Towards a Topological Theory of Cognition

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Abstract: - With the assumption that cognition is possible, this paper attempts to construct a mathematical model of cognition. It is argued that for a holistic model of cognition, one should take care of the idea of *closeness*. In turn, this gives rise to a topological model of the agent, who is capable of cognition. Some possibilities such as computing and etcetera in such a model are discussed for assessing the implications of this visualization.

Key-Words: - Cognition, Mathematical Model, Topology, Computation

1 Introduction

There had been many attempts at modeling cognition. The central assumption of these models is the so called cognitivism [1,2,3,7]. Here, the assumption is, besides other plausible hypotheses, that the brain is doing some kind of computation in a modular fashion. The input modules present symbolic representations of the input, the perceptions. Perhaps, a central module is responsible for the main cognitive process, and then the output modules convert the symbolic output of the central module into motor actions and speech.

Cognition involves a person, say, the agent, a world or circumstance where the agent is, say, the world, and a certain model, a dynamic model of the world that the agent constructs. We will be mainly concerned with the model of the agent taking into consideration all these artifacts. How the agent discovers the structure of the reality in which he exists and lives will, of course, be dependent on what basic fabric of his model consists of. In order to present all these in a unified manner, we will make yet another model of the whole situation.

Our model is a bit different from the connectionistic learning models such as neural networks. This model as an organized representation system capable of parameterizing the states of the world according to a coherent preconceived coordinate system. The coordinatization resembles the psychological models such as the Perceptual Control Theory [5] addressing some concerns of systematicity as raised by Fodor and Pylyshyn [2].

2 Modeling the Core

Is it better to model the perceiving agent first or the the world in which he perceives and acts? Of course, modeling the world will encompass a macroscopic model of the agent. This would drag us into the objective versus subjective plausibilities of perception and knowledge. I would rather try from the view point of an (individual) agent and visualize how he models the world he perceives. This would then necessitate representing its own capabilities. Since, on the scale of subjectivity, there will be an unbounded varieties of issues, differing from agents to agents, we cannot possibly talk of *the agent*. However, this is what exactly we want to describe of an agent, the commonality of all the perceiving agents, or so to say, the core of an agent so that the model may be modified later as a case demands.

An agent is capable of perceiving the world in a certain sense, though not in its entireness. He also acts in the world, according to a situation. Hence we assume that an agent has perceptual capabilities to gather information about the immediate environment and also he has motor capabilities (including speech etc.) to initiate or take suitable (as he thinks) actions.

The immediate environment of an agent changes from time to time and from place to place. However, for an abstract model, what is of concern, is the space of all possible states of the immediate environments. For example, if cognition takes place in a two dimensional world where action and perception are limited to change of position only, the state space of immediate environments is $\mathbb{R}^2$, the plane. It is the structure of this state space which we call as the (possible) world of the agent. Knowledge of this structure is the knowledge of the reality with which the agent may interact. The basic model can also be any mathematical space. However, we may employ the notion of closeness in such a space, since our familiar worlds have this property, or, at least when we perceive and act in it. We may emphasize that the world here is not the four dimensional physical space, but the (possibly imaginary) world of the agent.

For example, when an agent is only capable of motion in a two dimensional world, his state space is the plane, and his perceptual functions include all continuous functions on that space. We take here continuous functions since the agent does not vanish and reappear while moving in his world. If the agent is limited to perceiving and acting through colors or temperatures, then his state space may include the whole spectrum and the possible temperature states. In fact, the state space here, will be the product space of these two. If the agent is concerned with introspection about his marital happiness, its state space may be the space of all his moods and a felicity function may be defined on this space taking, say, real values for specifying degree of happiness etc. However, there are certain moods which are closer to each other and may vary too much from yet others on a psychological scale. So, to keep our assumptions on the state space at minimum, we take the state space S as a topological space. In a topological space, closeness of two points are measured by the family of open sets which contain both the points.

Suppose that the agent perceives two states s ant t. Curiously (as it happens for him), he finds that for all his perceptual functions f, the values of s and t match. That is, in whatever way he can perceive, he perceives both s and t as same; $f(s) = f(t)$ for every perceptual function f. Then, it is reasonable on his part to assume that s and t are nothing but the same state. Such a requirement, stated otherwise, imposes a condition on the world and also on the perceptual functions on the space. It says that given any two distinguishable states of the world, there are distinguishable perceptual functions to distinguish them. Such a phenomenon does not occur in any arbitrary topological space; but this is guaranteed in a normal Hausdorff topological space [4]. Further, in our familiar physical world, most properties such as color, temperature, resistance, pressure etc seem to vary continuously, and our perceptual functions deal with such continuous change. Perceptual functions *seem to keep closeness invariant*, or rather, we require this invariance property. Invariance of closeness amounts to assuming that points close to each other in *reality* are not to be separated by our perception. This assumption may, somewhat loosely, be described as *consistency of cognition*. Thus, as a first approximation we may restrict the perceptual functions to continuous functions on a selected topological space.

Here is a problem in modeling the agent. If the best suitable state space and a set of all plausible perceptual functions are fixed, then the agent is *the wisest* of all possible agents, because this is the best that can be achieved under the situation. Thus the modeling does not exactly model any reasonable agent. If that is what we require, then the static model of a fixed S and a fixed A of all continuous functions on S (all perceptual functions) would do. But looking at the variations in capabilities of the agents, this would be an impractical model. In reality, the reasoning process of an agent is involved in dynamically fixing (or discovering) the state space S. We may endow the agent (S, A) with a reasoning process, or a collection of such strategies. Alternatively, we may visualize our model to be a dynamic model, where the agent is only endowed with the perceptual functions in A and is discovering slowly the structure of S by its interactions with the world via these perceptual functions. The *increasing* S should not be confused with a monotonically increasing S with respect to its extension. This increase in S may be non-monotonic and intensional, in the sense of gathering more and more information on the structure of S. This amounts to an updating of the topological space S with more elements of the state space or with a finer or coarser topology on it. It may also happen that the basic set of the state space decreases in size but the topology on it becomes finer as restricted to the smaller set.

At any given point of time, the agent assumes, by default, that he knows all that there is to know. This leads to deliberate action. At the same time, only through action the agent learns that there is a need for belief revision. This gradual updating of the state space S amounts to the assumption that S is unknown to the agent, and the only way of accessing S is through the evaluation of the perceptual functions. Using these, the agent discovers as much as possible of S. Further *as much as possible of S* can also be forced to be limited by certain other rules, if one wants to model a specific type of agents; for example, a deranged personality. In general, our model assumes that an agent would find out its current state of immediate environment, the possible states closer to it, all possible states gathered so far, and how to take action in the given state of environment through evaluation of perceptual functions in as controlled manner as possible, at any given moment.

We see here that gradual updating of S takes care of the behavior called learning. I believe that cognition and learning are inseparable so far as a practical model of cognition is in view. In contrast, if one fixes a static model, then the agent is the wisest agent and he needs no learning. However, the dynamics of the gradual discovery in S can also be embedded in a static model, as each agent can be embedded in the wisest; each agent can have a certain amount of wisdom at or up to any particular moment, but he is not wiser than the wisest. In such a case, we may think of a bigger topological space S and the agent is probably working in a subspace at any moment. This setting is debatable as to who fixes the wisest agent, the biggest possible state space S? This is specifically interesting since one who fixes it is also an agent, and his wisdom may not be enough to entitle him *the wisest*.

We are entering the age old question of whether we are justified in looking at *above the neck* with the same eyes as we look at *below the neck*. Can a subject look at himself as an object? And of what good and reliable his observation would be? As we know, we will only end up in paradoxical situations in exploring the consequences of any attempt to answer these questions. I would like to slip through the horns by suggesting that the state space is unknown but gets gradually fixed or discovered by the agent through evaluation of his perceptual functions. Thus the agent can be formally taken as a pair (S, A), where S is a normal Hausdorff topological space, and A is a set of continuous maps defined on S. The range of the maps may be any other set with or without structures, such as a two element set of *true*, or *false*, or $\mathbb{R}$, the set of real numbers. The agent may be identified with A alone *categorically*, as S is yet to be fixed by evaluations of mappings in A at or up to any moment on the immediate state of environment. It looks somewhat puzzling to have the perceptual functions without knowing exactly what are their respective domains and ranges. However, this is the case, since the agent does not have any other source than the perceptual functions for perceiving the world. His world, the domains and ranges of the perceptual functions, are dynamically upgraded through his actions.

3 Assimilation

While discovering the state of environment and imposing an appropriate structure on it, the agent takes help from the perceptual functions. Thus implicitly, the perceptual functions involve certain kind of reasoning process. The reasoning strategies may also be specified explicitly so that maneuvering the perceptual functions in a model would become easier. In such a case, one may like to define an agent as a triple (S, A, R), where the additional R is a set of reasoning strategies for assimilating the structures on the available states of S. The reasoning strategies would require a specification language for representing the observation of an agent, using them to update S, and then take appropriate actions.

The specification language may be any language; but it would be computationally advantageous to have a formal language of logic. In case, the agent is capable of representing and reasoning with his as well as other possible states of environments, he may use a modal language for specification. For example, a case of belief revision may better be represented by a shift in possible worlds. However, For simplicity, and without loosing much generality, we will take the language of first order logic as the specification language of the agent. The language will be used to note down the specific observations, facts reasoned out from the observations, and the action statements which may be translated to real actions with the assumed motor capabilities of the agent.

To take a simple example, suppose that the agent, while being present in a certain place, during certain period of time, observes that the temperature varies within some limits. The place may be fixed by choosing the latitudes and longitudes, time may be fixed by a range of values using some unit, and the temperature range may be fixed by choosing again a unit. If the perceptual functions for latitude, longitude, time, temperature with chosen units are denoted by f_1, f_2, f_3, f_4, say, then a specification of the observation may look like:

$$(\exists x \in S)((a_1 \leq f_1(x) \leq b_1) \wedge (a_2 \leq f_2(x) \leq b_2) \wedge (a_3 \leq f_3(x) \leq b_3) \wedge (a_4 \leq f_4(x) \leq b_4))$$

For specifying complex observations, such simple observation statements can be compounded using logical apparatus of first order logic.

The agent does not stop with the observations only. Just as in any scientific enterprise, the agent tries to categorize observations and comes to a conclusion in the form of a fact statement. The fact statements involve universal quantifiers instead of the existential. For example, after observing many instances (whether directly or through a belief revision), an agent concludes that "in such a place during such a period of time, the temperature can only vary between such and such degree Celsius". This may be specified by the sentence:

$$(\forall x \in S)((a_1 \leq f_1(x) \leq b_1) \wedge (a_2 \leq f_2(x) \leq b_2) \wedge (a_3 \leq f_3(x) \leq b_3) \rightarrow (a_4 \leq f_4(x) \leq b_4))$$

Again, the fact statements can be combined with the help of operators.

We note however, that though an observation statement does not necessitate analysis of truth, a fact statement does, since expression of a fact itself assumes the validity of arriving at such a conclusion from the observations. But it is, by far, a difficult problem to decide the truth of a fact statement. If the state space S is infinite, there is, in general, no hope towards evaluating the perceptual functions at each state so as to conclude the truth of a fact statement. One way is to look at the particular domain, the concerned state (sub-)space. Possibly, its structure can be used for verifying the truth. For example, in mathematics, there are ample situations, where a universal statement can be deduced from a particular observation. Suppose that a function f is continuous in an open interval with a bounded derivative. Then from the sentence

$$(\exists x \in (a, b))(c < f(x) < d)$$

one concludes

$$(\forall x \in (a, b)(c' < f(x) < d')$$

From the observation that a particular $\mathbb{NP}$-complete problem is polynomial time solvable, one concludes that every $\mathbb{NP}$-complete problem is polynomial time solvable. This is easy because mathematics uses only logical procedures, apparently, having no connection with the real world.

In a less imaginary enterprise than mathematics, one may have to use extra-logical procedures. For example, in science the requirement of such a passage from particular to universal is the so called *repeatability* of an observation. However, it is too obvious that science cannot demand repeatability under the same circumstance so far as space-time fabric is intact. What it demands is that repeatability is a requirement under sufficiently similar circumstances. There comes the scientific method of setting standards for observations. All it amounts to developing *reliable* procedures for best possible accuracy in measurements. These development of procedures or methodology in any particular scientific field gives rise to measuring conventions; which is a humbler way of expressing a fact statement. We may say that the conclusion obtained by such observations is *truth by convention*, a weaker form of a fact statement. This is in contrast with laws, where often an artifact is imposed on the reality and the reality is interpreted accordingly.

4 Restructuring

It is not clear whether a human agent has in its disposal, all the necessary perceptual functions to know the world. However, whatever it can know, it can only know through its perceptual functions, which include all the sensory receptors and all possible motor actions by which it may change some states of its world. Thus it is reasonable to assume in this context, that the world is the way it can be perceived. This perception of the world includes all the perceptual functions direct or indirect such as the capability to theorize or to create models of the reality. Obviously, the perceived world is a reconstruction of the real world. However, an agent may not have the totality of the *best possible* perceived world in his state space. To make things simpler, we assume that S is the best possible perceived world for any agent, under a context. The problem for an agent is to discover S gradually by evaluating its perceptual functions. S is unknown to the agent; gradual updation of S makes the agent more and more knowledgeable. This process of restructuring the state space also involves discovering possible connections between perceptual functions. This is often met in daily life as one quips, "Oh, this smells like burnt milk". We will consider a simple mathematical example.

Imagine S to be the plane, $\mathbb{R}^2$, with its usual Cartesian coordinates. Suppose that the agent can only move on it. We equip the agent with three perceptual functions

$$\begin{aligned} f(x,y) &= 2x + 3y \\ g(x,y) &= x - 2y \\ h(x,y) &= (3x+y)e^{7y} + e^{-x-5y} + \sin(4x - 9y/2) \end{aligned}$$

The agent does not know what S is. It tries to discover S or restructures S by taking actions and evaluating its perceptual functions. Note that even if the agent is living in a three dimensional world, it observes or movesonly in two dimensions, as its perceptual functions show; here we are using our assumption that the agent's world is the way he perceives it. For, the same agent could have been modeled in a three dimensional world just by adding a neutral variable z to the domain of the perceptual functions. However, as external observers, we know that the agent does not know it as such, without any action or an act of cognition. For such a knowledge, it acts and infers, or *restructures* the state space S.

Assuming that S is two dimensional, and the agent knows that it is so, the next problem is to see how the perceptual functions are related. In fact, the agent does not know that the perceptual functions are given by any mathematical equations. He can only take actions to discover them in their originality, if possible. This happens when we ask "How is it possible that we see a thing with its color?". The procedure the agent follows is to take actions; here he moves in S, and attempts to find these perceptual functions in order to find the structure of S as related to himself, to restructure S. The agent knows that there are certain perceptual functions f, g, h. The agent learns to control f by moving in many directions. Having achieved this, it might try to discover the form of g keeping his motion restricted in such a way that effect of f remains unchanged. Once this goal is achieved, he knows that the functions f and g are independent. He then tries to fix h, may be, in a similar way. But no matter what way he tries, he is unable to change the effect of h while keeping effects of f and g unaltered. This happens because h is expressible in terms of f and g as far as his notion of S is concerned. Explicitly,

$$h(x,y) = (f+g)e^{f-2g} + e^{g-f} + \sin(f/2 + 3g)$$

However, it is not so simple (always) a relation to be discovered by the agent. As external observers we know that h depends on f and g. Nonetheless, the agent has discovered as a *fact* that his world has only two degrees of freedom, since he cannot have any observable change in the evaluation of h without any change in f and g. His knowledge of this dependence of h on f and g is not precise.

By taking f and g as his coordinates, he coordinatizes the world S, his world, in fact. This is the *reconstruction* or restructuring of S through the agent's perceptual functions. After the restructuring of S, he would not like to through away the redundant perceptual function h, for it may help him to settle occasional errors in evaluating his other perceptual functions in some as yet unknown circumstances. Here of course, in coping with discovering f and g I had sidetracked an age old problem, exemplified by Hume; the problem of Induction.

Since all measurements are only approximate, it might quite well happen that upon refinement of his measurement procedures, the agent discovers that the perceptual functions as he had reconstructed do not behave as he thought. For example, by adding a small quantity of another independent variable z to all the perceptual functions, we may see that, f, g, and h are independent. Then the agent concludes that he is moving in three dimensions and not in two. This is how new unknown features of the reality may be gradually discovered by the agent and his restructuring of the state space may continue, say, towards perfection.

5 Expansions

As per the neurobiological knowledge we have today, we at least know that neurons are capable of directly handling perceptual data and generate motor signals in a coordinated fashion. Since the brain is showered with an ocean of sensory data every moment, this coordination is, indeed, intriguing. Somehow it filters the data received and considers only those that are relevant in the context. The data which are processed for after the elimination of the irrelevant *informs* the brain; they constitute information. Whenever the brain increases its ability to extract such information from the sensory data, we would say that the human agent has learnt something, cognition has taken place. There is a way an agent can identify information even before knowing what it is that it informs about. With many perceptual functions in A, the most prominent one is picked up by its value which of course is evaluated according to the state of immediate environment. Pragmatics tell us that there may be many significant perceptual functions instead of just one, that there might be a correlated set of some perceptual functions which is significantly informative about the state of the environment.

Since any such correlation cannot be trusted in general, the agent makes suitable motor actions to determine the genuineness of the correlation. Thus, cognition partly involves ways of analyzing the perceptual functions evaluated into a particular useful system. This useful way turns out to discovering many correlations between them and also devising possibly new perceptual functions from the old. Inventing new perceptual functions is an important activity of the agent as it gives him power to control the states of environment gradually. I may, at this point, hypothesize that our brain is computing new perceptual functions from the old ones which have their basis in sensory receptors and motor capabilities. This is how probably a child learns to act in the world. Such an activity may be termed as the expansion of the perceptual functions. This becomes all the more important for our model since the agent has no access to S other than through the perceptual functions. Thus assimilation depends upon expansion.

We may imagine the collection of all perceptual functions as a function space, and then see how this function space A gets expanded. We do not know exactly what are the modalities and operations for combining the perceptual functions for expansion. However, we may symbolize the operations in our mathematical model. Our assumptions are that these perceptual functions are continuous maps on the (topological) state space S.

Though we do not have a clear cut way of how to go about expanding A, in some restricted cases we may be able to say something. For example, we may restrict all our perceptual functions to be bounded maps; which is a plausible idea as human experiences suggest. In such a case, with S as a topological space, A can be identified with $C(S)$, the space of all bounded continuous functions on S. With multiplication of functions as the operation, $C(S)$ turns out to be a commutative Banach Algebra. Moreover, since S is a normal Hausdorff space, there are different functions in $C(S)$ which can distinguish distinct states in S. Further, if S is assumed to be compact, then it can be recovered completely from the structure of $C(S)$. There is, in fact, a constructive way [6] to recover (reconstruct) S along with its topology from $C(S)$.

If the space S is not assumed to be compact, we may think of a compactification of S. In general, a compactification of S is a compact topological space X such that S is a subspace of X and that there is no proper compact subspace of X containing S. A compactification of a space may be thought of as adding to it a suitable collection of ideal points. A constructive way for compactification is the well known Stone-Čech compactification [4]. The consequence of this mathematical process amounts to the assumption that when an agent tries to restructure S through its perceptual functions in A, it is most likely that it will end up at a compact space, which has a copy of S in it, but possibly with many idealized points. If only a substructure of S could have been cognized by the agent, then he will end at a compactification of the subspace of S. This may lead him to live in a locally compact space, in some cases. The latter situation is very much like our feeling of the receding horizon. If the sky literally joins the ground, then our world is finite. Otherwise we will always see that the sky joins the ground though it does not, a compactification of our extent of vision; though we do not assume that our world is necessarily compact.

6 Conclusions

It is quite natural to take, as a first approximation, that the state space S is finite dimensional, say, $\mathbb{R}^n$, where n may be too large a number, in billions, say. Moreover, with the added hypothesis that at any moment the agent only deals with a finite number of perceptual functions, say, $f_1, \ldots, f_m$, the relevant structure for S requires determining a relation between these perceptual functions. Mathematically, it amounts to solving an equation of the type

$$F(f_1, \cdots, f_m) = 0$$

with the boundary conditions obtained from the observations of the agent while evaluating the perceptual functions $f_1, \ldots, f_m$ at the given state of the environment. This is clearly an intractable inverse problem due to its sheer size. In general, there is no unique solution to such inverse problems. In some cases, well motivated approximate unique solutions can be considered. Perhaps this is a reason that in practical life we do not usually have a universally agreed best judgment.

On the other hand, if we consider such a structural relation between m functions, then there are at least as many such relations as are ways of choosing m items out of n, since we are considering such relations in $\mathbb{R}^n$. That is, we may have to consider at least $n!/(m!(n-m)!)$ such relations. When n is in billions, and m possibly in hundreds, this is a huge task that the brain might face computationally. Clearly no brute force approach would be feasible. One may just fix the relevant perceptual functions by a suitable approximation scheme or a statistical method.

The theoretical possibility admits of these approximation procedures and for a choice between them. This explains the variations in personality types as regards cognition. It is quite possible that connectivism has still a role to play in order to get a better approximation in a shorter period of computation. When these goals are met computationally, perhaps robots can be endowed with the power of cognition in its entireness, or at least, with entireness in a considerably large but restricted domains. And when they are put to human use, in their actions, they would be silently reverberating "topologically yours"!

References

[1] E. Dietrich, Computationalism, *Social Epistemology*, Vol.4, No.2, 1990, pp.135-154.

[2] J. A. Fodor and Z. W. Pylyshyn, Connectioninsm and Cognitive Architecture, A Critical Analysis, *Cognition*, Vol.28, 1988, pp.3-71.

[3] J. Haugeland, The Nature and Plausibility of Cognitivism, *Behavioral and Brain Sciences*, Vol.1, 1978, pp.215-226.

[4] J. L. Kelly, *General Topology*, Van Nostrand, 1995.

[5] W. T. Powers, *Behaviors : The Control of Perception*, Aldive Pub. Co., 1973.

[6] C. E. Ricket, *General Theory of Banach Algebras*, Van Nostrand, 1960.

[7] J. R. Searle, *Minds, Brains and Science*, Harvard University Press, 1984.