to B.
> There exists a map from A onto B.

But there is a gap between the existence of a one-one map from A onto B and the conjunction of the two conditions above. It is quite possible that even if the above two conditions are satisfied, there may not exist a map which is both one-one and onto. It suggests to formulate another alternative of breaking $|A| = |B|$ into the following two parts:

> There exists a one-one map from A to B.
> There exists a one-one map from B to A.

Comparing these two formulations, we ask whether the second parts of them are equivalent or not. That is, whether the following are equivalent:

> There exists a map from A onto B.
> There exists a one-one map from B to A.

We see that if there exists a one-one map from B to A, then we have a bijection from B to $f(B) \subseteq A$. Then $f^{-1} : f(B) \to A$ is onto A. Now, if $A - f(B)$ is nonempty, then associate all elements in $A - f(B)$ to a particular element of B. This extension of the map f^{-1} is clearly a map from A onto B.

What about the converse? Suppose there exists a map g from B onto A. Then look at the reverse arrows of this g. For any $a \in A$, consider the set $g^{-1}(\{a\})$. Using Axiom of Choice, get an element $x_a \in g^{-1}(\{a\})$. The map that takes a to x_a is thus one-one from A to B. In the absence of the axiom of choice, it does not seem possible to prove this part.

A natural alternative is to take $|A| \leq |B|$ when there exists a one-one map from A onto a subset of B. However, this is equivalent to having a one-one map from A to B.

2 Uncountabilty

With these considerations, we compare cardinalities of sets as follows.

Definition 1. *Let A and B be sets.*
$|A| = |B|$ *if there exists a one-one map from A onto B.*
$|A| \leq |B|$ *if there exists a one-one map from A to B. $|A| \leq |B|$ is also written as $|B| \geq |A|$.*
$|A| < |B|$ *if $|A| \leq |B|$ but $|A| \neq |B|$. $|A| < |B|$ is also written as $|B| > |A|$.*

If $A \subseteq B$, then the identity map $I : A \to B$ is one-one. Hence $|A| \leq |B|$. The set E of all even positive numbers has cardinality less than or equal to that of $\mathbb{N}$. However, $|E| = |\mathbb{N}|$ since the map $f : \mathbb{N} \to E$ defined by $f(n) = 2n$ is a one-one and onto map. It is easy to see that for sets A, B, C, if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$ using composition of maps.

Let A be a set. A is said to be *finite* if eiher $A = \emptyset$ or $|A| = n = |\{1, 2, \ldots, n\}|$; A is called *denumerable* if $|A| = |\mathbb{N}|$; and A is *countable* if $|A| \leq |\mathbb{N}|$. All subsets of a countable set are countable; thus all supersets of uncountable sets are uncountable. It can be shown by induction that if A is any infinite set, then $|\mathbb{N}| \leq |A|$. Every finite set is countable and all infinite subsets of $\mathbb{N}$ are denumerable and thus countable. All elements of a denumerable set can be enumerated in a non-ending sequence of distinct elements:

$$x_1, x_2, \ldots, x_n, \ldots$$

The set $\mathbb{Z}$ of all integers is denumerable since the map $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(m) = \begin{cases} -1 - 2m & \text{if } m < 0 \\ 2(1+m) & \text{if } m \geq 0 \end{cases}$$

is a one-one and onto map. The set $\mathbb{Q}$ of all rational numbers is denumerable since it contains $\mathbb{Z}$ and the map $g : \mathbb{Q} \to \mathbb{Z}$ defined by

$$g\Big(\frac{p}{q}\Big) = \begin{cases} 2^p 3^q & \text{if } p \in \mathbb{N} \cup \{0\}, q \in \mathbb{N} \\ -2^p 3^q & \text{if } -p \in \mathbb{N}, q \in \mathbb{N} \end{cases}$$

is one-one; assuming that p/q is in reduced form. Wait! Have we exhibited a map from $\mathbb{Q}$ to $\mathbb{Z}$ which is one-one and onto? No! Then how do we conclude that $\mathbb{Q}$ is denumerable? By now, we have only proved that

$$|\mathbb{Q}| \le |\mathbb{Z}| \quad \text{and} \quad |\mathbb{Z}| \le |\mathbb{Q}|.$$

Does it follow that $|\mathbb{Q}| = |\mathbb{Z}|$? Yes, by Schröder-Bernstein theorem, which says the following:

> If there exist a one-map from A to B and a one-one map from B to A, then there exists a one-one map from A onto B.

We will prove this result in Section 3. What about the set $\mathbb{R}$ of real numbers? Recall that $\mathbb{R}$ is a (the) complete ordered field containing $\mathbb{Q}$. Along with the usual properties of addition, multiplication and of the order relation $\le$, it satisfies the following, called the *completeness principle*:

> Every subset of $\mathbb{R}$ which is bounded above, has a least upper bound (lub), and every subset of $\mathbb{R}$ which is bounded below, has a greatest lower bound (glb).

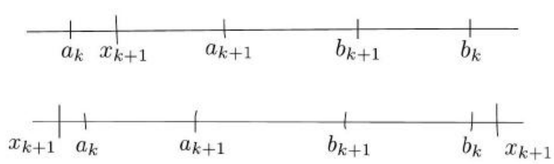
In fact, existence of lub guarantees the existence of glb and vice versa. Using this property, we give a proof of uncountability of $\mathbb{R}$.

Theorem 1. $\mathbb{R}$ *is uncountable.*

Proof. On the contrary, suppose $\mathbb{R}$ is countable. Then $[0,1]$ is countable since $[0,1] \subseteq \mathbb{R}$. But $[0,1]$ is not a finite set since $f : \mathbb{N} \to [0,1]$, defined by $f(n) = 1/n$, is one-one. Hence $[0,1]$ is denumerable. Then, let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of $[0,1]$. For each $n \in \mathbb{N}$, construct a sub-interval $[a_n, b_n]$ of $[0,1]$ that does not contain x_n, inductively, as in the following:

> Initially, set $a_0 := 0,\ b_0 := 1$.
> Suppose, for $k \ge 0,\ a_k, b_k$ have already been chosen. Choose $a_{k+1},\ b_{k+1}$ as follows:
> If $a_k < x_{k+1} < b_k$, then $y_k := x_{k+1}$, else, $y_k := a_k$.
> $a_{k+1} := y_k + (b_k - y_k)/3,\ b_{k+1} := y_k + 2(b_k - y_k)/3$.

The construction says that if x_{k+1} lies in the open interval (a_k, b_k), then choose the interval $[a_{k+1}, b_{k+1}]$ as the middle third of $[x_{k+1}, b_k]$; and if $x_{k+1} \le a_k$ or $x_{k+1} \ge b_k$, then choose $[a_{k+1}, b_{k+1}]$ as the middle third of $[a_k, b_k]$.



We observe that for each $n \in \mathbb{N}$, $[a_n, b_n] \ne \emptyset$ and $x_n \notin [a_n, b_n]$. Moreover,

$$0 < a_1 < a_2 \cdots < a_n < \cdots < \cdots < b_n < \cdots < b_2 < b_1 < 1.$$

Write $a = \text{lub}\,\{a_n : n \in \mathbb{N}\}$ and $b = \text{glb}\,\{b_n : n \in \mathbb{N}\}$. Then, $0 < a \le b < 1$. Thus $[a,b]$ is a nonempty sub-interval of $[0,1]$. Moreover,

$$x_n \notin [a_n, b_n] \supseteq [a, b] \quad \text{for each } n \in \mathbb{N}.$$

That is, no number in $[a,b]$ is enumerated in the assumed enumeration, a contradiction. $\square$

It can be shown that each real number has a decimal expansion; see [1]. This representation is not unique since any number having a finite number of digits after the decimal point does have exactly another representation having an infinite number of digits after the decimal point. For example, $0.12 = 0.11999\cdots$. We agree to always use the infinite one whenever a choice exists. The usual proof of uncountability of the semi-open interval $(0, 1]$ uses the *diagonalization method* of Cantor, which we now explain. Suppose, on the contrary, that $(0, 1]$ is countable. Then we have an enumeration of all numbers in $(0, 1]$ as

$$\begin{array}{c} 0.a_{11}\, a_{12}\, a_{13}\ \cdots\ a_{1n}\ \cdots \\ 0.a_{21}\, a_{22}\, a_{23}\ \cdots\ a_{2n}\ \cdots \\ \vdots \\ 0.a_{n1}\, a_{n2}\, a_{n3}\ \cdots\ a_{nn}\ \cdots \\ \vdots \end{array}$$

where each a_{ij} is a digit ranging from 0 to 9, and in which a finite decimal ends with 9s rather than 0s. It is of course immaterial whether we choose ending with 0s or 9s, but we must resort to one and not the other so that repetitions are avoided. We then construct a decimal number

$$0.b_1b_2b_3\cdots b_n\cdots$$

where $b_i = 0$ if $a_{ii} = 9$ else $b_i = a_{ii} + 1$. Then this new decimal number is in $(0, 1]$ but it differs from each in the list above. Therefore, no enumeration of numbers in $(0, 1]$ can have all the numbers in $(0, 1]$.

Another proof uses the representation of any real number as a binary decimal. Once again, we choose to use a binary decimal number with an infinite number of digits and discard one with finite number of digits after the decimal point, if such a choice exists. The uncountability of the closed interval $[0, 1]$ is accomplished by showing that $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$ and $[0, 1]$ are in one-one correspondence; and then resorting to Cantor's theorem that for any set A, $|\mathcal{P}(A)| > |A|$.

To see that $|\mathcal{P}(\mathbb{N})| \le |[0, 1]|$, define $f : \mathcal{P}(\mathbb{N}) \to [0, 1]$ as follows.

> Let $S \subseteq \mathbb{N}$. Let $n \in \mathbb{N}$. Then $f(S)$ is the decimal number $0.a_1a_2\cdots$ in base 10 such that its nth digit $a_n = 3$ if $n \in S$, and $a_n = 4$ if $n \notin S$.

For example, $f(\{1, 2, 3\}) = 0.3334444\cdots$. Notice that f is not an onto map since for no subset A of $\mathbb{N}$, $f(A) = 0.1$. Clearly f is one-one. Hence $|\mathcal{P}(\mathbb{N})| \le |[0, 1]|$.

For the other inequality, define $g : [0, 1] \to \mathcal{P}(\mathbb{N})$ by

> Let $x = 0.b_1b_2\cdots \in [0, 1]$, where $b_i \in \{0, 1\}$. Then $g(x) = \{i \in \mathbb{N} : b_i = 1\}$.

For example, $g(0.10111\cdots) = \{1, 3, 4, 5, \ldots\}$. Again, notice that g is not an onto map, since there is no binary decimal $a \in [0, 1]$ such that $g(a) = \{1, 2\}$. For, according to the definition of g, the only suitable number a would have been 0.11, which has been discarded in favour of $0.10111\cdots$. Obviously, g is one-one. Hence $|[0, 1]| \le |\mathcal{P}(\mathbb{N})|$.

By Schröder-Bernstein theorem, $|\mathcal{P}(\mathbb{N})| = |[0, 1]|$. It remains to prove Cantor's theorem.

Theorem 2. (Cantor) *For any set A, $|A| < |\mathcal{P}(A)|$.*

Proof. Let A be any set. The function $f : A \to \mathcal{P}(A)$ defined by $f(x) = \{x\}$ is a one-one map. Therefore, $|A| \leq |\mathcal{P}(A)|$. We next show that no function from A to $P(A)$ can be onto. On the contrary, suppose that $g : A \to \mathcal{P}(A)$ is an onto map. Notice that for any $x \in A$, $g(x) \subseteq A$. Let $B = \{x \in A : x \notin g(x)\}$. Since g is an onto map, there exists $y \in A$ such that $B = g(y)$. Then $y \in B$ iff $y \notin g(y)$ iff $y \notin B$, a contradiction. □

$\mathbb{R}$ and $\mathbb{N}$ are not in one-one correspondence. But $\mathbb{R}$ and the open interval $(0, 1)$ are in one-one correspondence, since the map $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \tan(-(\pi/2) + \pi\, x)$$

is one-one and onto. If you accept Schröder-Bernstein theorem, then it is trivial to guarantee the existence of a one-one map from $[0, 1]$ onto $\mathbb{R}$. Can you construct a one-one map from $[0, 1]$ onto $\mathbb{R}$?

Notice that Cantor's theorem establishes a hierarchy of infinities:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| <, \ldots$$

It raises the question whether there exists a set whose cardinality is inbetween those of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$? Cantor conjectured that there exists no such set, however he could not prove it. Later it was shown by Gödel that the truth of this conjecture is consistent with ZFC, a widely accepted formalization of set theory by Zermelo and Fraenkel. Cohen proved that the falsity of this conjecture is also consistent with ZFC. These two proofs established the independence of this conjecture from ZFC. Thus, the conjecture is considered a hypothesis, the *continuum hypothesis*. The generalized continuum hypothesis states that if A is any infinite set, then there does not exist a set B such that $|A| < |B| < |\mathcal{P}(A)|$. It is not yet known whether there exists a formalization of set theory in which (generalizd) continuum hypothesis becomes a theorem; see [5].

Another consequence of Cantor's theorem is that the collection of all sets is not a set. For, suppose that S is the set of all sets. Then $\mathcal{P}(S) \subseteq S$. In that case, $|\mathcal{P}(S)| \leq |S|$, contradicting Cantor's theorem. This also shows that we cannot build a set corresponding to every property. For otherwise, the property "x is a set" would define the set of all sets. In fact, as the history goes, this concerns gave rise to various formalizations of set theory, one of which is ZFC.

Central to this discussion is comparing cardinalities, which rests on Schröder-Bernstein Theorem.

3 Cantor-Schröder-Bernstein Theorem

As to the name of this section, Cantor first formulated the theorem and gave a proof relying on the well ordering principle; this principle, as we know, is equivalent to the axiom of choice. Schröder gave a proof without using the axiom of choice, in which there were gaps. The first published correct proof without using the axiom of choice was by Bernstein, though it is said that Dedekind proved it a bit earlier but did notpublish. Zermelo and König have also proved the theorem; see [2, 4]. All these proofs reveal partitions of the sets into a fixed finite number of parts each, where the individual parts of both the sets are in one-one correspondence. This is the content of Banch-Mapping theorem, where the sets are partitioned into two subsets. We prove this first.

Theorem 3. (Banach Mapping) *Let A, B be nonempty sets. Let $f : A \to B$ and $g : B \to A$ be functions. Then there exist subsets A_1, A_2 of A and subsets B_1, B_2 of B such that*

$$A_2 = A - A_1,\ B_2 = B - B_1,\ f(A_1) = B_1,\ g(B_2) = A_2.$$

Proof. Consider the collection

$$\mathcal{C} = \{D \subseteq A : g(B - f(D)) \subseteq A - D\}.$$

$f(\emptyset) = \emptyset$. $B - f(\emptyset) = B$. $g(B - f(\emptyset)) = g(B) \subseteq A = A - \emptyset$. Hence, $\emptyset \in \mathcal{C}$. That is, $\mathcal{C}$ is a nonempty collection. Define

$$E = \cup_{D\in\mathcal{C}} D.$$

Now,

$$\begin{aligned} g(B - f(E)) &= g(B - f(\cup_{D\in\mathcal{C}} D)) = g(B - \cup_{D\in\mathcal{C}} f(D)) = g(\cap_{D\in\mathcal{C}}(B - f(D))) \\ &\subseteq \cap_{D\in\mathcal{C}}(g(B - f(D))) \subseteq \cap_{D\in\mathcal{C}} (A - D) = A - \cup_{D\in\mathcal{C}} D = A - E. \end{aligned}$$

That is,

$$g(B - f(E)) \subseteq A - E.$$

We want to show that $g(B - f(E)) = A - E$. On the contrary, suppose $g(B - f(E)) \neq A - E$. Then there exists an $x \in A - E$ such that $x \notin g(B - f(E))$. The conditions $x \notin g(B - f(E))$ and $g(B - f(E)) \subseteq A - E$ imply that

$$g(B - f(E)) \subseteq A - (E \cup \{x\}).$$

Since $f(E) \subseteq f(E \cup \{x\})$, $\ B - f(E \cup \{x\}) \subseteq B - f(E)$. Then

$$g(B - f(E \cup \{x\})) \subseteq g(B - f(E)) \subseteq A - (E \cup \{x\}).$$

That is, $E \cup \{x\} \in \mathcal{C}$. As $E = \cup_{D\in\mathcal{C}} D$, $E \supseteq E \cup \{x\}$. That is, $x \in E$, This contradicts $x \in A - E$. We conclude that $g(B - f(E)) = A - E$.
Finally, take $E = A_1,\ A - E = A_2,\ B_1 = f(E)$ and $B_2 = B - f(E)$. □

Cantor-Schröder-Bernstein theorem can be derived from Theorem 2 by defining the map $h : A \to B$ with $h(x) = f(x)$ for $x \in A_1$ and $h(x) = g^{-1}(x)$ for $x \in A_2$. Notice that $f : A_1 \to B_1$ is one-one and onto; so is the map $g : B_2 \to A_2$. Therefore, $h : A \to B$ is one-one and onto.

The proof of Banach mapping theorem constructs the set $E \subseteq A$ satisfying

$$g(B - f(E)) = A - E.$$

That is, $E = A - g(B - f(E))$. It means that if $\phi(X) = A - g(B - f(X))$ for subsets X of A, then E is a fixed point of this map ϕ. But under what condition(s) a map ϕ taking subsets of A to subsets of A will have a fixed point? The proof of Theorem 2 suggests this condition.

Theorem 4. (Knaster Fixed Point) *Let $\mathcal{P}(A)$ denote the power set of a nonempty set A. Let a map $\psi : \mathcal{P}(A) \to \mathcal{P}(A)$ satisfy*

$$X \subseteq Y \text{ implies } \psi(X) \subseteq \psi(Y) \quad \text{for } X, Y \subseteq A.$$

Then there exists $G \subseteq A$ such that $\psi(G) = G$.

Proof. The collection $\mathcal{K} = \{X \subseteq A : X \subseteq \psi(X)\}$ of subsets of A is nonempty since $\emptyset \subseteq \psi(\emptyset)$. Define

$$G = \cup_{Y \in \mathcal{K}} Y.$$

It says that each set in $\mathcal{K}$ is a subset of G. Now,

$$G = \cup_{Y \in \mathcal{K}} Y \subseteq \cup_{Y \in \mathcal{K}} \psi(Y) = \psi(\cup_{Y \in \mathcal{K}} Y) = \psi(G).$$

If $G \neq \psi(G)$, then there exists $x \in \psi(G)$ such that $x \notin G$. Since $G \subseteq \psi(G)$ and $x \in \psi(G)$, we see that

$$G \cup \{x\} \subseteq \psi(G) \subseteq \psi(G \cup \{x\}).$$

That is, $G \cup \{x\} \in \mathcal{K}$. Hence $G \cup \{x\} \subseteq G$. That is, $x \in G$, a contradiction.
Therefore, $\psi(G) = G$. □

Generalization has made the things simpler. The proof of Theorem 3 looks simpler than that of Theorem 2. To derive Theorem 2 from Theorem 3, all that we do is take a particular map ψ which satisfies the required condition. For this purpose, define the map $\psi : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$\psi(X) = A - g(B - f(X)) \quad \text{for } X \subseteq A.$$

Now,

$$\begin{aligned} X \subseteq Y \quad &\Rightarrow \quad f(X) \subseteq f(Y) \Rightarrow B - f(X) \supseteq B - f(Y) \\ &\Rightarrow \quad g(B - f(X)) \supseteq g(B - f(Y)) \\ &\Rightarrow \quad A - g(B - f(X)) \subseteq A - g(B - f(Y)). \end{aligned}$$

That is, $X \subseteq Y$ implies $\psi(X) \subseteq \psi(Y)$. An application of Theorem 3 completes the proof of Theorem 2.

In fact, Knaster fixed point theorem holds true in a much more general setting. Suppose A is a partially ordered set with a partial order $\leq$. It is called a *complete lattice* if every subset of A has an infimum and a supremum with respect to the partial order. We say that a map $\phi : A \to A$ is *order preserving* if $\phi(x) \leq \phi(y)$ whenever $x \leq y$. Then Knaster-Tarski fixed point theorem can be stated as follows:

> Every order preserving map on a complete lattice has a fixed point.
> Further, the set of all such fixed points of the map is a complete lattice.

The proof of this result is similar to that of Theorem 3, but is to be formulated in terms of the generalized notions appropriate to a complete lattice. Moreover, the converse of Knaster-Tarski fixed point theorem for lattices holds. It states that a lattice, i.e., a partially ordered set in which every finite subset has a minimum and a maximum, is complete if each order preserving map has a fixed point.

The generalizations have helped in constructing an independent proof of Cantor-Schröder-Bernstein theorem, which we give below. Find out in the proof, how the ideas of fixed point and partition of the sets are in action.

Theorem 5. (Cantor-Schröder-Bernstein) *Let A and B be nonempty sets. Let $f : A \to B$ and $g : B \to A$ be one-one functions. Then there exists a function $h : A \to B$, which is both one-one and onto.*

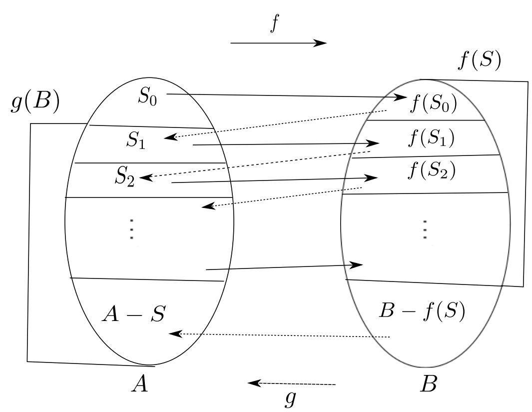
Proof. Define $S = (A - g(B)) \cup \cup_{n \in \mathbb{N}} (g \circ f)^n (A - g(B))$. See the figure below. Then $S = (A - g(B)) \cup (g \circ f)(S) \subseteq A$. Since $g(B) \subseteq A$ and $(g \circ f)(S) \subseteq A$, we obtain

$$A - S = A - ((A - g(B)) \cup (g \circ f)(S)) = (A - (A - g(B)) - (g \circ f)(S) = g(B) - g(f(S)) = g(B - f(S)).$$

The last equality follows since g is one-one. Hence $g : B - f(S) \to A - S$ is one-one and onto. Also, $f : S \to f(S)$ is one-one and onto. Hence the map $h : A \to B$ defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in S \\ g^{-1}(x) & \text{for } x \in A - S \end{cases}$$

is both one-one and onto. $\square$



$S_0 = A - g(B)$, $S_1 = g(f(S_0))$, $S_2 = g(f(S_1)), \ldots,$ $S = \cup S_i$

Compare this proof with the classical one in text books on Set Theory, for example, in [1]. For completeness, prove the facts contained in the following exercise.

Exercise Let $f : A \to B$ be a function where A and B are nonempty sets. Suppose that $A_1 \subseteq A$ and $A_2 \subseteq A$.

1. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

3. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Equality holds if f is one-one.

4. $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$. Equality holds if f is one-one.

5. Formulate and prove (2)-(3) when the operations of union and intersection are taken over a collection of subsets of A.

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