# Incompleteness Phenomena in Arithmetic

Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Chennai-600036, India
Email: asingh@iitm.ac.in

**Abstract**

Though the incompleteness results of Peano's arithmetic have been well appreciated, their proofs seem to be mysterious to many. In this paper, we present simple proofs of these results. We motivate the results with a ball game.

# 1 Introduction

Mathematicians have devised many puzzles and games to explain difficult and abstract results. We revisit a ball game invented by the American logician, R. M. Smullyan to illustrate König's lemma on infinite trees.

Assume that corresponding to each positive integer we have an infinite supply of balls labelled with that positive integer. There is a box having a finite number of labelled balls, where all balls are not labelled 1. We remove a ball from the box randomly and read its label. Suppose the label is $m$. Then we choose two numbers, $\ell < m$ and $n$. Next, we put into the box $n$ number of balls all labelled with $\ell$. This constitutes a single move in the game. We continue the game trying to prolong the game as far as possible.

Observe that given any positive integer $k$, we can continue the game for at least $k$ moves by choosing a suitable large $n$. However, whatever way we play the game, it will eventually end. A detailed discussion on the ball game may be found in [2]. Here, we give a short explanation.

A tree is called *finite* or *infinite* according as the number of vertices in it is finite or infinite. A tree is called a *rooted tree* if there is a designated vertex, called its *root*, which is not a child of any other vertex. A tree is called *finitely generated* if each vertex in it has a finite number of children. A *path* in a rooted tree is a sequence of vertices from the root to a leaf, where the $(j+1)$th vertex is a child of the $j$th vertex.

Notice that for a finitely generated rooted infinite tree, there may not exist a certain positive integer $n$ such that each vertex of the tree has at most $n$ number of children. For example, take a tree with 1 as its root, where 1 has two children; the right child 2 has three children; the rightmost child of 2, call it 3, has four children; and so on. *König's lemma* asserts that every finitely generated rooted infinite tree has an infinite path. Its contraposition, called the *Fan theorem*, is as follows:

If each path in a finitely generated rooted tree is finite, then the tree is finite.

For the ball game, take the box as the root with its children as the balls that are there initially. When we replace a ball with finitely many others, these new balls become the children of the removed ball. Each game thus corresponds to a finitely generated tree.

In such a tree, label the root, i.e., the box, with one plus the maximum of labels of all balls the box contains initially. All other vertices (the balls) are already labelled with positive integers. In this tree, the children of a vertex have smaller labels than that of the vertex. If the root has label $i$, then its children have labels less than $i$; the children of those children have labels less than $i-1$; and so on. Thus, any path in such a tree is finite. By the Fan theorem, the tree is finite. Therefore, each game will end. That is, given a box of balls initially, whatever way the game continues, it will eventually come to an end.

We thus conclude that given any positive integer $k$, we can play the game in such a way that it does not end in $k$ or fewer number of moves; and yet whatever way we play the game, it comes to an end. A similar phenomenon happens in Peano's arithmetic. It can be shown that there exists a property $P(\cdot)$ of natural numbers (in fact, numerals) such that none of $P(1), P(2), \ldots, P(k), \ldots$ is provable, but $\exists k P(k)$ is provable.

Taking $Q(\cdot)$ as the negation of the property $P(\cdot)$, it would then follow that all of $Q(1), Q(2), \ldots,$ $Q(k), \ldots$ are provable, but $\forall k Q(k)$ is not provable. Observe that existence of such a property $Q(\cdot)$ does not lead to a contradiction, because proofs are finite in length, and it is quite possible that we have proofs of $Q(1),\ Q(2), \ldots,\ Q(k), \ldots,$ but we do not know how to combine all these proofs to obtain a proof of $\forall k Q(k)$.

Instead of provability, if we consider truth, then the statement "All of $Q(1), Q(2), \ldots, Q(k), \ldots$ are true, but $\forall k Q(k)$ is not true." leads to a contradiction. Whereas with *provability*, there is no contradiction. This possibility suggests a possible gap between truth and provability.

In the rest of the paper, we focus on the proofs of these results in Peano's arithmetic.

## 2 The Arithmetic

We consider a first-order axiomatization of the natural number system, and call it *PA*, *Peano's Arithmetic* or *Arithmetic*, for short. It has a constant 0, and a unary function symbol $s$, three binary operations $+$, $\cdot$, $\hat{}$, and one binary relation $<$ . These symbols stand, respectively, for the number 0, the successor function, addition, multiplication, exponentiation, and the relation 'less than', as usual. Further, it has the following axioms:

1. $\forall x(\neg(0 = s(x))$
2. $\forall x(\neg(x = 0) \to \exists y(x = s(y))$
3. $\forall x \forall y((x = y) \leftrightarrow (s(x) = s(y)))$
4. $\forall x((x + 0) = x)$
5. $\forall x \forall y((x + s(y)) = s(x + y))$

6. $\forall x((x \cdot 0) = 0)$

7. $\forall x \forall y((x \cdot s(y)) = ((x \cdot y) + x))$

8. $\forall x((x\hat{}0) = s(0))$

9. $\forall x \forall y((x\hat{}s(y)) = ((x\hat{}y) \cdot x))$

10. $\forall x \neg(x < 0)$

11. $\forall x \forall y((x < s(y)) \leftrightarrow (x < y) \vee (x = y))$

12. $\forall x \forall y((x < y) \vee (x = y) \vee (y < x))$

13. *Induction Scheme*: For each formula $Y(\cdot)$ having exactly one free variable, the sentence $Y(0) \wedge \forall x(Y(x) \to Y(s(x))) \to \forall x Y(x)$ is an axiom.

We have not been economic in formulating the axioms of *PA*. It is known that operations of addition, multiplication, exponentiation, and the binary relation of 'less than' can be defined using only the axioms (1)-(3) and the Induction scheme. Instead of diverting our attention to this, we prefer to continue with the above thirteen axioms. Further, we are free to use all the laws of first-order logic.

The standard model of *PA* is the structure $\omega$; its domain is the set of natural numbers $\{0, 1, 2, \ldots, \}$, with standard interpretation of the symbols 0, $s$, $+$, $\cdot$, $\hat{}$ and $<$ as mentioned earlier. The expressions $s(0), s^2(0), \ldots$ in *PA* are called *numerals*. The numeral $s^n(0)$ is interpreted as the natural number $n$ in $\omega$. To simplify notation we will write $s^n(0)$ as $n$. The numeral $n$ will mean the expression $s^n(0)$; and the number $n$ will mean the natural number $n$ in $\omega$.

As usual in any first-order axiomatic theory, a *proof* in *PA* is a finite sequence of formulas, generated from the axioms by following the laws of first-order logic. A formula of *PA* is called *provable* in *PA* if it is the last formula of a proof. A proof proves its last formula. Further, a *sentence* in *PA* is a formula with no free variables. In the following, we lay out the required assumptions.

*Assumptions*:

1. Each sentence of $\omega$ can be written down as a sentence in *PA*.

2. If $S$ is any sentence in *PA*, then either $S$ is true in $\omega$ or $\neg S$ is true in $\omega$.

3. *PA* is *sound*: if $S$ is any provable sentence in *PA*, then $S$ is true in $\omega$.

4. *PA* is *consistent*: if $S$ is any sentence in *PA*, then both $S$ and $\neg S$ are not provable in *PA*.

The assumptions are reasonable since the first-order theory *PA* is supposed to be an axiomatization of $\omega$. Moreover, with so many years of work we have not succeeded in finding a contradiction in *PA*.

# 3 Gödel numbering

The symbols, strings of symbols, and finite sequences of strings of symbols of *PA* are commonly referred to as *expressions*. The set of all expressions of *PA* are countable; thus they can be encoded as numerals. However, we wish to have an encoding where the numeral associated with an expression of *PA* can be computed by an algorithm. Such an encoding is called *Gödel numbering* following K. Gödel. We will use a simplified numbering scheme. We start with the symbols used in the theory *PA*. Since a comma is also a symbol, we use blank space to separate the symbols of *PA* in the following list:

$$, \quad ) \quad ( \quad x \quad | \quad \neg \quad \wedge \quad \vee \quad \to \quad \leftrightarrow \quad \forall \quad \exists \quad = \quad 0 \quad s \quad + \quad \cdot \quad \hat{} \quad < \quad \top \quad \bot$$

Instead of using infinite number of variables, $x_0, x_1, x_2, \ldots$, we will use the symbols $x$ and $|$ to generate them. That is, $x$ followed by $n$ number of $|$s will be abbreviated to $x_n$. Recall that the numerals $s(0)$, $s(s(0))$ and $s(s(s(0)))$ are abbreviated to 1, 2 and 3, respectively. If $\beta$ is any of the digits 1 or 2, we abbreviate $\beta$ written $n$ times to $\beta^n$. For instance 222222 will be written as $2^6$; you should not confuse $2^6$ here with 64. We write the encoding as $g(\cdot)$, and start with the symbols:

$$g(,) = 1,\ g())=11,\ g(()=111,\ g(x)=1^4,\ \ldots,\ g(\top)=1^{20},\ g(\bot)=1^{21}.$$

Next, a string of symbols $\sigma_1 \cdots \sigma_k$ is encoded as

$$g(\sigma_1 \cdots \sigma_k) = 2\,g(\sigma_1)\,2 \cdots 2\,g(\sigma_k)\,2.$$

We keep the digit 2 as a separator; it will help us in decoding the symbols. Notice that each string of symbols begins with a 2 and also ends with a 2. Next, if $s_1, s_2, \ldots, s_m$ are strings of symbols from the language of *PA*, then we define

$$g(s_1, s_2, \ldots, s_m) = 2\,g(s_1)\,2\,g(s_2)\,2 \cdots 2\,g(s_m)\,2.$$

Finally, for an expression $w$ of *PA*, if $g(w)$ is $a_k a_{k-1} \cdots a_1 a_0$, where each $a_i$ is either 1 or 2, we read $g(w)$ as the value of the following expression in *PA*:

$$g(w) = a_k 3^k + a_{k-1} 3^{k-1} + \cdots + a_1 3^1 + a_0.$$

We observe that $g$ is a one-to-one function from the set of expressions of *PA* to the set of numerals. We will call any numeral $n$ a *Gödel number* iff there exists an expression $w$ such that $n = g(w)$. Further, we see that there exist an algorithm specified in the language of *PA* that computes the numeral $g(w)$ corresponding to each expression $w$ of *PA*; and also there exists an algorithm specified in the language of *PA* that computes the expression $g^{-1}(n)$ corresponding to each Gödel number $n$.

The encoding of expressions of *PA* as numerals in *PA* makes self-reference possible. This is reflected in the following result.

**Theorem 1.** (Diagonalization Lemma)
*Corresponding to each formula $B(\cdot)$ of PA with a single free variable, there exists a sentence S in PA such that $S \leftrightarrow B(g(S))$ is provable in PA.*

*Proof.* Let $x_1$ be the Gödel number of a formula with a single free variable $x$. Then such a formula is given by $g^{-1}(x_1)$. The Gödel numbering provides an algorithm for computing $g^{-1}(x_1)$, and also an algorithm for computing $g(w)$ for any expression $w$ of *PA*. These algorithms are specified by formulas of *PA*. Thus we have an algorithm specified by formulas of *PA* that takes $x_1$ as an input and gives the formula $g^{-1}(x_1)$ as its output. Then $(g^{-1}(x_1))[x/x_1]$ is a formula of *PA*; and the expression $g((g^{-1}(x_1))\,[x/x_1])$ evaluates to a numeral.

Let $B(\cdot)$ be a formula with a single free variable. Then $B(g((g^{-1}(x_1))\,[x/x_1]))$ is a well defined formula of *PA*. We construct a formula $H(x)$ in *PA* as follows:

> For each Gödel number $x_1$ of a formula with a single free variable $x$, we take $H(x_1)$ as the formula $B(g((g^{-1}(x_1))\,[x/x_1]))$; and for each $x_2$ which is not the Gödel number of a formula with the free variable $x$, we take $H(x_2)$ as $\bot$.

We take $H(x_2)$ as $\bot$, which is always *false*, since $g^{-1}(x_2)$ for such a numeral $x_2$ may become meaningless. Thus, in *PA*, we have a formula $H(x)$ so that for each Gödel number $x_1$ of a formula with the only free variable $x$,

$$H(x_1) \leftrightarrow B\big(g\big((g^{-1}(x_1))\,[x/x_1]\big)\big) \text{ is provable in } PA.$$

In particular, with $x_1$ as $k = g(H(x))$, we have

$$H(k) \leftrightarrow B\big(g\big((g^{-1}(k))\,[x/k]\big)\big) \text{ is provable in } PA.$$

Now, $k = g(H(x))$ implies that $H(k) = H(x)[x/k] = (g^{-1}(k))\,[x/k]$. Therefore,

$$H(k) \leftrightarrow B\big(g\big(H(k)\big)\big) \text{ is provable in } PA.$$

This sentence $H(k)$ serves the purpose of $S$ as required. □

Given a formula $B(\cdot)$ with a single free variable, the sentence "My Gödel number satisfies the formula $B(\cdot)$" is such a sentence provided by the Diagonalization lemma. It shows that the Arithmetic is a rich axiomatic theory, where this self-referring sentence can be expressed and proved.

# 4 Incompleteness in Arithmetic

Notice that each expression of *PA* has a Gödel number. In particular, each formula and each proof has a Gödel number. Further, a number is or is not the Gödel number of some expression of *PA* can be written as a formula of *PA*. Further, from the Gödel number of a proof, the Gödel number of the formula it proves can be determined by an algorithm. Therefore, the following formula with its meaning as given, is a formula of *PA* with two free variables $x$ and $x_1$:

$Pr(x, x_1)$**:** $x$ is the Gödel number of a formula of *PA* and $x_1$ is the Gödel number of a proof of that formula

That is, $Pr(x, x_1)$ is true in $\omega$ iff $x_1$ is the Gödel number of a proof of a formula $F$ whose Gödel number is $x$. Such a formula $F$ in the definition of $Pr(x, x_1)$ is $g^{-1}(x)$. Next, write

$P(x)$**:** $\exists x_1 Pr(x, x_1)$

As a formula of *PA*, $P(x)$ is true in $\omega$ iff there exists a natural number $x_1$ which is the Gödel number of a proof of $g^{-1}(x)$ iff the formula $g^{-1}(x)$ is provable in *PA*. In particular, let $Y$ be any sentence, and let $x = g(Y)$. Then $g^{-1}(x) = Y$. We conclude that for any sentence $Y$ in *PA*,

$$Y \text{ is provable in } PA \text{ iff } P(g(Y)) \text{ is true in } \omega. \tag{1}$$

Using the Diagonalization lemma on the formula $\neg P(x)$, we obtain a sentence $S$ in *PA* that satisfies the following:

$$S \leftrightarrow \neg P(g(S)) \text{ is provable in } PA. \tag{2}$$

Suppose that $S$ is provable in *PA*. By (1), $P(g(S))$ is true in $\omega$. Whereas by (2), $\neg P(g(S))$ is provable in *PA*. By the soundness of *PA*, $\neg P(g(S))$ is true in $\omega$. We thus obtain the contradiction that both $P(g(S))$ and $\neg P(g(S))$ are true in $\omega$. Therefore, $S$ is not provable in *PA*.

Assume that $\neg S$ is provable in *PA*. By (2), $P(g(S))$ is provable in *PA*. Due to the soundness of *PA*, $P(g(S))$ is true in $\omega$. By (1), $S$ is provable in *PA*. This contradicts the consistency of *PA*. Therefore, $\neg S$ is not provable in *PA*.

Further, since $S$ is not provable in *PA*, due to (1), $P(g(S))$ is not true in $\omega$. Due to our assumption on the theory $\omega$, the sentence $\neg P(g(S))$ is true in $\omega$. Due to the soundness of *PA* and (2), we know that $S \leftrightarrow \neg P(g(S))$ is true in $\omega$. Therefore, $S$ is true in $\omega$.

We thus conclude the following:

$S$ is true in $\omega$, but neither $S$ nor $\neg S$ is provable in *PA*.

We refer to the existence of such a sentence $S$ in *PA* by telling that *PA* is *negation incomplete.* This is the celebrated *First incompleteness theorem* of K. Gödel, which says that Peano's Arithmetic is negation incomplete. Notice that $\neg P(g(S))$ means $S$ is not provable. Thus the sentence $S$ in (2) asserts that "I am not provable". The sentence $\neg P(g(S))$ is also a sentence in *PA* which is true in $\omega$, but neither it nor its negation is provable in *PA*.

Write the formula $\neg Pr(g(S), x_1)$ as $Q(x_1)$. Let $n$ be any numeral. Now, $Q(n) \equiv \neg Pr(g(S), n)$. The sentence $\neg Pr(g(S), n)$ means that $n$ is not the Gödel number of a proof of $S$. Since we have proved that $S$ is not provable, we have a proof of the sentence "$n$ is not the Gödel number of a proof of $S$." That is, we have a proof of $\neg Pr(g(S), n)$, or of $Q(n)$. Therefore, for each numeral $n$, $Q(n)$ is provable in *PA*.

On the other hand,

$$\forall x_1 Q(x_1) \equiv \forall x_1 \neg Pr(g(S), x_1) \equiv \neg \exists x_1 Pr(g(S), x_1) \equiv \neg P(g(S)).$$

As $S$ is not provable in $PA$, by (2) it follows that $\neg P(g(S))$ is not provable in $PA$. Hence the sentence $\forall x_1 Q(x_1)$ is not provable in $PA$. We conclude that

All of $Q(0),\ Q(1),\ldots,\ Q(x_1),\ldots$ are provable in $PA$ but $\forall x_1 Q(x_1)$ is not provable in $PA$.

This formula $Q(x_1)$ works like "the ball game does not end in $x_1$ or fewer moves". Existence of such a predicate in $PA$ is referred to as saying that $PA$ is *$\omega$-incomplete.*

Moreover, since all $Q(0),\ Q(1),\ldots,\ Q(x_1),\ldots$ are provable in $PA$, they are true in $\omega$. Therefore, $\forall x_1 Q(x_1)$ is true in $\omega$. So, the sentence $\forall x_1 Q(x_1)$ is a sentence in $PA$ which is true in $\omega$ but not provable in $PA$. Thus $\omega$-incompleteness of $PA$ implies the negation incompleteness of $PA$.

To have an existential version of $\omega$-incompleteness, abbreviate the formula $Q(x_1) \to \forall x_1 Q(x_1)$ to $R(x_1)$. We see that

$$\exists x_1 R(x_1) \equiv \exists x_1 (Q(x_1) \to \forall x_1 Q(x_1)) \equiv \forall x_1 Q(x_1) \to \forall x_1 Q(x_1) \equiv \top.$$

Therefore, $\exists x_1 R(x_1)$ is provable in $PA$. On the other hand, if for some numeral $x_1$, $R(x_1)$ is provable in $PA$, then $Q(x_1) \to \forall x_1 Q(x_1)$ is provable in $PA$. We know that $Q(x_1)$ is provable in $PA$. Then $\forall x_1 Q(x_1)$ will be provable in $PA$. But it is not. Therefore, for no numeral $x_1$, $R(x_1)$ is provable. We have thus shown the following:

$\exists x_1 R(x_1)$ is provable in $PA$, but none of $R(0),\ R(1),\ldots,\ R(x_1),\ldots$ is provable in $PA$.

R. M. Smullyan names the existence of such a predicate $R(x_1)$ in $PA$ by telling that $PA$ is *$\exists$-incomplete.* It was described by P. R. Halmos in a funny way: A child searches for a specific chocolate. After finding one, she shows it to her mother. Her mother says, no it is not this, but it is there, find it out. The child finds another; the mother says, no it is not this, but it is there, find it out. The process goes on.

# 5 Conclusion

In this paper, we have re-looked at a ball game. A vague analogy suggests that there is a possible gap between truth and provability in Arithmetic. Employing the diagonalization lemma, it is seen that there exists a sentence in Arithmetic which is true, but neither the sentence nor its negation can be proved in Arithmetic. Further, we find that there exists a property of numerals which can be proved for each numeral, but the sentence that "for each numeral the property holds" cannot be proved in Arithmetic. Similarly, we obtain a property of numerals which cannot be proved for any particular numeral, yet the sentence that "there exists a numeral which satisfies the property" can be proved in Arithmetic.

In summary, the Arithmetic is negation-incomplete, $\omega$-incomplete, and $\exists$-incomplete. We have not touched upon the second incompleteness theorem of K. Gödel connected with the unprovability of consistency of Arithmetic. For a proof of this result, one may see [1, 3].

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