Functions of One Real Variable

A Survival Guide

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Contents

Chapter 1

Limit and Continuity

1.1 Preliminaries

We use the following notation:

$\emptyset$ = the empty set.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of integers.

$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N}\}$, the set of rational numbers.

$\mathbb{R}$ = the set of real numbers.

$\mathbb{R}_+$ = the set of all positive real numbers.

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$. The numbers in $\mathbb{R} - \mathbb{Q}$ is the set of irrational numbers.
Examples are $\sqrt{2}$, $3.10110111011110\cdots$ etc.

Along with the usual laws of $+,\ \cdot, <$, $\mathbb{R}$ satisfies the **Archimedian property**:

If $a > 0$ and $b > 0$, then there exists an $n \in \mathbb{N}$ such that $na \geq b$.

Also $\mathbb{R}$ satisfies the **completeness property**:

Every nonempty subset of $\mathbb{R}$ having an upper bound has a least upper bound (**lub**) in $\mathbb{R}$.

Explanation: Let A be a nonempty subset of $\mathbb{R}$. A real number u is called an upper bound of A if each element of A is less than or equal to u. An upper bound ℓ of A is called a least upper bound if all upper bounds of A are greater than or equal to ℓ.

Notice that $\mathbb{Q}$ does not satisfy the completeness property. For example, the nonempty set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ has an upper bound, say, 2. But its least upper bound is $\sqrt{2}$, which is not in $\mathbb{Q}$.

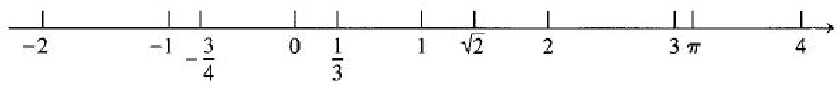
Similar to lub, we have the notion of glb, the greatest lower bound of a subset of $\mathbb{R}$. Let A be a nonempty subset of $\mathbb{R}$. A real number v is called a lower bound of A if each element of A is greater than or equal to v. A lower bound m of A is called a greatest lower bound if all lower bounds of A are less than or equal to m. The completeness property of $\mathbb{R}$ implies that

Every nonempty subset of $\mathbb{R}$ having a lower bound has a greatest lower bound (**glb**) in $\mathbb{R}$.

The lub acts as a maximum of a nonempty set and the glb acts as a minimum of the set. In fact, when the $\mathrm{lub}(\mathrm{A}) \in \mathrm{A}$, this lub is defined as the **maximum of** A and is denoted as $\max(\mathbf{A})$. Similarly, if the $\mathrm{glb}(\mathrm{A}) \in \mathrm{A}$, this glb is defined as the **minimum of** A and is denoted by $\min(\mathbf{A})$.

Moreover, both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are **dense** in $\mathbb{R}$. That is, if $x < y$ are real numbers then there exist a rational number a and an irrational number b such that $x < a < y$ and $x < b < y$.

We may not explicitly use these properties of $\mathbb{R}$ but some theorems, whose proofs we will omit, can be proved using these properties. These properties allow $\mathbb{R}$ to be visualized as a number line:



Let $a, b \in \mathbb{R},\ a < b$.

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, the closed interval $[a, b]$.

$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, the semi-open interval $(a, b]$.

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, the semi-open interval $[a, b)$.

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$, the open interval (a, b).

$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$, the closed infinite interval $(-\infty, b]$.

$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$, the open infinite interval $(-\infty, b)$.

$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$, the closed infinite interval $[a, \infty)$.

$(a, \infty) = \{x \in \mathbb{R} : x \leq b\}$, the open infinite interval (a, ∞).

$(-\infty, \infty) = \mathbb{R}$, both open and closed infinite interval.

We also write $\mathbb{R}_+$ for $(0, \infty)$ and $\mathbb{R}_-$ for $(-\infty, 0)$. These are, respectively, the sets of all positive real numbers, and the set of all negative real numbers.

A **neighborhood** of a point c is an open interval $(c - \delta, c + \delta)$ for some $\delta > 0$.

A **deleted neighborhood** of a point c is $(c - \delta, c + \delta) - \{c\} = (c - \delta, c) \cup (c, c + \delta)$ for some $\delta > 0$.

Let $D \subseteq \mathbb{R}$. Let $c \in D$. We say that c is an **interior point** of D if there exist an open interval (a, b) such that $c \in (a, b) \subseteq D$.

Notice that c in interior point of D if there exists a neighborhood of c which is contained in D.

For example, 0.1 is an interior point of $[0, 1)$. The point 0 is not an interior point of $[0, 1)$.

In contrast, we say that a is a **left end-point** of the intervals $[a, b)$ and of $[a, b]$.

Similarly, b is a **right end-point** of the intervals $(a, b]$ and of $[a, b]$.

The **absolute value** of $x \in \mathbb{R}$ is defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x| = \sqrt{x^2}$. And $|-a| = a$ or $a \geq 0$; $|x - y|$ is the distance between real numbers x and y.

Moreover, if $a, b \in \mathbb{R}$, then

$$|-a| = |a|, \quad |ab| = |a|\,|b|, \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ if } b \neq 0, \quad |a+b| \le |a| + |b|, \quad |\,|a| - |b|\,| \le |a-b|.$$

Let $x \in \mathbb{R}$ and let $a > 0$. The following are true:

1. $|x| = a$ iff $x = \pm a$.
2. $|x| < a$ iff $-a < x < a$ iff $x \in (-a, a)$.
3. $|x| \le a$ iff $-a \le x \le a$ iff $x \in [-a, a]$.
4. $|x| > a$ iff $-a < x$ or $x > a$ iff $x \in (-\infty, -a) \cup (a, \infty)$ iff $x \in \mathbb{R} - [-a, a]$.
5. $|x| \ge a$ iff $-a \le x$ or $x \ge a$ iff $x \in (-\infty, -a] \cup [a, \infty)$ iff $x \in \mathbb{R} - (-a, a)$.

Therefore, for $a \in \mathbb{R}, \delta > 0$,

$$|x - a| < \delta \text{ iff } a - \delta < x < a + \delta.$$

The following statements are useful in proving equalities from inequalities:

Let $a, b \in \mathbb{R}$.

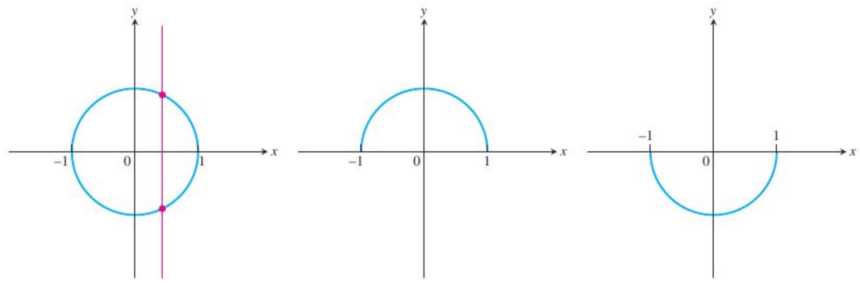
1. If for each $\epsilon > 0$, $|a| < \epsilon$, then $a = 0$.
2. If for each $\epsilon > 0$, $a < b + \epsilon$, then $a \le b$.

Let $A, B \subseteq \mathbb{R}$. A **function** f from A to B is a rule that assigns each point in A to some point in B in a unique way. The *unique* here means that if $f(a) \neq f$, then $a \neq b$; that is, $f(a)$ cannot be sometimes a number b and sometimes a number c. The *unique* does not necessarily mean that all points in A are associated to the same point in B. We write such a function f as $f : A \to B$. If x is a generic point in A and if f assigns this x to y in B, then we write $f(x) = y$. Here, y is the **image** of x under f and x is one of the **pre-images** of y. Often we say that $f(x)$ is a function, loosely, or even as $y = f(x)$ is a function. The set A is called the **domain** and the set B is called the **co-domain** of f. The **range** of f is the set of actual values that have been assigned to in B. That is, the range of f is the set $\{f(x) : x \in A\}$. Mostly, we will be interested in functions with domains as intervals. When a function is given in the form $y = f(x)$, we do not say explicitly what its domain is. rather, we find out a largest subset of $\mathbb{R}$ which may serve as its domain. For example,

$y = x^2$ has domain as $\mathbb{R}$ and range as $[0, \infty)$.
$y = \sqrt{x}$ has domain and range as $[0, \infty)$.
$y = 1/x$ has domain and range as $\mathbb{R} - \{0\}$.
$y = \sqrt{2 - x}$ has domain as $(-\infty, 2]$ and range as $[0, \infty)$.
$y = \sqrt{3 - x^2}$ has domain as $[-\sqrt{3}, \sqrt{3}]$ and range as $[0, \sqrt{3}]$.

The **graph** of a function $y = f(x)$ is the set $\{(x, f(x)) : x \in \text{domain of } f\}$. We plot the points $(x, f(x))$ in the plane $\mathbb{R}^2$. In this plotting, the convention is to regard x as the independent variable and $y = f(x)$ as the function. Since $f(a)$ is uniquely defined for each a in the domain of f, if you draw a vertical line on the graph of f, then it never crosses the graph in more than one point.

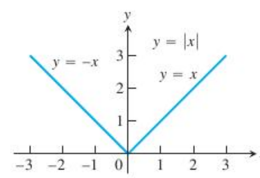
For example, the unit circle in the plane does not represent a function but the semicircles above the x-axis represent functions. The semicircles to the right or left of the y-axis are not functions.



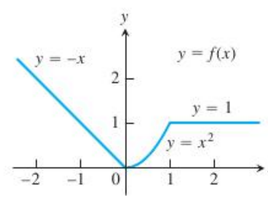
(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

Graphs of some known functions are as follows:

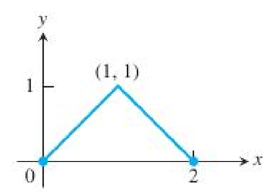
1. $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



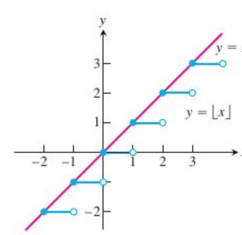
2. $y = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$



3. $y = f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$



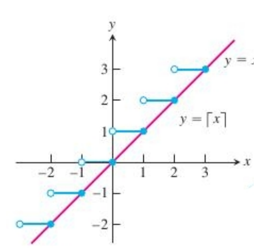
4. $y = \lfloor x \rfloor = n$ if $n \leq x < n + 1$ for $n \in \mathbb{N}$. It is the largest integer less than or equal to x.



The largest integer function or the **floor** function.

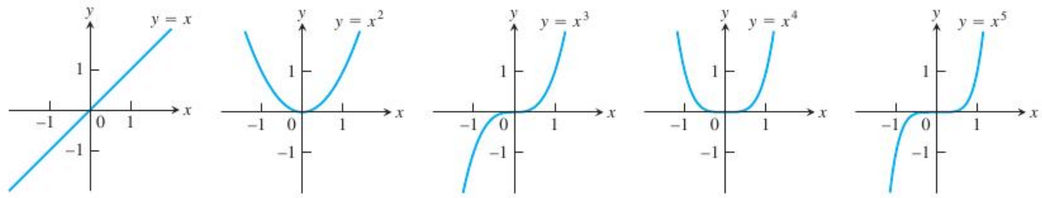
Sometimes we write $\lfloor\ \rfloor$ as $[\ ]$.

5. $y = \lfloor x \rfloor = n + 1$ if $n < x \leq n + 1$ for $n \in \mathbb{N}$. It is the smallest integer greater than or equal to x.

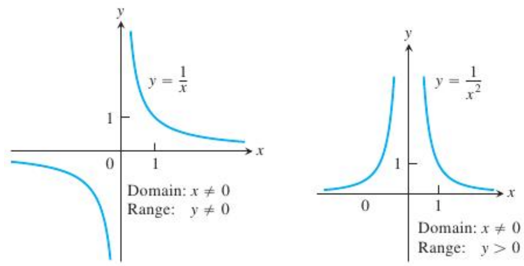


The smallest integer function or the **ceiling** function.

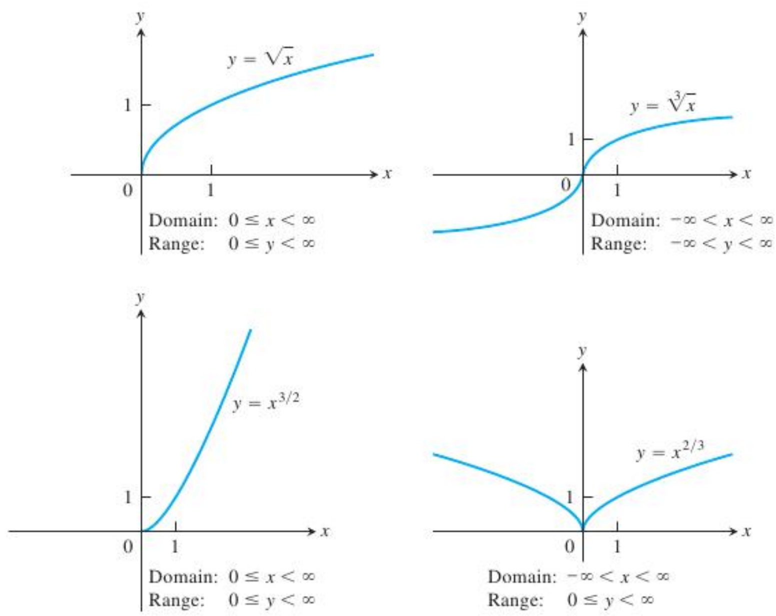
6. The **power function** $y = x^n$ for $n = 1, 2, 3, 4, 5$ look like



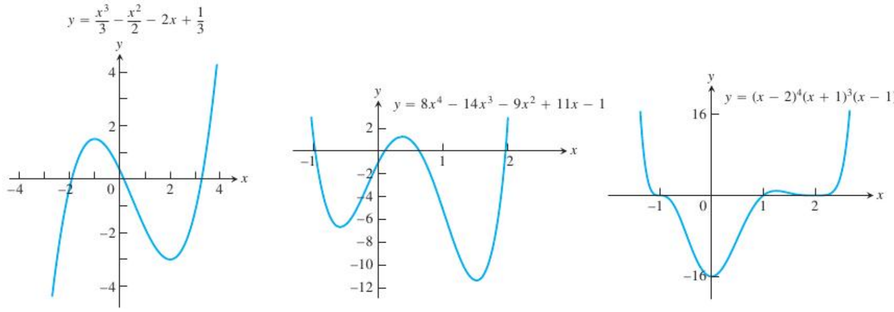
7. The power function $y = x^n$ for $n = -1$ and $n = -2$ look like



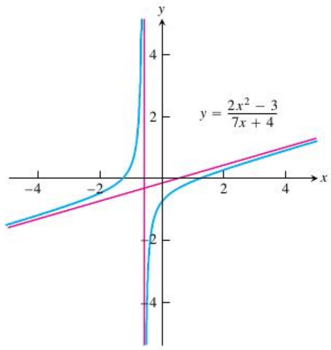
8. The graphs of the power function $y = x^a$ for $a = \frac{1}{2},\ \frac{1}{3},\ \frac{3}{2}$ and $a = \frac{2}{3}$ are

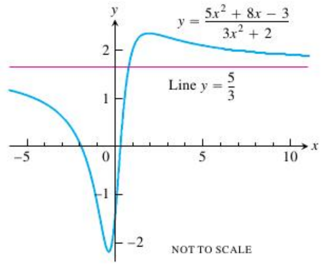


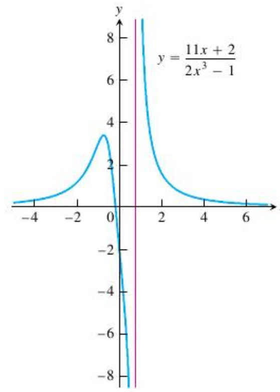
9. **Polynomial** functions are $y = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$ for some $n \in \mathbb{N} \cup \{0\}$. Here, the coefficients of powers of x are some given real numbers $a_0, \ldots, a_n$ and $a_n \neq 0$. The highest power n in the polynomial is called the **degree** of the polynomial. Graphs of some polynomial functions are as follows:



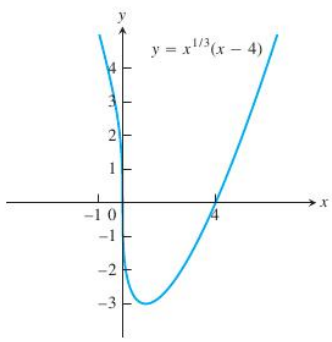
10. A **rational function** is a ratio of two polynomials; $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, may or may not be of the same degree. Graphs of some rational functions are as follows:

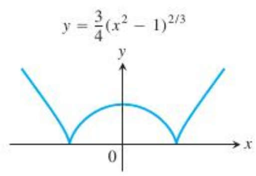


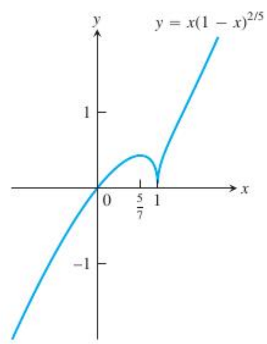




11. **Algebraic functions** are obtained by adding subtracting, multiplying, dividing or taking roots of polynomial functions. Rational functions are special cases of algebraic functions. Some graphs of alhebraic functions:







12. **Trigonometric functions** come from the ratios of sides of a right angled triangle. The angles are measured in radian. The trigonometric functions have a period. That is, $f(x+p) = f(x)$ happens for some $p > 0$. The **period** of $f(x)$ is the minimum of such p. The period for $\sin x$ is 2π.

The functions $\cos x$ and $\sec x$ are **even functions** and all others are **odd functions**. Recall that $f(x)$ is even if $f(-x) = f(x)$ and it is odd if $f(-x) = -f(x)$ for each x in the domain of the function. Some of the useful inequalities are

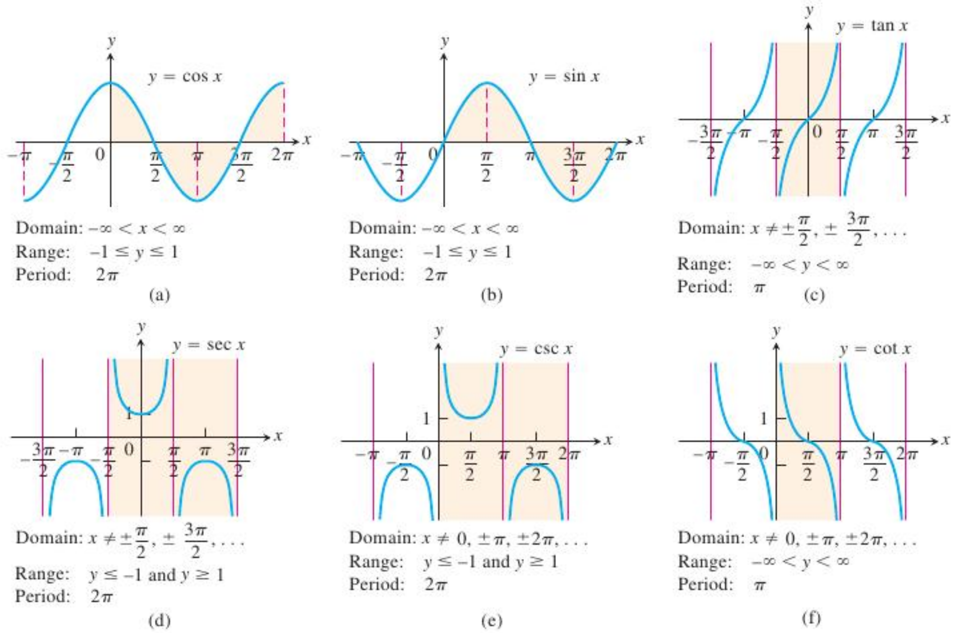
$$-|x| \le \sin x \le |x| \text{ for all } x \in \mathbb{R}.$$

$$-1 \le \sin x,\ \cos x \le 1 \text{ for all } x \in \mathbb{R}.$$

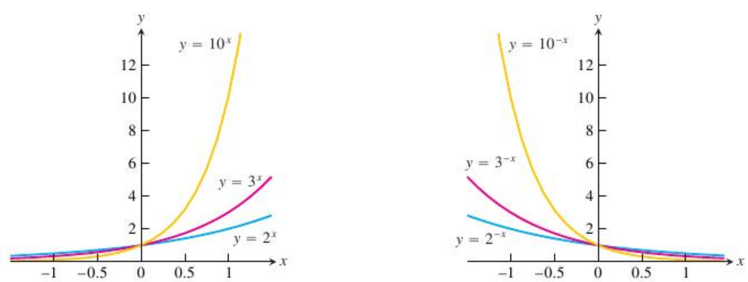
$$0 \le 1 - \cos x \le |x| \text{ for all } x \in \mathbb{R}.$$

$$\sin x \le x \le \tan x \text{ for all } x \in (0, \pi/2).$$

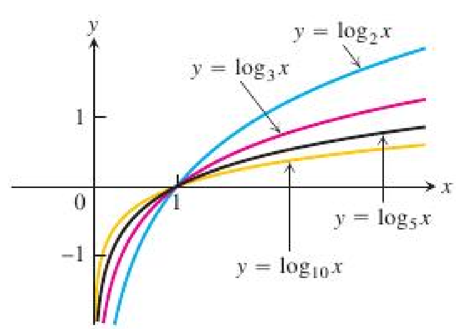
In fact, if $x \neq 0$, then $\sin x < |x|$. Some examples:



13. **Exponential functions** are in the form $y = a^x$ for some $a > 0$ and $a \neq 1$. All exponential functions have domain $(-\infty, \infty)$ and co-domain $(0, \infty)$. They never assume the value 0. Graphs of some exponential functions:



14. **Logarithmic functions** are inverse of exponential functions. That is, $a^{\log_a x} = \log_a(a^x) = x$. Some examples:



Functions that are not algebraic are called transcendental functions. Trigonometric functions, exponential functions and logarithmic functions are examples of transcendental functions.

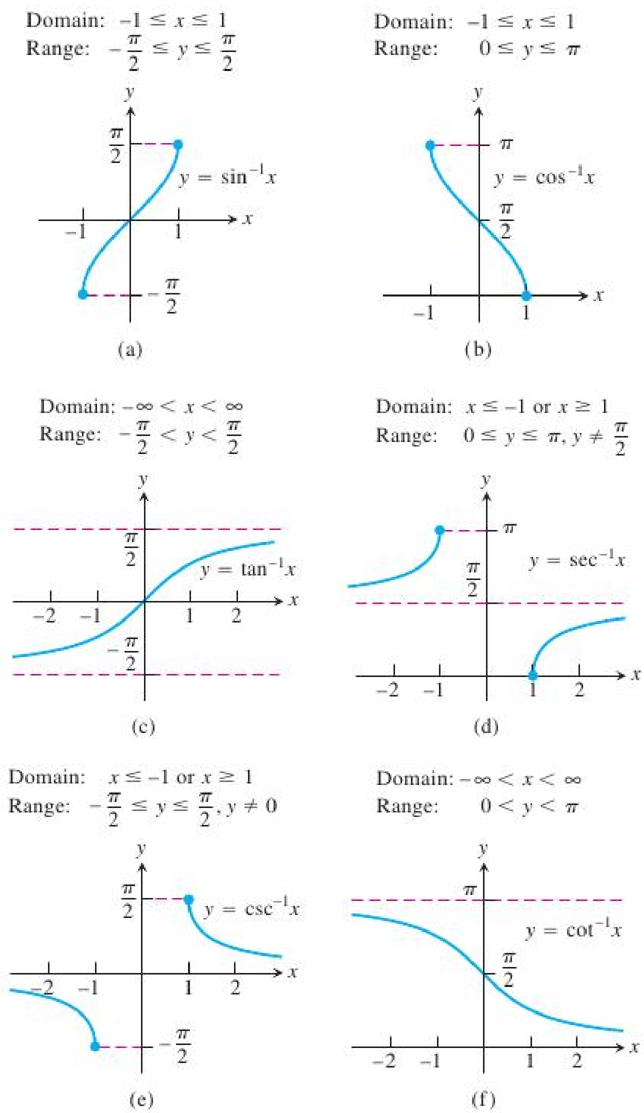
We will come back to exponential functions and logarithmic functions later.

Two functions can be combined to give another function. If $f(x)$ and $g(x)$ have the same domains, then $f \pm g(x)$ are new functions defined by $(f \pm g)(x) = f(x) \pm g(x)$.

Similarly, $(fg)(x) = f(x)g(x)$ and $(f/g)(x) = f(x)/g(x)$ provided that $g(x) \neq 0$. Also, you can have $h(x) = f(x)^{g(x)}$ as another function provided that the latter is meaningful.

Another way is to obtain a function by taking **composition**. If the range of $f(x)$ is a subset of the domain of $g(x)$, then $g(f(x))$ is meaningful. The composite function $(g \circ f)$ is then defined as $(g \circ f)(x) = g(f(x)$.

Notice that the inverse of a function $f(x)$ is defined through the composition as f^{-1} is that function which satisfies $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. In that case, f must be a one-one function onto its co-domain. Sometimes, we restrict the domain of a function so that it may be one-one and onto so that its inverse can be defined. For example, restricting the domain of $\sin x$ to $[-\pi/2, \pi/2]$, makes it one-one and onto the range $[-1, 1]$. Then its inverse $\sin^{-1}(x)$ is a function from $[-1, 1]$ to $[-\pi/2, \pi/2]$ and this is also one-one and onto. This means that $\sin^{-1} x$ is that number $y \in [-\pi/2, \pi/2]$ for which $\sin y = x$. Similarly, for other trigonometric functions their inverse functions are defined. Warning: $\sin^{-1} x$ is not the same as $(\sin x)^{-1}$. Inverse trigonometric functions are given below:



1.2 Limit

The idea is to find how a function behaves near a point in its domain. Suppose $y = f(x)$ is a function, say, defined on an open interval (a, b). Let $c \in (a, b)$. Under what condition can we say that when $x \in (a, b)$ is taken near c, the values $f(x)$ stay near $f(c)$?

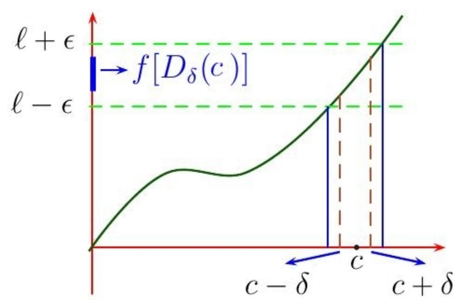
To tackle the vague notion of nearness, suppose I quantify it. Suppose $\epsilon > 0$. I say that two numbers r and s are ϵ-near if their difference is bounded by ϵ; That is, when $|r - s| < \epsilon$. But the amount of nearness in x to c need not be the same amount in $f(x)$. However, there should be some connection between the two. I may say that whatever nearness I choose between $f(x)$ and $f(a)$, I find a certain nearness in x and c such that if x lies near c, then $f(x)$ lies near $f(a)$. If this condition is satisfied, we say that the function f is *continuous at* c.

If f is not defined at c but at every point in an open interval around c, then it is not meaningful to write $f(c)$. But still there may exist some real number ℓ such that $f(x)$ lies near ℓ. In such a case, we say that f has limit ℓ at c. We state it formally.

Let $a < c < b$. Let $f : D \to R$ be a function whose domain D contains the union $(a, c) \cup (c, b)$. Let $\ell \in \mathbb{R}$. We say that **the limit of** $f(x)$ **as** x **approaches** c **is** ℓ and write it as

$$\lim_{x \to c} f(x) = \ell$$

iff for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each $x \in (a, c) \cup (c, b)$ with $0 < |x - c| < \delta$, we have $|f(x) - \ell| < \epsilon$.



This is the formal definition of the notion of limit of a function. In this definition, δ may depend on ϵ and also on the point c. Notice that given a function our definition does not lead us to find ℓ. it is a guess work. Once a guess has been taken, we can only verify whether the guess is correct.

The definition says that each ϵ-neighborhood of ℓ must contain the f-image of some δ-neighborhood of c. Writing $D_\delta(c) = (c - \delta, c + \delta)$ for a δ-neighborhood of c, it demands that each $D_\epsilon(\ell)$ must contain the f-image of some $D_\delta(c)$. To put it another way, write the inverse image of f of a subset A of $\mathbb{R}$ as $\{x \in \text{dom}(f) : f(x) \in A\}$. Then

$\lim_{x \to c} f(x) = \ell$ iff inverse image of each neighborhood of ℓ is contained in some neighborhood of c.

We will not always use this formal definition; rather we will work with our intuitive understanding of the notion that $f(x)$ has a limit ℓ at c provided the following condition holds:

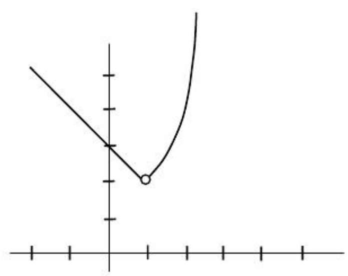
if x is near c, then $f(x)$ is near ℓ

If a certain case poses problem, we may come back to the formal definition.

Example 1.1. Find $\lim_{x \to 0} |x|$.

From the graph of $|x|$, we guess that this limit is 0. To show this, let $\epsilon > 0$. Choose $\delta = \epsilon$.
(Why? Do not ask me; see whether it works.)
If $|x - 0| = |x| < \delta = \epsilon$, then $|f(x) - 0| = |x| < \epsilon$. Hence $\lim_{x \to 0} |x| = 0$.

Example 1.2. Find $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} 3 - x & \text{if } x < 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$.



On the left side of $x = 1$ the function is $3 - x$. If x is near 1, we see that $f(x)$ is near 2. On the right side, if x is near 1, then we see that $x^2 + 1$ lies near 2. Therefore, $\lim_{x \to 1} f(x) = 2$.
We can check it formally. Let $\epsilon > 0$. Choose $\delta = \min\{\epsilon, \sqrt{1 + \epsilon} - 1\}$.
(Again, do not ask me how I got it; but see how it works.)
Suppose $1 - \delta < x < 1 + \delta; x \neq 1$. Break into two cases.
(a) Let $x < 1$. That is, $1 - \delta < x < 1$, Then $|f(x) - 2| = |3 - x - 2| = 1 - x < \delta \leq \epsilon$.
(b) Let $x > 1$. That is, $1 < x < 1 + \delta$, Then

$$|f(x) - 2| = |x^2 + 1 - 2| = x^2 - 1 < (1 + \delta)^2 - 1 \leq (1 + \sqrt{1 + \epsilon} - 1)^2 - 1 = 1 + \epsilon - 1 = \epsilon.$$

Therefore, $|x - 1| < \delta$ implies $|f(x) - 2| < \epsilon$.

We see that it is often difficult to follow the definition. However, the definition can be used to prove certain properties of limits which help us in the intuitive understanding and also in determining the limit of a function at a point. Before stating the theorem, let us consider some more examples. Remember that the nearness in $f(x)$ to ℓ includes $f(x)$ being equal to ℓ whereas the nearness in x to c, $x \neq c$.

Example 1.3. Let $f(x) = 5$ for $x \neq 1$; and $f(1) = 1$. What is $\lim_{x \to 1} f(x)$?

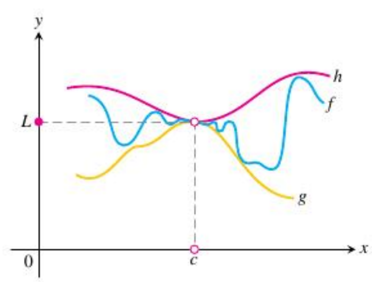
When x is near 1 but not equal to 1, $f(x) = 5$ or that $f(x)$ is near 5. Therefore, $\lim_{x \to 1} f(x) = 5$. It does not matter whether f is defined at 1 or not. It also does not matter what value $f(1)$ has.

Example 1.4. Let $f(x) = \dfrac{x^2 - 25}{x - 5}$ for $x \neq 5$. What is $\lim_{x \to 5} f(x)$?

When x is near 5, the numerator $x^2 - 25$ is near 0. So is the denominator $x - 5$. It looks that the limit is undefined. However, it is wrong, since for $x \neq 5$, $f(x) = \frac{x^2 - 25}{x - 5} = x + 5$. Now, $x + 5$ stays near 10 when x is near 5. Hence $\lim_{x \to 5} f(x) = 10$.

Theorem 1.1. (The Limit Properties) *Let k be a constant; or a constant function.*

1. $\lim_{x\to c} k = k$ *and* $\lim_{x\to c} x = c$.
2. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$.
3. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$.
4. $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \ \lim_{x\to c} g(x)$.
5. $\lim_{x\to c}[f(x)/g(x)] = \left[\lim_{x\to c} f(x)\right] / \left[\lim_{x\to c} g(x)\right]$ *if* $\lim_{x\to c} g(x) \neq 0$.
6. $\lim_{x\to c}(f(x))^r = (\lim_{x\to c} f(x))^r$ *if taking powers are meaningful.*
7. $\lim_{x\to c} f(x)$ *is a unique real number if it exists.*
8. *If* $\lim_{x\to c} g(x) = 0$, *and* $\lim_{x\to c}[f(x)/g(x)]$ *exists, then* $\lim_{x\to c} f(x) = 0$.
9. *(**Sandwich**) Let f, g, h be functions whose domain include $(a, c) \cup (c, b)$ for $a < c < b$. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \in (a, c) \cup (c, b)$. If $\lim_{x\to c} g(x) = \ell = \lim_{x\to c} h(x)$, then $\lim_{x\to c} f(x) = \ell$.*
10. *(**Domination Limit**) Let f, g be functions whose domains include $(a, c) \cup (c, b)$ for $a < c < b$. Suppose that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. If $f(x) \leq g(x)$ for all $x \in (a, c) \cup (c, b)$, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.*



It follows from Theorem 1.1 that if $p(x) = a_0 + a_1 x + \cdots a_n x^n$ is a polynomial, then $\lim_{x\to c} p(x) = p(c)$. Also, if $q(x)$ is another such polynomial with $q(c) \neq 0$, then $\lim_{x\to c}(p(x)/q(x)) = p(c)/q(c)$. Also, the (8) implies that

If $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c}(f(x)/g(x))$ does not exist.

For example, $\lim_{x\to -3} \dfrac{x^2+9}{x+3}$ does not exist.

Caution: In (10), if $f(x) < g(x)$ happens for all x, then we may not be able to conclude that $\lim_{x\to c} f(x) < \lim_{x\to c} g(x)$. The first limit may become equal to the second limit. For example, consider the function $f(x) = |x|$ with the domain $\mathbb{R} - \{0\}$. We have $0 < f(x)$ for all x in this domain. Then $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} |x|$.

Example 1.5. Evaluate
(a) $\lim_{x\to 0} |x|\sin x$ (b) $\lim_{x\to 0} \sin x$ (c) $\lim_{x\to 0} \cos x$ (d) $\lim_{x\to c} f(x)$, given that $\lim_{x\to c} |f(x)| = 0$.

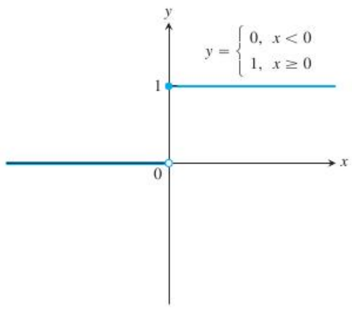
(a) Since $-1 \le \sin x \le 1$, we have $-|x| \le |x|\sin x \le |x|$. We know that $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. By Sandwich theorem, $\lim_{x\to 0} |x|\sin x = 0$.

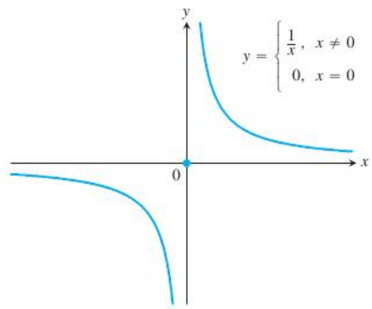
(b) Since $-|x| \le \sin x \le |x|$, Sandwich theorem gives $\lim_{x\to 0} \sin x = 0$.

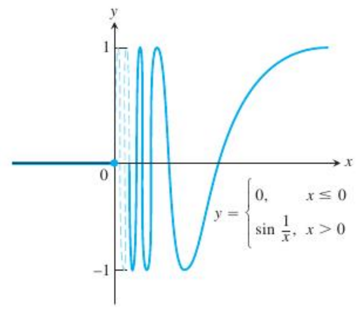
(c) Since $0 \le 1 - \cos x \le |x|$, it follows that $\lim_{x\to 0}(1 - \cos x) = 0$. This gives $\lim_{x\to 0} \cos x = 1$.

(d) $-|f(x)| \le f(x) \le |f(x)|$. Since $\lim_{x\to c} |f(x)| = 0$, by Sandwich theorem, $\lim_{x\to c} f(x) = 0$.

Functions may not have a limit at a given point because there may be a jump at that point, or it may be growing too large near that point, or even it may be oscillating too often near the point.







1.3 One-sided Limits and Asymptotes

One-sided limits can be defined in an analogous manner. We write $\lim_{x\to c-} f(x) = \ell$ iff whenever x lies near c and remains less than c, we have that $f(x)$ lies near ℓ. That is, in the $\epsilon - \delta$ definition, we simply consider all x satisfying all those conditions along with x being less than c. This is called the **left hand limit** of $f(x)$ at $x = c$. Similarly, we write $\lim_{x\to c+} f(x) = \ell$ iff whenever x lies near c and remains greater than c, we have that $f(x)$ lies near ℓ. This is the **right hand limit** of $f(x)$ at $x = c$.

It thus follows that $\lim_{x\to c} f(x) = \ell$ iff $\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x) = \ell$.

Example 1.6. Evaluate the left hand and right hand limits if they exist:
(a) $f(x) = x/|x|$ at $x = 0$ (b) $g(x) = \sqrt{4 - x^2}$ at all the points in its domain $[-2, 2]$.

(a) For the left hand limit, $x < 0$. It gives $f(x) = x/|x| = x/(-x) = -1$. Thus $\lim_{x\to 0-} f(x) = -1$.
For the right hand limit, $x > 0$. Then $\lim_{x\to 0+} (x/|x|) = \lim_{x\to 0+} (x/x) = \lim_{x\to 0+} (1) = 1$.
Thus $\lim_{x\to 0}(x/|x|)$ does not exist.

(b) We cannot speak of the left hand limit of $g(x)$ at $x = -2$ since the domain does not include an open interval to the left of $x = -2$. Similarly, we cannot speak of right hand limit of $g(x)$ at $x = 2$.
But $\lim_{x\to -2+} \sqrt{4 - x^2} = 0 = \lim_{x\to 2-} \sqrt{4 - x^2}$. Hence $\lim_{x\to 2} \sqrt{4 - x^2} = 0$.

At all (other) points $c \in (-2, 2)$, $\lim_{x \to c} g(x) = \sqrt{4 - c^2}$.

All the properties in Theorem 1.1 hold true for one-sided limits.

Example 1.7. Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Using the inequality $\sin x < x < \tan x$ for $0 < x < \pi/2$, we have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \text{Or} \quad \cos x < \frac{\sin x}{x} < 1.$$

By Sandwich theorem,

$$\lim_{x \to 0+} \frac{\sin x}{x} = 1.$$

When $x < 0$, write $x = -t$ for $t > 0$. Then $\sin x = -\sin t$.

$$\lim_{x \to 0-} \frac{\sin x}{x} = \lim_{t \to 0+} \frac{-\sin t}{-t} = 1.$$

Hence $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

We also bring in ∞ in discussing limits. These are referred to as **limits at** ∞. The formal definitions follow:

Let I be (a, ∞) or $[a, \infty)$ for some $a \in \mathbb{R}$. Let $f : I \to \mathbb{R}$. Let $\ell \in \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = \ell$ if for each $\epsilon > 0$, there exists an $m > 0$ such that if x is any real number greater then m, then $|f(x) - \ell| < \epsilon$.

Let I be $(-\infty, a)$ or $(-\infty, a]$ for some $a \in \mathbb{R}$. Let $f : I \to \mathbb{R}$. Let $\ell \in \mathbb{R}$. We say that $\lim_{x \to -\infty} f(x) = \ell$ if for each $\epsilon > 0$, there exists an $m > 0$ such that if x is any real number less than $-m$, then $|f(x) - \ell| < \epsilon$.

Limits at ∞ or at $-\infty$ have all the same properties as the limit at any point c.

Example 1.8. Show that $\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$.

Let $\epsilon > 0$. Choose $m = 1/\epsilon$. If $x > m$, then $x > 0$ and $|1/x - 0| = 1/x < \epsilon$. Hence $\lim_{x \to \infty} \frac{1}{x} = 0$.

For the other limit, let $\epsilon > 0$. Choose $m = 1/\epsilon$. If $x < -m$, then $0 > 1/x > -1/m = -\epsilon$. So, $|1/x - 0| = -1/x < \epsilon$. Hence $\lim_{x \to -\infty} \frac{1}{x} = 0$.

As earlier, we will not use this $\epsilon - \delta$ definition often in our calculations. These definitions are not new provided that you consider any neighborhood of $-\infty$ as any interval $(-\infty, -a)$ and any neighborhood of ∞ is of the form (a, ∞) for $a > 0$. We rather depend on our intuitive understanding. For the limit at ∞ to be equal to ℓ, we understand that as x increases without bound, $f(x)$ remains near ℓ. Similarly, for the limit at $-\infty$ to be equal to ℓ means that as x decreases without bound, $f(x)$ lies near ℓ.

Geometrically, it means that at ∞ or at $-\infty$, the line $y = \ell$ merges with the curve $y = f(x)$.

We say that the line $y = \ell$ is an **asymptote of** $y = f(x)$ **at** ∞ if $\lim_{x\to\infty} f(x) = \ell$.

Similarly, the line $y = \ell$ is an **asymptote of** $y = f(x)$ **at** $-\infty$ if $\lim_{x\to-\infty} f(x) = \ell$.

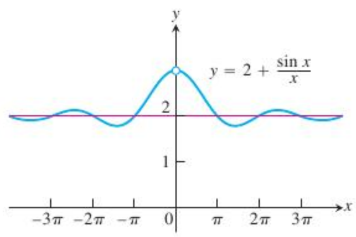
Both the asymptotes at ∞ and at $-\infty$ are called **asymptotes** of $y = f(x)$.

An asymptote $y = \ell$ is also called a **horizontal asymptote**.

An asymptote can intersect the curve as the following example shows.

Example 1.9. Find the asymptotes of $y = f(x) = 2 + \dfrac{\sin x}{x}$.

Since $0 \le \left|\dfrac{\sin x}{x}\right| \le \left|\dfrac{1}{x}\right|$, by Sandwich theorem, $\lim_{x\to\pm\infty} \dfrac{\sin x}{x} = 0$. Hence $\lim_{x\to\pm\infty} \big(2 + \dfrac{\sin x}{x}\big) = 2$. The line $y = 2$ is the asymptote to $y = 2 + \dfrac{\sin x}{x}$ at ∞ and also at $-\infty$.



In general, a straight line $y = ax + b$ is an **asymptote** to $y = f(x)$ if $\lim_{x\to\infty}(f(x) - (ax+b)) = 0$ or if $\lim_{x\to-\infty}(f(x) - (ax+b)) = 0$, respectively at ∞ and at $-\infty$. Such asymptotes are called **slanted asymptotes**.

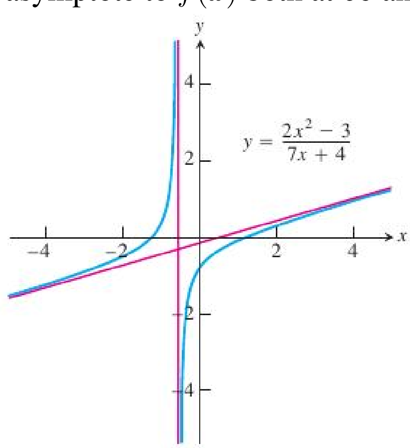
Usually, when a rational function $p(x)/q(x)$ has the degree of the numerator $p(x)$ one more than the degree of the denominator $q(x)$, then a slanted asymptote exists.

Example 1.10. Find an asymptote of $y = f(x) = \frac{2x^2-3}{7x+4}$.

Using long division write $f(x) = \dfrac{2x^2-3}{7x+4} = \left(\dfrac{2}{7}x - \dfrac{8}{49}\right) + \dfrac{-115}{49(7x+4)}$.

Notice that $\lim_{x\to\pm\infty}\left\{f(x) - \left(\dfrac{2}{7}x - \dfrac{8}{49}\right)\right\} = \lim_{x\to\pm\infty}\left\{\dfrac{-115}{49(7x+4)}\right\} = 0$.

Hence the line $y = \frac{2}{7}x - \frac{8}{49}$ is an asymptote to $f(x)$ both at ∞ and at $-\infty$.



Sometimes it is easier to calculate the limits at infinity by substituting the independent variable by its reciprocal. We see that if the limits at infinity exist, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0+} f(1/x) \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to 0-} f(1/x).$$

Consider two functions $f(x)$ and $g(x)$ with the same domain $(0, \infty)$. Let $f(x) = 1$ for all positive rationals and $f(x) = 0$ for all positive irrationals. In contrast, take $g(x) = 1/x$ for all positive x. Limits for both as x approaches 0 from the right hand side, do not exist. For $f(x)$, the limit does not exist because the values $f(x)$ when x is near 0 fluctuates between 0 and 1, and never remains near either of them. Whereas as x approaches 0, $g(x)$ increases without bound, so does not remain near any real number. We want to separate the case of $g(x)$ from the rest. We say that $g(x)$ has an **infinite limit** as x approaches 0. Formal definitions in various cases follow.

1. Let $f(x)$ have a domain containing (c, b). Then $\lim_{x\to c+} f(x) = \infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $c < x < c + \delta$, we have $f(x) > m$.

   Again we work with an intuitive understanding that

   $$\lim_{x\to c+} f(x) = \infty \text{ iff, “as } x \text{ decreases to } c,\ f(x) \text{ increases without bound”.}$$

2. Let $f(x)$ have a domain containing (a, c). Then $\lim_{x\to c-} f(x) = \infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $c - \delta < x < c$, we have $f(x) > m$.

   That is, $\lim_{x\to c-} f(x) = \infty$ iff, “as x increases to c, $f(x)$ increases without bound”.

3. Let $f(x)$ have a domain containing (c, b). Then $\lim_{x\to c+} f(x) = -\infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $c < x < c + \delta$, we have $f(x) < -m$.

   That is, $\lim_{x\to c+} f(x) = -\infty$ iff, “as x decreases to c, $f(x)$ decreases without bound”.

4. Let $f(x)$ have a domain containing (a, c). Then $\lim_{x\to c-} f(x) = -\infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $c - \delta < x < c$, we have $f(x) < -m$.

   That is, $\lim_{x\to c-} f(x) = -\infty$ iff, “as x increases to c, $f(x)$ decreases without bound”.

   As earlier, we write $\lim_{x\to c} f(x) = \infty$ if $\lim_{x\to c+} f(x) = \infty = \lim_{x\to c-} f(x)$.

   Similarly, we write $\lim_{x\to c} f(x) = -\infty$ if $\lim_{x\to c+} f(x) = -\infty = \lim_{x\to c-} f(x)$.

5. Let $f(x)$ have a domain containing (a, ∞). Then $\lim_{x\to\infty} f(x) = \infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $x > \delta$, we have $f(x) > m$.

   That is, $\lim_{x\to\infty} f(x) = \infty$ iff, “as x increases without bound, $f(x)$ also increases without bound”. Notice that $\lim_{x\to\infty} f(x) = \infty$ iff $\lim_{x\to 0+} f(1/x) = \infty$.

6. Let $f(x)$ have a domain containing (a, ∞). Then $\lim_{x\to\infty} f(x) = -\infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $x > \delta$, we have $f(x) < -m$.

That is, $\lim\limits_{x\to\infty} f(x) = -\infty$ iff, "as x increases without bound, $f(x)$ decreases without bound". Again, $\lim\limits_{x\to\infty} f(x) = -\infty$ iff $\lim\limits_{x\to 0+} f(1/x) = -\infty$.

7. Let $f(x)$ have a domain containing $(-\infty, a)$. Then $\lim\limits_{x\to-\infty} f(x) = \infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $x < -\delta$, we have $f(x) > m$.

   That is, $\lim\limits_{x\to-\infty} f(x) = \infty$ iff, "as x decreases without bound, $f(x)$ increases without bound". Here, $\lim\limits_{x\to-\infty} f(x) = \infty$ iff $\lim\limits_{x\to 0-} f(1/x) = \infty$.

8. Let $f(x)$ have a domain containing $(-\infty, a)$. Then $\lim\limits_{x\to-\infty} f(x) = -\infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $x < -\delta$, we have $f(x) < -m$.

   That is, $\lim\limits_{x\to-\infty} f(x) = -\infty$ iff, "as x decreases without bound, $f(x)$ also decreases without bound". Once more, $\lim\limits_{x\to-\infty} f(x) = -\infty$ iff $\lim\limits_{x\to 0-} f(1/x) = -\infty$.

If you understand the principle behind limits at infinity and infinite limits, then you will not have to remember each one separately.

Caution: When limit of a function at a point, or at infinite, becomes infinite, it does not mean that the limit exists. It only says that this non-existence of limit is special, in the sense of growing too large or becoming smaller and smaller without bound. In fact, if the limit of a function exists at $x = c$, then it is necessarily a real number, which is never equal to ∞ or $-\infty$.

The algebra with the symbols ∞ and $-\infty$ are as follows:

Let $a \in \mathbb{R},\ b > 0$ and $c < 0$. Then

$$a + \infty = \infty,\ a - \infty = a + (-\infty) = -\infty,\ a - (-\infty) = \infty,\ a/\infty = a/(-\infty) = 0.$$

$$b \cdot \infty = \infty,\ b \cdot (-\infty) = -\infty,\ c \cdot \infty = -\infty,\ c \cdot (-\infty) = \infty.$$

$$\infty + (-\infty),\ 0 \cdot \infty,\ 0 \cdot (-\infty),\ \pm\infty/\pm\infty \text{ are indeterminate.}$$

All the properties of limits hold provided we do not end up at indeterminate forms. For example, if $\lim\limits_{x\to c} f(x) = d > 0$ and $\lim\limits_{x\to c} g(x) = \infty$, then $\lim\limits_{x\to c} f(x)g(x) = \infty$.

Example 1.11. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_n \neq 0$. Let $b > 1$. Show the following;

(a) $\lim\limits_{x\to\infty} \dfrac{p(x)}{x^n} = a_n$. (b) $\lim\limits_{x\to\infty} p(x) = \infty$ if $a_n > 0$ (c) $\lim\limits_{x\to\infty} p(x) = -\infty$ if $a_n < 0$.

(a) $\lim\limits_{x\to\infty} \dfrac{p(x)}{x^n} = \lim\limits_{x\to\infty} \left(\dfrac{a_0}{x^n} + \dfrac{a_1}{x^{n-1}} + \cdots + \dfrac{a_{n-1}}{x} + a_n\right) = a_n$.

(b-c) From (a), $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} \dfrac{p(x)}{x^n} \lim\limits_{x\to\infty} x^n = a_n \cdot \infty$. Now the answers follow.

This example says that as x approaches ∞, $p(x)$ behaves as its leading term.

Infinite limits allow us to define vertical asymptotes.

A line $x = c$ is called a **vertical asymptote** of the function $y = f(x)$ if one of the following happens:

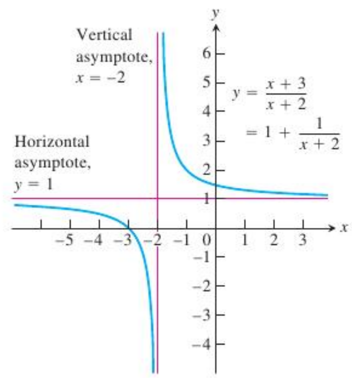
$$\lim_{x\to c+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to c-} f(x) = \pm\infty.$$

Example 1.12. Find the horizontal and vertical asymptotes of $y = f(x) = \dfrac{x+3}{x+2}$.

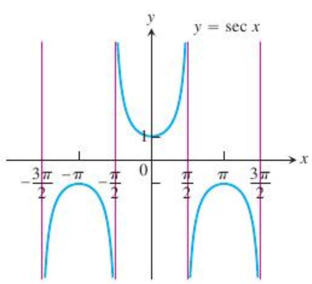
We see that $\lim\limits_{x \to -2-} f(x) = -\infty$ Also, $\lim\limits_{x \to -2+} = \infty$. And, for no other point c, limit of $f(x)$ is $\pm\infty$ as x approaches c. So, $x = -2$ is a vertical asymptote.

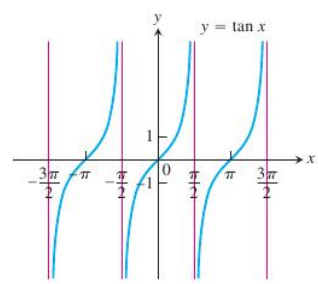
Next, $\lim\limits_{x \to \infty} f(x) = 1$. Also, $\lim\limits_{x \to -\infty} f(x) = 1$. So, $y = 1$ is the horizontal asymptote.

Alternatively, write $f(x)$ as $f(x) = 1 + \dfrac{1}{x+2}$.
Now, both horizontal and vertical asymptotes are visible from this expression!
As $x \to -2$, from left or from right, $f(x) \to \mp\infty$. And as $x \to \pm\infty$, $f(x) \to 1$.



There are curves with infinitely many asymptotes. For example, $y = \sec x$ and $y = \tan x$ have vertical asymptotes at points $(2n+1)\pi/2$ for each $n \in \mathbb{N}$.





1.4 Continuity

Let $f : D \to \mathbb{R}$ be a function. Let c be an interior point of D. We say that $f(x)$ is **continuous at** c if $\lim\limits_{x \to c} f(x) = f(a)$.

If $D = [a, b)$ or $D = [a, b]$, then $f(x)$ is called continuous at the left end-point a if $\lim\limits_{x \to a+} f(x) = f(a)$.

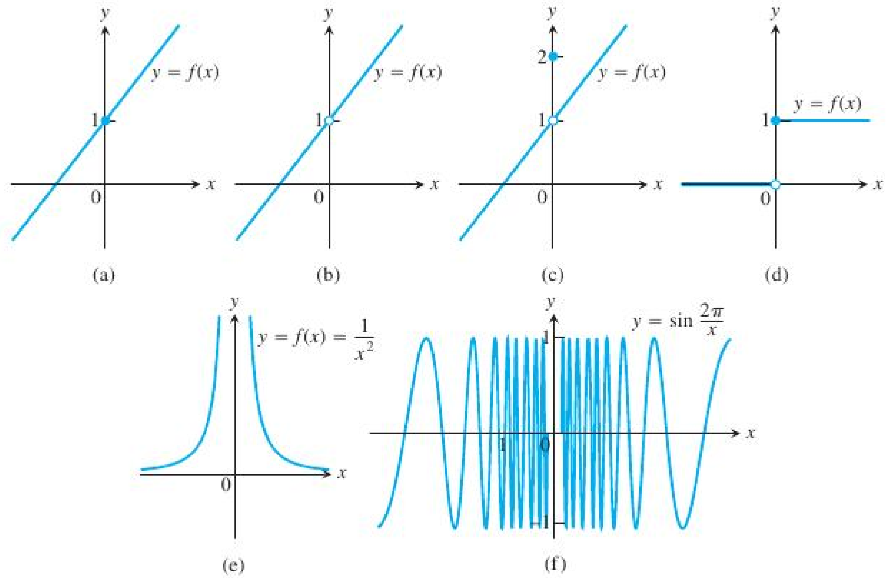
If $D = (a, b]$ or $D = [a, b]$, then $f(x)$ is called continuous at the right end-point b if $\lim\limits_{x \to b-} f(x) = f(a)$.

$f(x)$ is called **continuous** if it is continuous at each point of its domain D.

For domains of the type $(0, 1] \cup [2, 3)$, continuity can be defined similarly at $x = 1$ and at $x = 2$, but we will not need them.

A function with domain as in interval if found to be continuous at all points, then you can draw its graph without lifting the pen.

In the following figure, the functions in (a) is continuous at $x = 0$. Others are not continuous at $x = 0$, though the types of discontinuity are different. Write out what kinds of discontinuity you observe.



Example 1.13. Where is the function defined below continuous?

$$f(x) = \begin{cases} \frac{1}{x-3} & \text{if } x < 4 \\ 2x+3 & \text{if } x \geq 4 \end{cases}$$

At any point $c < 3$, $f(x) = 1/(x-3)$ is continuous since $\lim_{x \to c} f(x) = 1/(c-3)$.

At $x = 3$, $f(x)$ is not defined; so we cannot speak of its continuity at $x = 3$.

For $3 < c < 4$, $f(x) = 1/(x-3)$. It is again continuous.

At $x = 4$, $\lim_{x \to 4-} f(x) = \lim_{x \to 4-} \{1/(x-3)\} = 1$. And $\lim_{x \to 4+} f(x) = \lim_{x \to 4+} (2x+3) = 11$. Hence limit of $f(x)$ does not exist at $x = 4$. That is, $f(x)$ is not continuous at $x = 4$.

For $c > 4$, clearly $f(x) = 2x + 4$ is continuous at $x = c$.

Due to the properties of limits, the following theorem is obvious.

Theorem 1.2. *Let $f(x)$ and $g(x)$ be functions continuous at $x = c$. Let $k \in \mathbb{R}$. Then*

1. $f(x) + g(x),\ f(x) - g(x),\ f(x) \cdot g(x),\ k \cdot f(x)$ are continuous at $x = c$.

2. $f(x)/g(x)$ is continuous at $x = c$ provided that $g(c) \neq 0$.

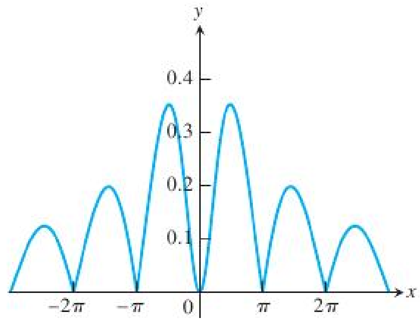
3. $[f(x)]^k$ is continuous at $x = c$ provided $[f(x)]^k$ is defined in an interval around c.

4. In addition, if $h(x)$ is continuous at $f(c)$, then $(h \circ f)(x)$ is continuous at c.

Every polynomial and every rational function is continuous. The trigonometric functions are continuous on their respective domains. The absolute value function $f(x) = |x|$ is continuous. The function $f(x) = 1/x$ is continuous on $\mathbb{R} - \{0\}$, which is its domain. So, $f(x) = 1/x$ is a continuous function.

Example 1.14. Is the function $f(x) = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ continuous?

$f_1(x) = x$ is continuous, $f_2(x) = \sin x$ is continuous. Thus $f_3(x) = x \sin x$ is continuous. $f_4(x) = x^2 + 2$ is continuous and is never 0. So, $f_5(x) = f_3(x)/f_4(x)$ is continuous. The function $f_6(x) = |x|$ is continuous. So, $f(x) = (f_6 \circ f_5)(x)$ is continuous. Its graph looks like:



The function $f(x) = \dfrac{\sin x}{x}$ is not continuous at $x = 0$ since it is not defined at $x = 0$. However, the function $g(x) = \dfrac{\sin x}{x}$ for $x \neq 0$; and $g(0) = 1$ is continuous everywhere since $\lim_{x \to 0} \dfrac{\sin x}{x} = 1 = g(0)$.

1.5 Continuous functions on Closed Intervals

The formal definition can be used to prove a nice fact about continuous functions.

Theorem 1.3. *Let $f(x)$ be continuous at $x = c$, where the domain of $f(x)$ includes a neighborhood of c. If $f(c) > 0$, then there exists a neighborhood $(c - \delta, c + \delta)$ such that $f(x) > 0$ for each point $x \in (c - \delta, c + \delta)$.*

Proof: Suppose that $f(c) > 0$. Let $\epsilon = f(c)/2$. Since $f(x)$ is continuous at $x = c$, we have a $\delta > 0$ such that for each $x \in (c - \delta, c + \delta)$, $|f(x) - f(c)| < f(c)/2$. That is, $f(c)/2 < f(x) < 3f(c)/2$. As $f(c) > 0$, we see that for each $x \in (c - \delta, c + \delta)$, we have $f(x) > 0$. □

Continuous functions on intervals enjoy another nice property:

The image of a closed bounded interval under a continuous function is a closed bounded interval.

We write it formally as follows.

Theorem 1.4. *Let $f(x)$ be a continuous function, whose domain contains $[a, b]$ for $a < b$. Then there exist $\alpha,\ \beta \in \mathbb{R}$ such that $\{f(x) : x \in [a, b]\} = [\alpha, \beta]$.*

Proof of this result requires the completeness property of $\mathbb{R}$ and we omit it. Of course, the image as an interval may degenerate into a single point, for example, when $f(x)$ is a constant function.

We will discuss two important corollaries of Theorem 1.4.

If $f(x)$ is not known to be continuous, then the real numbers α and β given by

$$\alpha = \text{glb}\{f(x) : x \in [a, b]\} \quad \text{and} \quad \beta = \text{lub}\{f(x) : x \in [a, b]\}$$

may or may not be in the image of $[a, b]$ under f. However, if $f(x)$ is continuous, then α and β must be in the image of $[a, b]$ under f. This is the content of our first corollary.

Theorem 1.5. (Extreme Value Theorem) *Let $f(x)$ be continuous on a closed bounded interval $[a, b]$. Then there exist numbers $c, d \in [a, b]$ such that $f(c) \le f(x) \le f(d)$ for each $x \in [a, b]$.*

Proof: By Theorem 1.4, we have $\alpha, \beta \in \mathbb{R}$ such that $\{f(x) : x \in [a, b]\} = [\alpha, \beta]$. Then $\alpha \le f(x)$ for each $x \in [a, b]$ and $\alpha \in \{f(x) : x \in [a, b]\}$. That is, there exists $c \in [a, b]$ such that $\alpha = f(c)$. Similarly, $\beta \ge f(x)$ for each $x \in [a, b]$ and there exists $d \in [a, b]$ such that $\beta = f(d)$. $\square$

The Extreme Values Theorem (**EVT**) says that the quantities

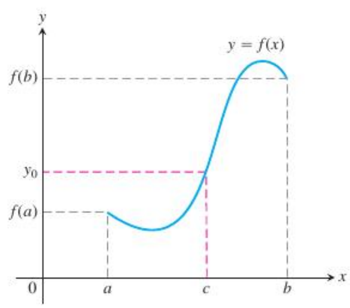
$$\min\{f(x) : x \in [a, b]\} \quad \text{and} \quad \max\{f(x) : x \in [a, b]\}$$

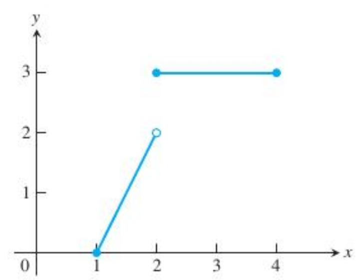
are well-defined. It is informally stated as

A continuous function on a closed bounded interval achieves its minimum and maximum.

The other important corollary of Theorem 1.4 is as follows:

Theorem 1.6. (Intermediate Value Theorem) *Let $f(x)$ be continuous on a closed bounded interval $[a, b]$. Let d be a number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = d$.*





Proof: By Theorem 1.4, we have $\alpha, \beta \in \mathbb{R}$ such that $\{f(x) : x \in [a, b]\} = [\alpha, \beta]$. Then

$$\alpha \le f(a) \le \beta \quad \text{and} \quad \alpha \le f(b) \le \beta.$$

If d lies between $f(a)$ and $f(b)$, then $\alpha \le d \le \beta$. Therefore, there exists $c \in [a, b]$ such that $f(c) = d$. $\square$

Intermediate Value Theorem (**IVT**) says that a function continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. It helps in bracketing a **zero of a function**. A number α is called a zero of a function if $f(\alpha) = 0$. Given a continuous function $f(x)$, if $f(a)$ and $f(b)$ have opposite signs, then the interval (a, b) must contain a point c where $f(c) = 0$; IVT confirms this conclusion.

Remark: We have derived IVT and EVT from Theorem 1.4. However, IVT and EVT together prove Theorem 1.4. In such an approach, we would require independent proofs of IVT and EVT; and these proofs use the completeness principle of $\mathbb{R}$.

Example 1.15. Let $ABCD$ be a square. Draw a curve lying inside $ABCD$ and joining the diagonally opposite points A and C. Draw a curve joining B and D and again lying inside $ABCD$. Prove that the two curves intersect somewhere inside the square. (Though a curve is much more general, here we assume that they are represented by continuous functions of x.)

Let the vertices of the square be $A(0,0),\ B(1,0),\ C(1,1)$, and $D(0,1)$ in the plane. The curves are (assumed to be) continuous functions $f, g : [0,1] \to [0,1]$. We see that $f(0) = 0,\ f(1) = 1$ and $g(0) = 1,\ g(1) = 0$. Then $h(x) = f(x) - g(x)$ is a continuous function from $[0,1]$ to $[-1,1]$. And $h(0) = -1,\ h(1) = 1$. Therefore, by IVT, there is a point $c \in [0,1]$ such that $h(c) = 0$. At this point, $f(c) = g(c)$, that is, the curves intersect.

Example 1.16. Prove that every polynomial of odd degree has a real root.

Without loss of generality let $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, where n is odd. Now, $\lim_{x\to-\infty} p(x) = -\infty$. So, there exists a point $a < 0$ such that $p(a) < -1$. Also, $\lim_{x\to\infty} p(x) = \infty$. So, there exists a point $b > 0$ such that $p(b) > 1$. Consider p as a function from $[a,b]$ to $\mathbb{R}$. Since $0 \in [p(a), p(b)]$, by IVT, we conclude that there exists a point $c \in [a,b]$ such that $p(c) = 0$.

If a function is not continuous, then the conclusion of IVT may or may not hold.

Example 1.17. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = x$ if $x \neq 1$ and $f(1) = 0$. Then $f(x)$ is not continuous at $x = 1$. But it satisfies the conclusion of IVT, even on any closed subinterval of $[0,1]$. On the other hand, let $g : [0,2] \to \mathbb{R}$ be given by $g(x) = x$ for $0 \le x \le 1$ and $g(x) = x + 2$ for $1 < x \le 2$. Here, g is not continuous at $x = 1$, $g(0) = 0,\ g(2) = 4$. For no $c \in [0,2],\ \ g(c) = 2$.

1.6 Review Problems

Problem 1: (a) Find $\ell = \lim_{x\to 0} \sqrt{x}\sin\dfrac{1}{x}$.

$0 \le |\sqrt{x}\sin\frac{1}{x}| \le \sqrt{x}$. By Sandwich theorem, $\ell = 0$.

Same argument shows that $\lim_{x\to 0} x^n \sin\dfrac{1}{x} = 0$ for any $n \in \mathbb{N}$.

(b) Find $\ell = \lim_{x\to 0} \sin\dfrac{1}{x}$.

When x is near 0, we may take x as $(m\pi/2)^{-1}$ for a large m or we may choose x as $(n\pi)^{-1}$ for a large n. Then $\sin\frac{1}{x}$ in the first case, will be equal to 1 and in the seond case, will be equal to 0. Notice that in any neighborhood of $x = 0$, there are numbers of the first form and also of the second form for some large m and n. Therefore, this limit does not exist.

Problem 2: Find the limits $\lim_{x\to a} f(x),\ \lim_{x\to a} g(x)$ where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin Q, \end{cases} \qquad g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin Q. \end{cases}$$

In any neighborhood of $x = a$, there are rational as well as irrational numbers. Thus, for any rational number close to $x = a$, $f(x) = 1$ where as for irrational numbers close to $x = a$, $f(x) = 0$. Therefore, the first limit does not exist.

For $g(x)$, if $a = 0$, and x is any rational number close to $a = 0$, then $f(x) = x$ which is close to 0. On the other hand, when x is any irrational number close to 0, $f(x)$ is equal to 0. Therefore, $\lim_{x\to 0} g(x) = 0$.

For $x \neq 0$, if x is any rational number close to a, then $f(x) = x$ is close to a, and not to 0. Whereas if x is an irrational number close to a, then $f(x)$ is equal to 0. Therefore, limit of $g(x)$ as x approaches a does not exist, for $a \neq 0$.

Problem 3: Discuss the continuity of $f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 1 + \lfloor x \rfloor + \sin x & \text{if } 0 \le x \le \pi/2 \\ 3 & \text{if } x \ge \pi/2. \end{cases}$

For $x < 0,\ f(x) = 1 + x$, which is a continuous function. At $x = 0$,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (1+x) = 1, \quad \lim_{x\to 0+} f(x) = \lim_{x\to 0+} (1 + \lfloor x \rfloor + \sin x) = 1, \quad f(0) = 1.$$

Hence $f(x)$ is continuous at $x = 0$.

For $0 < x < 1$ and $1 < x < \pi/2$, $f(x) = 1 + \lfloor x \rfloor + \sin x$ is continuous. At $x = 1$,

$$\lim_{x\to 1-} (1 + \lfloor x \rfloor + \sin x) = \lim_{x\to 1-} (1 + 0 + \sin x) = 1 + \sin 1.$$

$$\lim_{x\to 1+} (1 + \lfloor x \rfloor + \sin x) = \lim_{x\to 1-} (1 + 1 + \sin x) = 2 + \sin 1 = f(1).$$

So, $f(x)$ is not continuous at $x = 1$; it is only right-continuous at $x = 1$.

For $x > \pi/2,\ f(x) = 3$, which is continuous. At $x = \pi/2$,

$$\lim_{x\to\pi/2-} f(x) = \lim_{x\to\pi/2-} (1 + \lfloor x \rfloor + \sin x) = 3 = \lim_{x\to\pi/2+} f(x) = f(\pi/2) = 3.$$

So, $f(x)$ is continuous at $x = \pi/2$.

In summary, $f(x)$ is continuous everywhere except at the point $x = 1$.

Problem 4: Let $f(x)$ be an unbounded function, i.e., for each $m \in \mathbb{N}$, there exists an x such that $|f(x)| > m$. Does it follow that $\lim_{x\to\infty} f(x) = \pm\infty$?

No. For example, let $f(x) = x \sin x$. Let $m \in \mathbb{N}$. Consider $x = m\pi/2$. Now, $|\sin x| = 1$. Consequently, $|x \sin x| = m\pi/2 > m$. But $\lim_{x\to\infty} x \sin x \neq \pm\infty$. Reason: If x is large and in the form $m\pi$, then $x \sin x = 0$.

Problem 5: Find the largest integer m such that

$$\lim_{x\to 1} \left[\frac{m(1-x) + \sin(x-1)}{x - 1 + \sin(x-1)} \right]^{1+x} = \frac{1}{4}.$$

We are interested in finding the limit of the given function when x is near 1 but x is not equal to 1.

If $m = 0$, then using $\lim_{x\to 1} \dfrac{\sin(x-1)}{x-1} = 1$, we find that the limit is equal to $(\frac{1}{2})^2 = \frac{1}{4}$. Thus $m = 0$ is a possible value. There can be others.

If $m \geq 1$ and x is near 1, then $\dfrac{m(1-x)+\sin(x-1)}{x-1+\sin(x-1)} < 0$. Its power is not defined.

Therefore, $m = 0$. **Exercise 1**: Let $f(x) = \begin{cases} x^{2n}\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that $f'(0), f''(0), \ldots, f^{(n)}(0)$ exist, $f^{(n)}(x)$ is not continuous at $x = 0$.

Exercise 2: Let $g(x) = \begin{cases} x^{2n+1}\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that $g'(0), g''(0), \ldots, g^{(n)}(0)$ exist, $g^{(n)}(x)$ is continuous at $x = 0$, but $g^{(n)}(x)$ is not differentiable at $x = 0$.

Chapter 2

Differentiation

2.1 Derivative

Let $f(x)$ be a function whose domain includes an open interval (a, b). Let $c \in (a, b)$. If the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists, we say that $f(x)$ is **differentiable at** $x = c$; and we write the limit as $f'(c)$ and call it the derivative of $f(x)$ at $x = c$.

We also write $f'(c)$ as $\left.\frac{df}{dx}\right|_{x=c}$, and also as $\frac{df}{dx}(c)$. Throughout, we will use the prime ($'$) notation for differentiation with respect to the concerned independent variable. For example, if $y = f(x)$, then both y' and $f'(x)$ denote $\frac{dy}{dx}$, whereas if $x = g(y)$, then both x' and $g'(y)$ will denote $\frac{dx}{dy}$.

Example 2.1. Is $f(x) = x^3 - x^2 + x - 2$ differentiable at $x = 1$?

$$\lim_{h \to 0} \frac{\{(1+h)^3 - (1+h)^2 + (1+h) - 2\} - \{1^3 - 1^2 + 1 - 2\}}{h} = 2.$$

Hence $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$.

Example 2.2. Is $f(x) = |x|$ differentiable at $x = 0$?

$$\lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

does not exist since the left hand limit is -1 whereas the right hand limit is 1.
Therefore, $|x|$ is not differentiable at $x = 0$.

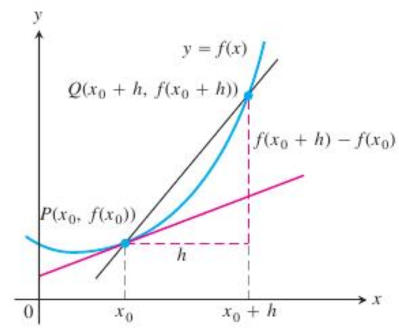
Example 2.3. Is $f(x) = x^{1/3}$ differentiable at $x = 0$?

$\lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} h^{-2/3}$ does not exist. Hence $x^{1/3}$ is not differentiable at $x = 0$.

Let $y = f(x)$ be a curve. Let $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ be two points on the curve. The line joining these points is a secant to the curve. The slope of the secant is

$$\frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}.$$

Then its limit as $h \to 0$ gives the slope of the tangent to $y = f(x)$ at the point $x = x_0$. That is, $f'(x_0)$ is the slope of the tangent to the curve $y = f(x)$ at $x = x_0$.



We should be cautious in using the word *tangent*. A tangent to a curve may be defined criptically as a unique straight line that has exactly one point in common with the curve at identically two points. That is, at a point, a tangent is unique and if we change its slope a little bit, then it has more than one point in common with the curve. For example, there are many straight lines that touch the curve $y = |x|$ at the point $x = 0$ and therefore, this curve has no tangent at $x = 0$. Leaving out the horizontal and vertical tangents, we may take the following view: if both $\frac{dy}{dx}$ and $\frac{dx}{dy}$ do not exist, or if both $\frac{dy}{dx} = \frac{dx}{dy} = 0$ at a point, then a tangent to the curve is not defined at that point. If both $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are nonzero at a point, then a tangent to the curve exists at that point.

If a curve is given parametrically by $x = x(t),\ y = y(t)$, and at a point $t = t_0$, both $x'(t_0)$ and $y'(t_0)$ are nonzero, then the straight line

$$(y - y(t_0))x'(t_0) - (x - x(t_0))y'(t_0) = 0$$

is the tangent to the curve at $t = t_0$. Similarly, if the curve is given implicitly by $f(x, y) = 0$ and both $m = \frac{dy}{dx}(x_0, y_0)$ and $\overline{m} = \frac{dx}{dy}(x_0, y_0)$ are nonzero at the point (x_0, y_0), then the tangent at (x_0, y_0) is given by

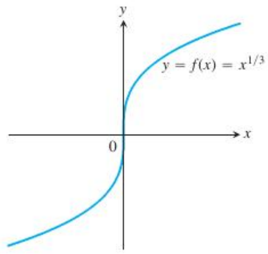
$$y - y_0 = m(x - x_0) \quad \text{and also by} \quad x - x_0 = \overline{m}(y - y_0).$$

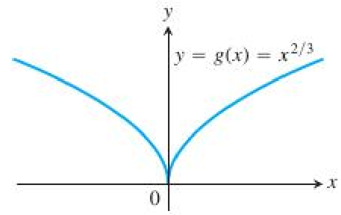
However, differentiability says something more than the existence of a tangent at a point. For example, the tangent to the curve $y = x^{1/3}$ at $x = 0$ is the y-axis whereas $y = x^{1/3}$ is not differentiable at $x = 0$. Because,

$$\lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} h^{-2/3} = \infty.$$

The curve has a vertical tangent at $x = 0$, which intersects the curve only at that point.

We thus say that a curve $y = f(x)$ has a **vertical tangent** at $x = c$ if $f'(c)$ is either equal to ∞ or equal to $-\infty$.





Caution: Vertical tangents are not covered in the usual interpretation of derivative as the slope of the tangent.

What about the curve $y = x^{2/3}$? Compute its derivative:

$$\lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} h^{-1/3}.$$

We get the left hand limit as $-\infty$ and the right hand limit as ∞. That is, this curve does not even have a vertical tangent.

We say that a function $f(x)$ is **differentiable** if it is differentiable at every point of its domain. Leaving the pathological cases aside, we will write $f'(x)$ as a new function obtained from $f(x)$. Then $f'(c)$ is the value of $f'(x)$ at $x = c$.

Let $\phi(t)$ be a differentiable function on an open interval (a, b). Suppose that $\phi(t)$ represents the position of a moving body at time t. Let $c \in (a, b)$. Then

$$\phi'(c) = \text{instantaneous velocity of the moving body at } t = c.$$

Theorem 2.1. *Each function differentiable at $x = c$ is continuous at $x = c$.*

Proof: Let c be any point in the domain of $f(x)$. Given that $f(x)$ is differentiable at $x = c$, we have a real number $f'(c)$ and $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$. Now,

$$\begin{aligned} \lim_{x\to c}\{f(x) - f(c)\} &= \lim_{h\to 0}\{f(c+h) - f(c)\} = \lim_{h\to 0}\left[\frac{f(c+h) - f(c)}{h} \cdot h\right] \\ &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h = f'(c) \cdot 0 = 0. \end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = f(c)$. And the function $f(x)$ is continuous at $x = c$. □

It only says that a differentiable function is continuous on its domain; it does not say that the derivative of a function is continuous.

Example 2.4. Let $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

We see that $f'(x) = g(x) = 2x\sin(1/x) - \cos(1/x)$ for $x \neq 0$.
Also, $\lim_{x\to 0} g(x)$ does not exist. Hence $f'(x)$ is not continuous at $x = 0$.

The derivative of a function need not be continuous, but it satisfies the conclusion of IVT. Its proof again uses the completeness principle of $\mathbb{R}$.

Theorem 2.2. (Darboux Theorem) *Let f be a differentiable function whose domain includes an interval $[a, b]$. If d is between $f'(a)$ and $f'(b)$, then there exists a point $c \in [a, b]$ such that $d = f'(c)$. That is, $f'(x)$ takes on every value between $f'(a)$ and $f'(b)$.*

Since derivatives are defined through the limiting process, the usual laws of limits apply to derivatives, with some twists.

Theorem 2.3. *Let $f(x), g(x)$ be differentiable functions. Let $k \in \mathbb{R}$. Then*

1. $(f(x) + g(x))' = f'(x) + g'(x)$.
2. $(k\, f(x))' = k\, f'(x)$.
3. $(f(x)g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$.
4. $(g(f(x)))' = g'(f(x))\, f'(x)$.
5. $(f(x)/g(x))' = [f'(x)g(x) - f(x)g'(x)]/(g(x)^2)$.
6. $(f^{-1}(x))' = \dfrac{1}{f'(f^{-1}(x))}$.

The fourth one, called the **Chain Rule** is written more suggestively as

$$\frac{dg(f(x))}{dx} = \frac{dg(f(x))}{df(x)} \cdot \frac{df(x)}{dx}.$$

We prove (3), (4) and (6).

Proof: (3)

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}.$$

Taking limit as $h \to 0$ proves (3).

For (4), consider $x = a$; define

$$\phi(h) = \begin{cases} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} & \text{if } f(a+h) \neq f(a) \\ g'(f(a)) & \text{if } f(a+h) = f(a). \end{cases}$$

Now, ϕ is continuous at 0. And $\lim\limits_{h \to 0} \phi(h) = \phi(0) = g'(f(a))$. Also,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h)\, \frac{f(a+h) - f(a)}{h}.$$

Taking limit as $h \to 0$ gives $(g \circ f)'(a) = g'(f(a))\, f'(a)$.

(6) Let $g(x) = f^{-1}(x)$ be the inverse of the function $f(x)$. We assume that $f'(x)$ is not zero in the domain of $f(x)$ which is an interval I. Differentiating $f(g(x)) = x$ we have

$$f'(g(x))g'(x) = 1 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=b} = g'(b) = (f'(x))^{-1}\Big|_{x=f^{-1}(b)}.$$

This proves (6) $\square$

We will not use the formula in (6) explicitly. We will rather differentiate the equations $g(f(x)) = x$ or $f(g(x)) = x$ as appropriate.

For example, differentiating $\sin(\sin^{-1} x) = x$, we obtain $\cos(\sin^{-1} x)(\sin^{-1} x)' = 1$.
Since $\sin(\sin^{-1} x) = x$, we have

$\cos(\sin^{-1} x) = \sqrt{1 - (\sin(\sin^{-1} x))} = \sqrt{1 - x^2}$. Therefore, $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1 - x^2}}$.

The derivative of inverse of a function in Theorem 2.3(6) is also stated in a different way. For a function $y = f(x)$, we may write $x = f^{-1}(y)$ provided that f^{-1} exists. In this case,

$$\frac{dy}{dx}\,\frac{dx}{dy} = 1.$$

This formulation helps in differentiating a function given in parametric form or even given implicitly. If a function $y = f(x)$ is given parametrically, say, as $x = g(t)$ and $y = h(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1}.$$

If $y = f(x)$ is given implicitly by an equation $g(x, y) = 0$, then we differentiate $g(x, y)$ as a function of x, treating y also as a function of x. Since the derivative of 0 is 0, we get an equation in y'. Then we solve this new equation in determining y'.

For example, if $y = f(x)$ is given implicitly by $x^2 + \sin(xy) + y^2 = 0$. Then differentiating the *equation* we get:

$$2x + \cos(xy)(y + xy') + 2y\,y' = 0.$$

Solving this, we obtain

$$y' = -\frac{2x + y\cos(xy)}{2y + \cos(xy)}.$$

Analogous to one-sided continuity at the end-points of a semi-closed or closed interval, we also define one-sided derivatives.

Let the domain of a function $f(x)$ be either $[a, b)$ or $[a, b]$. If the limit

$$\lim_{h\to 0+} \frac{f(a+h) - f(a)}{h}$$

exists, then this limit is the **right hand derivative** of $f(x)$ at $x = a$; and we denote it as $f'_+(a)$.

Similarly, suppose the domain of a function $f(x)$ is either $(a, b]$ or $[a, b]$. The **left hand derivative** of $f(x)$ at $x = b$ is defined as

$$f'_-(b) = \lim_{h\to 0-} \frac{f(b+h) - f(b)}{h}$$

provided this limit exists. Notice that $f'_-(b) = \lim_{h\to 0+} \frac{f(b)-f(b-h)}{h}$.

Also, notice that $f'_+(a)$ need not be the same as $\lim\limits_{h\to 0+} f'(a+h)$, and $f'_-(a)$ need not be equal to $\lim\limits_{h\to 0+} f'(a+h)$. These two equalities say something about the continuity of $f'(x)$ at the end-points.

To extend the vocabulary a bit, suppose $f(x)$ is differentiable on (a, b) and it has right hand derivative at $x = a$, then we say that $f(x)$ is **differentiable on** $[a, b)$.

Similarly, if $f(x)$ is differentiable on (a, b) and it has a left derivative at $x = b$, then $f(x)$ is called **differentiable on** $(a, b]$.

Suppose the domain of $f(x)$ includes $[a, b]$. If $f(x)$ is differentiable on (a, b) and has both right derivative at $x = a$ and left derivative at $x = b$, then we say that $f(x)$ is **differentiable on** $[a, b]$.

Obviously, at an interior point, if both the left hand derivative and the right hand derivative are equal, then that quantity is the derivative at the point.

The absolute value function $|x|$ is differentiable on $[0, \infty)$. It is also differentiable on $(-\infty, 0]$. But it is not differentiable on $(-\infty, \infty)$. Reason: the right hand derivative of $|x|$ at $x = 0$ is not equal to the left hand derivative at $x = 0$.

The function $f(x) = \sqrt{x}$ for $x \geq 0$, is not differentiable at $x = 0$. Reason?

$$\lim_{h \to 0+} \frac{\sqrt{0 + h} - 0}{h} = \infty.$$

The Right hand derivative at $x = 0$ does not exist.

Common functions and their derivatives

Function	Derivative	Where
k	0	
x^r	rx^{r-1}	$r \neq 0$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	$x \neq n\pi/2,\ n \in \mathbb{Z}$
$\sec x$	$\sec x \tan x$	$x \neq n\pi/2,\ n \in \mathbb{Z}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$x \neq n\pi,\ n \in \mathbb{Z}$
$\cot x$	$-\operatorname{cosec}^2 x$	$x \neq n\pi,\ n \in \mathbb{Z}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\|x\| < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	
$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$	$\|x\| > 1$

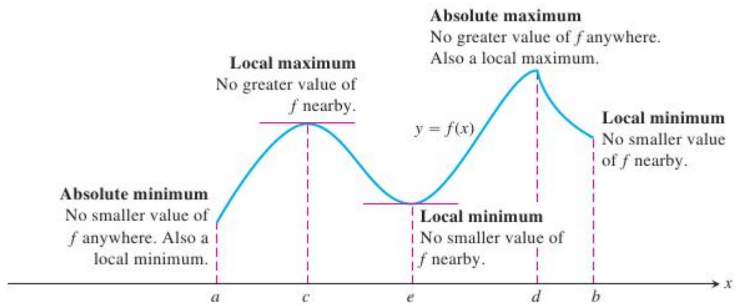
Use the formulas: $\cos^{-1} x = \pi/2 - \sin^{-1} x,\ \cot^{-1} x = \pi/2 - \tan^{-1} x,\ \operatorname{cosec}^{-1} x = \pi/2 - \sec^{-1} x$ for their derivatives.

2.2 Maxima Minima

Let a function $f(x)$ have domain D. The function $f(x)$ has an **absolute maximum** at a point $d \in D$ if $f(x) \leq f(d)$ for every $x \in D$. in such a case, we also say that the point $x = d$ is a **point of absolute maximum** of the function $f(x)$.

Similarly, $f(x)$ has an **absolute minimum** at $b \in D$ if $f(b) \leq f(x)$ for every $x \in D$. In this case, we say that the point $x = b$ is a **point of absolute minimum** of the function $f(x)$.

The points of absolute maximum and absolute minimum are commonly referred to as **absolute extremum points**; and the function is said to have **absolute extrema** at those points.



In the figure, $x = d$ is a point of absolute maximum and $x = a$ is a point of absolute minimum. But the point $x = c$ is also some sort of maximum. It is a point where $f(x)$ achieves its maximum compared with values of $f(x)$ at all the neighboring points.

Let a function $f(x)$ have domain D. The function $f(x)$ has a **local maximum** at a point $d \in D$ if $f(x) \leq f(d)$ for every x in some neighborhood of d contained in D. in such a case, we also say that the point $x = d$ is a **point of local maximum** of the function $f(x)$.

Similarly, $f(x)$ has an **local minimum** at $b \in D$ if $f(b) \leq f(x)$ for every x in some neighborhood of b contained in D. In this case, we say that the point $x = b$ is a **point of local minimum** of the function $f(x)$.

The points of local maximum and local minimum are commonly referred to as **local extremum points**; and the function is said to have **local extrema** at those points.

As we know from Theorem 1.5, a continuous function defined on a closed interval achieves its extremum points. These are the absolute extrema. If differentiability of $f(x)$ is assumed, then something can be said about local extremum points.

Theorem 2.4. *Let $f(x)$ with domain D have a local extremum at $x = c$. Let c be an interior point of D. Suppose that $f(x)$ is differentiable at $x = c$. Then $f'(c) = 0$.*

Proof: Suppose that $f(x)$ has a local minimum at $x = c$. There exists a neighborhood $(c - \delta, c + \delta)$ of c such that for every $x \in (c - \delta, c + \delta)$, we have $f(c) \leq f(x)$. Now, for all such points x we obtain:

$$\frac{f(x) - f(c)}{x - c} \leq 0 \text{ for } x < c, \quad \text{and} \quad \frac{f(x) - f(c)}{x - c} \geq 0 \text{ for } x > c.$$

Taking the limit as $x \to c$, the first inequality says that $f'(c) \leq 0$ and the second inequality says that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

When $f(x)$ has a local maximum, the inequalities above get changed. Then also we conclude that $f'(c) = 0$. □

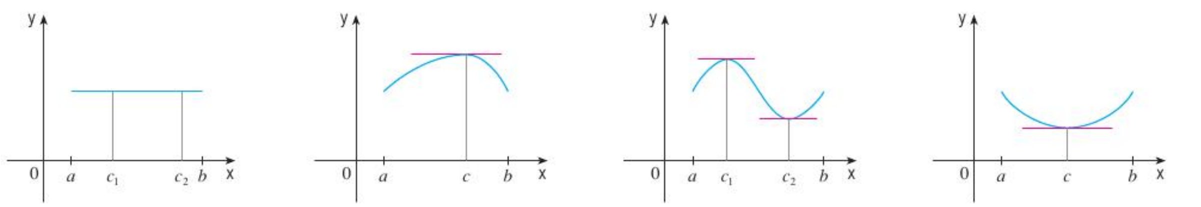
Notice that there can be points where a function can have its *extreme value* where $f(x)$ is not differentiable or such a point is not an interior point. In such scenario, Theorem 1.5 is not applicable.

Let $f(x)$ have domain D. A point $c \in D$ is called a **critical point** of $f(x)$ if c is not an interior point of D, or if $f(x)$ is not differentiable at $x = c$, or if $f'(c) = 0$.

Then Theorem 2.4 says that if $f(x)$ has an extremum at $x = c$, then c is a critical point of $f(x)$.

A critical point need not be an extremum point. For example, the function $f(x) = x^3$ defined on $\mathbb{R}$ has a critical point at $x = 0$ but this function does not achieve an extreme value at $x = 0$. Reason: to the left of $x = 0$, the values of x^3 are less than $0 = f(0)$ and to the right of $x = 0$, the values of x^3 are greater than 0. When is a critical point an extremum point? To answer this question we discuss some results which are geometrically appealing.

Theorem 2.5. (Rolle's Theorem) *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous,* $f(x)$ *is differentiable on* (a, b)*, and* $f(a) = f(b)$*. Then* $f'(c) = 0$ *for some* $c \in (a, b)$*.*



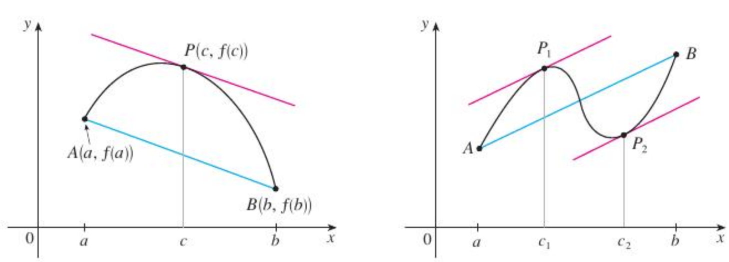
Proof: If $f(x)$ is a constant function, then there is nothing to prove. Assume that $f(x)$ is not a constant function. By Theorem 1.5, $f(x)$ has a maximum and a minimum in $[a, b]$. Since $f(x)$ is not a constant, at least one of these extreme values is different from $f(a)$, and also different from $f(b)$. Suppose such an extreme value is achieved at $x = c$. Then $c \in (a, b)$, is an interior point of $[a, b]$. By Theorem 2.4, $f'(c) = 0$. $\square$

Example 2.5. Show that $x^3 + ax + b$ has a unique real root if $a > 0$.

Since $f(x) = x^3 + ax + b$ is a polynomial of odd degree, it has a real root. Suppose $c < d$ are two ral roots of $f(x)$. Then by Rolle's theorem, we have $r \in (c, d)$ such that $f'(r) = 0$. But $f'(x) = 3x^2 + a > 0$.

Rolle's theorem can be generalized.

Theorem 2.6. (Mean value Theorem) *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous and* $f(x)$ *is differentiable on* (a, b)*. Then there exists* $c \in (a, b)$ *such that* $f(b) - f(a) = f'(c)(b - a)$*.*



Proof: Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$.

Then $g(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = 0 = g(b)$. By Rolle's theorem, we have $c \in (a, b)$ such that $g'(c) = 0$. That is, $f(b) - f(a) = f'(c)(b - a)$. $\square$

The conclusion of the Mean value theorem is stated in another way:

$$f(a+h) = f(a) + hf'(a+\theta h) \quad \text{for some } \theta \in (0,1).$$

It follows from Theorem 2.6 by writing $b = a + h$.

We know that the derivative of a constant function is zero. Its converse follows from the Mean value theorem provided the domain is an interval. We do not consider degenerate intervals of the form $[a, a]$ etc.

Theorem 2.7. *Let I be an interval containing at least two points. Let $f : I \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for each $x \in I$, then $f(x)$ is a constant function.*

Proof: Let $s < t \in I$. Then $f(x)$ is continuous on $[s, t]$ and is differentiable on (s, t). By the Mean value theorem, $f(t) - f(s) = f'(c)(t - s)$. If $f'(x) = 0$ for all $x \in I$, then $f'(c) = 0$; consequently, $f(b) = f(a)$. □

It thus follows that functions whose derivatives are the same can only differ by a constant.

In addition to the assumptions of the Mean value theorem if $m \le f'(x) \le M$ for some real numbers m and M, then It follows that

$$\boxed{m(b-a) \le f(x) \le M(b-a).}$$

This is sometimes called as the **Mean value inequality.**

MVT helps in determining whether a function is increasing or decreasing on an interval, and consequently it leads to a test for maximum/minimum.

Let $f(x)$ be a function defined on an interval I.
We say that $f(x)$ is **increasing on** I if for all $s < t \in I,\ f(s) < f(t)$.
Similarly, we say that $f(x)$ is **decreasing on** I if for all $s < t \in I,\ f(s) > f(t)$.
A **monotonic** function on I is one which either increases on I or decreases on I.

For example, $f(x) = x^2$ increases on $[0, \infty)$ and decreases on $(-\infty, 0]$. It is monotonic on both the intervals $(-\infty, 0]$ and on $[0, \infty)$. But it is not monotonic on $\mathbb{R}$, since it neither increases on $\mathbb{R}$ nor decreases on $\mathbb{R}$.

Theorem 2.8. *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) > 0$ on (a, b), then $f(x)$ is increasing on $[a, b]$.*

2. *If $f'(x) < 0$ on (a, b), then $f(x)$ is decreasing on $[a, b]$.*

Proof: Let $s < t \in [a, b]$. By MVT, $f(t) - f(s) = f'(c)(t - s)$ for some $c \in (s, t)$. Since $f'(c) > 0$, $f(t) > f(s)$. Therefore, $f(x)$ is increasing on $[a, b]$. This proves (1). Proof of (2) is similar. □

The point $x = c$ is a point of local maximum for the function $f(x)$ iff in the left neighborhood of c, $f(x)$ is increasing and in the right neighborhood of c, $f(x)$ is decreasing. The increasing and decreasing nature of $f(x)$ are determined from the sign of $f'(x)$.

If $f'(c) = 0,\ f'(x) > 0$ on an interval to the immediate left of $x = c$, and $f'(x) < 0$ on an interval to the immediate right of $x = c$, then we say that $f'(x)$ **changes sign from** $+$ **to** $-$ **at** $x = c$.

Similarly, $f'(x)$ **changes sign from** $-$ **to** $+$ means that $f'(c) = 0$, $f'(x) < 0$ on an interval to the immediate left of $x = c$, and $f'(x) > 0$ on an interval to the immediate right of $x = c$.

Thus the following are true:

If $f'(x)$ changes sign from $-$ to $+$ at $x = c$ then $f(x)$ has a local minimum.
If $f'(x)$ changes sign from $+$ to $-$ at $x = c$ then $f(x)$ has a local maximum.

Look at $f'(x)$ as a function and we can take its derivative once more. Suppose $f''(c) < 0$. Then $f'(x)$ is decreasing on a neighborhood of $x = c$. Since $f'(c) = 0$, we see that this decrease must be from $+$ to $-$. In that case, $x = c$ is a point of local maximum.

Analogous arguments hold for a point of local minimum where $f'(x)$ changes sign from $-$ to $+$ at $x = c$.

2.3 Tests for Maxima-Minima

We summarize the discussion.

Test for Local Extrema:

Let c be an interior point of the domain of $f(x)$ with $f'(c) = 0$.

$f'(x)$ changes sign from $+$ to $-$ at $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.

If $f''(c) < 0$, then $x = c$ is a point of local maximum of $f(x)$.

$f'(x)$ changes sign from $-$ to $+$ at $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

If $f''(c) > 0$, then $x = c$ is a point of local minimum of $f(x)$.

Let $x = c$ be a left end-point of the domain of $f(x)$.

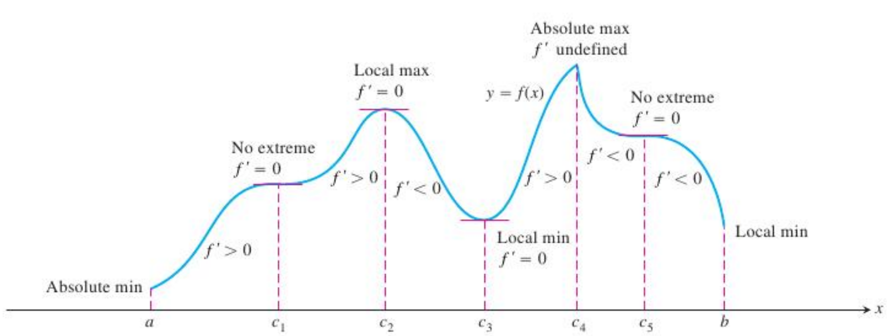
$f'(x) < 0$ on the immediate right of $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.

$f'(x) > 0$ on the immediate right of $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

Let $x = c$ be a right end-point of the domain of $f(x)$.

$f'(x) > 0$ on the immediate left of $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.

$f'(x) < 0$ on the left of $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.



Example 2.6. Find all points of local extrema of the function $f(x) = x^3 - 3x^2 - 24x + 5$.

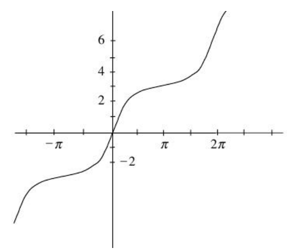
The domain of $f(x)$ is $-(\infty, \infty)$. Here, $f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4)$.
The critical points are $x = -2$ and $x = 4$. Now, $f''(x) = 6x - 6$.

Then $f''(-2) = -18 < 0$ says that $x = -2$ is a local maximum point and the local maximum value of $f(x)$ at $x = -2$ is $f(-2) = 33$.

$f''(4) = 18 > 0$. Then $x = 4$ is a local minimum point and the local minimum value of $f(x)$ at $x = 4$ is $f(4) = 75$.

Example 2.7. Find all local maxima/minima for the function $g(x) = x + \sin x$.

$g'(x) = 1 + \cos x$ vanishes for $x = (2n+1)\pi$, where $n \in \mathbb{Z}$. Now, $g''(x) = -\sin x$ is also zero at all these critical points. So, it does not help. We look for change of sign of $g'(x)$. Now, $g'(x) > 0$ for all values of x except the critical points. So, there is no change in sign of $g'(x)$ at the critical points. Therefore, $g(x)$ has no local maxima/minima.



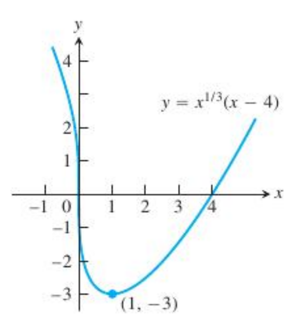
Example 2.8. Find the critical points of $f(x) = x^{1/3}(x-4)$ defined on the interval $[-1, 2]$. Identify the intervals in which $f(x)$ is increasing or decreasing. Find the absolute and local extreme points and values of $f(x)$.

$$f'(x) = \frac{1}{3}x^{-2/3}(x-4) + x^{1/3}(1) = \frac{x - 4 + 3x}{3x^{2/3}} = \frac{4(x-1)}{3x^{2/3}}.$$

It is zero at $x = 1$ and is undefined at $x = 0$. The end-points of the domain of $f(x)$ are -1 and 2. Thus $x = -1, 0, 1, 2$ are its critical points.

(a) For absolute extrema: $f(-1) = 5,\ f(0) = 0,\ f(1) = -3,\ f(2) = -2^{4/3}$.
Comparing these values, we see that the absolute minimum value of $f(x)$ is -3 and the point of absolute minimum is $x = 1$.
The point of absolute maximum is $x = -1$ and the absolute maximum value is 5.

(b) For the increasing and decreasing nature of $f(x)$, look for the signs of $f'(x)$.
For $x < 0$, $f'(x)$ is $-$ve. Thus $f(x)$ is decreasing on $[-1, 0]$.
For $0 < x < 1$, $f'(x)$ is $-$ve. Thus $f(x)$ is decreasing on $[0, 1]$.
For $x > 1$, $f'(x)$ is $+$ve. Thus $f(x)$ is increasing on $[1, 2]$.



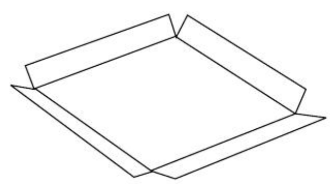
(c) The change of sign of $f'(x)$ says that $f(x)$ does not have a local extremum at $x = 0$.
And $f(x)$ has a local minimum at $x = 1$. The local minimum value at $x = 1$ is $f(1) = 3$.
At the end-points, since $f(x)$ is decreasing on $[-1, 0]$, $x = -1$ is a local maximum point, the local maximum value is $f(-1) = 5$. Similarly, $x = 2$ is also a local maximum point, and the local maximum value at $x = 2$ is $f(2) = -2^{4/3}$.

In summary,

$x = -1$ is a local maximum point and also an absolute maximum point.
$x = 0$ is not an extremum point.
$x = 1$ is a local minimum point as well as an absolute minimum point.
$x = 2$ is a local maximum point.

Notice that $\lim_{x \to 0} f'(x) = -\infty$. Thus $y = f(x)$ has a vertical tangent at the origin.

Example 2.9. A box is to be made from a sheet of cardboard that measures 12×12. The construction will be achieved by cutting a square from each corner of the sheet and then folding up the sides. What is the box of greatest volume that can be constructed in this fashion?



Let x be the side length of the squares that are to be cut from the sheet of cardboard. Then the side length of the resulting box will be $12 - 2x$; the height of the box will be x. As a result, the volume of the box will be

$$f(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3.$$

This function is to be maximized. The critical points are

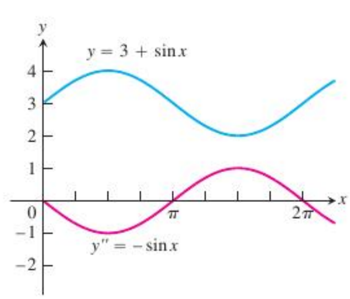
$$f'(x) = 0 \Rightarrow 144 - 96x - 12x^2 = 0 \Rightarrow x = 2, 6.$$

Now, $f''(x) = -96 + 24x$. Since $0 \le x \le 6$; comparing $f(0), f(2), f(6)$, we see that $f(2)$ is maximum.

Also, notice that $f''(2) = -48 < 0$ and $f''(6) = 48 > 0$. Therefore, $f(x)$ has a local maximum at $x = 2$ and a local minimum at $x = 6$.

The greatest volume is $f(2) = 2(12 - 4)^2 = 128$.

The second derivatives help us to know the shape of a curve.



Here, the graph of the function $f(x) = 3 + \sin x$ is concave down on $(0, \pi)$ and is concave up from $(\pi, 2\pi)$. The concavity changes at $x = \pi$.

The graph of a function $y = f(x)$ is **concave up** on an open interval I if $f'(x)$ is increasing on I. The graph of $y = f(x)$ is **concave down** on an open interval I if $f'(x)$ is decreasing on I.
A **point of inflection** is a point where $y = f(x)$ has a tangent and the concavity changes.

Now, $f'(x)$ increases on I when $f''(x) > 0$ on I. Similarly, $f''(x) < 0$ on I implies that $f'(x)$ decreases on I. We thus have the following:

Second derivative test for concavity:

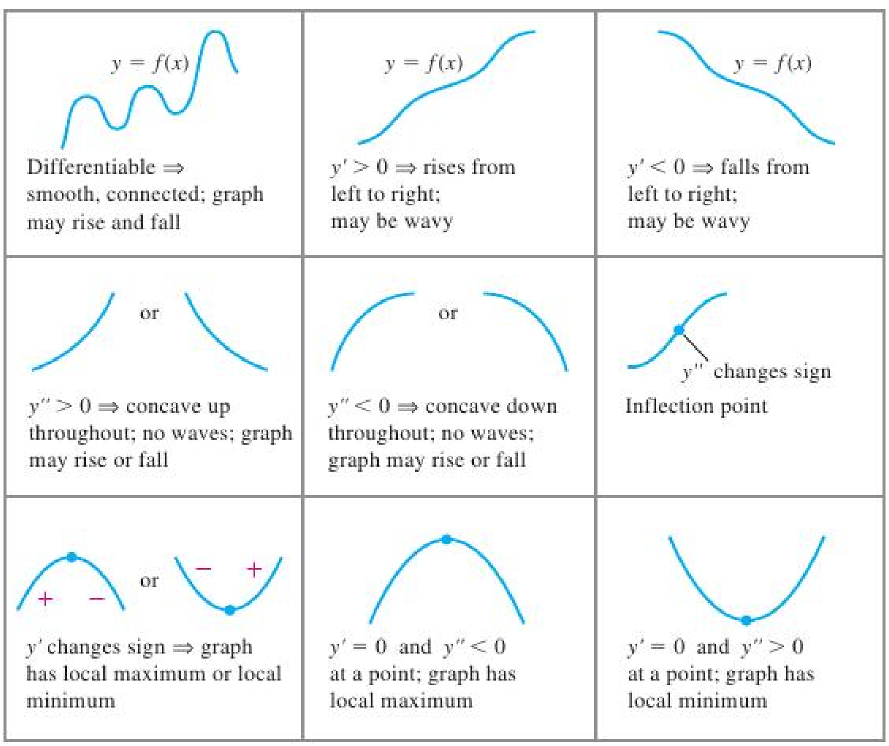
Let $y = f(x)$ be twice differentiable on an interval I.

If $f''(x) > 0$ on I, then the graph of $y = f(x)$ is concave up over I.

If $f''(x) < 0$ on I, then the graph of $y = f(x)$ is concave down over I.

If $f''(x)$ is positive on one side of $x = c$ and negative on the other side, then the point $(c, f(c))$ on the graph of $y = f(x)$ is a point of inflection.

At a point of inflection, either f'' does not exist or it vanishes. Notice that in the latter case, $f'(x)$ has a local extremum.
Here is a summary of how the derivatives give information regarding the shape of a curve:

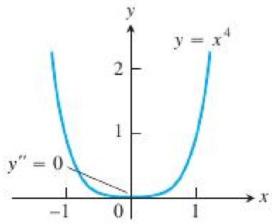


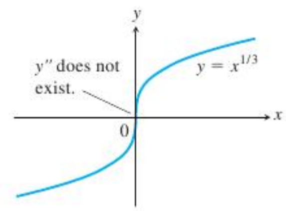
Example 2.10.
(a) The curve $y = x^2$ is concave up everywhere. Because, $(x^2)'' = 2 > 0$ at all points.

(b) The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $f''(x) = 6x < 0$. It is concave up on $(0, \infty)$, where $f''(x) = 6x > 0$. The point $x = 0$ is its inflection point.

(c) The curve $y = f(x) = x^4$ has $f''(x) = 12x^2 = 0$ at $x = 0$. But the point $(0, 0)$ is not a point of inflection since $f''(x)$ does not change sign at $x = 0$.

(d) For the curve $y = f(x) = x^{1/3}$, $f''(x) = -(2/9)x^{-5/3}$ for $x \neq 0$. And $f''(x)$ does not exist at $x = 0$. However, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. That is, $f''(x)$ changes sign at $x = 0$. Therefore, the point $(0, 0)$ is a point of inflection.





Example 2.11. Give a rough sketch of the graph of $y = f(x) = \dfrac{(1+x)^2}{1+x^2}$.

1. The domain of $f(x)$ is $(-\infty, \infty)$. There are no symmetries.

2. $f(x) = \dfrac{(1+x)^2}{1+x^2}$, $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2}$, $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$.

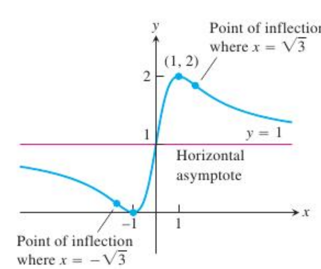
3. The critical points are $x = -1,\ 1$.
   At $x = -1$, $f''(-1) = 1 > 0$. So, $y = f(x)$ has a local minimum at $x = -1$.
   At $x = 1$, $f''(1) = -1 < 0$. So, $y = f(x)$ has a local maximum at $x = 1$.

4. On the interval $(-\infty, -1)$, $f'(x) < 0$, so, $f(x)$ is decreasing.
   On the interval $(-1, 1)$, $f'(x) > 0$, so, $f(x)$ is increasing.
   On the interval $(1, \infty)$, $f'(x) < 0$, so, $f(x)$ is decreasing.

5. $f''(x) = 0$ for $x = -\sqrt{3},\ 0, \sqrt{3}$. On the interval $(-\infty, -\sqrt{3})$, $f''(x) < 0$. On the interval $(-\sqrt{3}, 0)$, $f''(x) > 0$. On the interval $(0, \sqrt{3})$, $f''(x) < 0$. And on the interval $(\sqrt{3}, \infty)$, $f''(x) > 0$. Thus, each of these points is a point of inflection. Moreover, on $(-\infty, -\sqrt{3})$, $y = f(x)$ is concave down; on $(-\sqrt{3}, 0)$, it is concave up; on $(0, \sqrt{3})$ it is concave down, and on $(\sqrt{3}, \infty)$, it is concave up.

6. Rewrite $f(x) = \dfrac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}$. Thus $f(x)$ decreases to 1 as $x \to \infty$ and $f(x)$ increases to 1 as $x \to -\infty$. Thus $y = 1$ is a horizontal asymptote to $y = f(x)$.

   For every $c \in \mathbb{R}$, limit of $f(x)$ as $x \to c$ remains finite. Thus there are no vertical asymptotes. Since the curve decreases to reach the point $(-1, 0)$ and then increases up to the point $(1, 2)$, and then decreases towards its asymptote, the local minimum at $x = -1$ is an absolute minimum. And also the local maximum at $x = 1$ is an absolute maximum.



7. The curve thus looks like :

2.4 L' Hospital's Rule

Throughout, we have used the MVT for deciding about the increasing and decreasing nature of functions. The MVT can be generalized a bit.

Theorem 2.9. (Cauchy Mean Value Theorem) *Let $f(x)$ and $g(x)$ be continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0$ on (a,b), then there exists $c \in (a,b)$ such that*

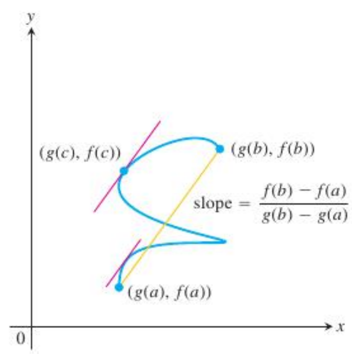
$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof: If $g(b)-g(a) = 0$, then by MVT, $g'(d) = 0$ for some $d \in (a,b)$, contradicting our assumption. Hence $g(b)-g(a) \neq 0$. Define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} \{g(x)-g(a)\}.$$

Now, $h(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Also, $h(a) = h(b)$. By Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. That is,

$$f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0. \qquad \square$$



Theorem 2.10. (L'Hospital's Rule) *Let $f(x)$ and $g(x)$ be differentiable on a neighborhood of a point $x = a$. Suppose $f(a) = g(a) = 0$ but $g(x) \neq 0$, $g'(x) \neq 0$ in the deleted neighborhood of $x = a$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.*

Proof: First, consider the interval $[a,x]$ to the right of a, which is contained in the given neighborhood of a. Both f and g are continuous on $[a,x]$, and differentiable on (a,x). Also, $g' \neq 0$ on (a,x). By Cauchy MVT, there exists a point $\theta \in (a,x)$ such that

$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)}.$$

When $x \to a$, we have $\theta \to a$. Therefore,

$$\lim_{\theta\to a+} \frac{f'(\theta)}{g'(\theta)} = \lim_{x\to a+} \frac{f(x)}{g(x)}.$$

Similarly, considering an interval $[t, a]$ to the left of the point a, we conclude that

$$\lim_{\tau \to a-} \frac{f'(\tau)}{g'(\tau)} = \lim_{x \to a-} \frac{f(x)}{g(x)}.$$

Combining the two above completes the proof. $\square$

Of course, when only one sided limit is relevant, our proof shows that L' Hospital's Rule is still applicable. The method can also be used for evaluating limits in the indeterminate forms such as $\frac{\pm\infty}{\pm\infty}$, $\pm\infty \cdot 0$, and $\infty - \infty$.

Example 2.12. (a) $\displaystyle\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ in $\frac{0}{0}$ form.

$\displaystyle= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$ in $\frac{0}{0}$ form.

$\displaystyle= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = \frac{1}{8}.$

(b) $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{1 - \cos x}{x} \frac{1}{1+x} = \lim_{x \to 0} \frac{\sin x}{1} = 0.$

But $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$

What is wrong? The second limit in the second calculation is not in $\frac{0}{0}$ form. In fact, the limit there is equal to 0 as it should be.

(c) $\displaystyle\lim_{x \to 0+} \frac{\sin x}{x^2} = \lim_{x \to 0+} \frac{\cos x}{2x} = \infty.$

(d) $\displaystyle\lim_{x \to 0-} \frac{\sin x}{x^2} = \lim_{x \to 0-} \frac{\cos x}{2x} = -\infty.$

(e) For the limit $\displaystyle\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$, notice that both the numerator and denominator are discontinuous at $x = \pi/2$. We consider only one-sided limits.

$\displaystyle\lim_{x \to (\pi/2)-} \frac{\sec x}{1 + \tan x} = \lim_{x \to (\pi/2)-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)-} \sin x = 1.$ Similarly,

$\displaystyle\lim_{x \to (\pi/2)+} \frac{\sec x}{1 + \tan x} = 1.$ Hence $\displaystyle\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x} = 1.$

(f) $\displaystyle\lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x \to \infty} \frac{1 - 4x}{6x + 5} = \lim_{x \to \infty} \frac{-4}{6} = -\frac{2}{3}.$ $\frac{\pm\infty}{\pm\infty}$

(g) $\displaystyle\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{t \to 0+} \frac{1}{t} \sin t = \lim_{t \to 0+} \cos t = 1.$ $\infty \cdot 0$

(h) For the limit $\displaystyle\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$, we see that as $x \to 0+$, it is in $\infty - \infty$ form and as $x \to 0-$, it is in $-\infty + \infty$ form. We need $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. So, we rearrange.

$\displaystyle\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + \cos x} = \lim_{x \to 0} \frac{\sin x}{\cos x - \sin x} = 0.$

Let c be an arbitrary point in the domain of a differentiable function $y = f(x)$. Suppose Δx

represents the change in x near c. Then the change in y near c is $f(c+\Delta x) - f(c)$. The derivative $f'(x)$ at c is $f'(c) = \lim_{\Delta x \to 0} \dfrac{f(c+\Delta x) - f(c)}{\Delta x}$. Therefore, near c, the change in y is approximately equal to $f'(c)\Delta x$. In abstract terms, we express this as differentials.

The change dx at any arbitrary point x is an independent variable called the **differential of** x. It is the same as Δx. The corresponding change in y is a dependent variable and is called the **differential of** y. The differential of y is denoted by dy. We define this differential of y as

$$dy = f'(x)\,dx.$$

Then the differential dy is an approximation of the increment Δy in y near c. Notice that in this view, the differentials give a proper notation in viewing $\frac{dy}{dx}$ as a ratio.

This helps. For example, the derivative of a product

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

can be stated in terms of differentials as $d(fg) = df\,g + f\,dg$.

For $y = f(x)$, the differential $dy = f'(c)dx$ gives an approximation to the increment in y near c. Thus

$$\Delta y = f(c+dx) - f(c) \approx dy \Rightarrow f(c+dx) \approx f(c) + dy.$$

It helps in approximately computing the function value at a neighboring point from the value at that point and the differential near the point.

Example 2.13. The radius of a circle of radius 10 increases to 10.1. Estimate the change in area.

Since $y = f(x) = \pi x^2$, the differential formula gives the estimate as

$$\Delta y \approx dy = 2\pi x\,dx = \pi(10)(0.2) = 2\pi.$$

Of course, the change in area is $\pi\{(10.1)^2 - 10^2\} = \pi(0.1 \times 20.1) = 2.01\,\pi$.

2.5 Curvature

The curvature of a curve at a point quantifies how much the curve is bent at that point. For example, consider two circles, C_1 of radius 1 and C_2 of radius 2. Intuitively, the bent on any circle is independent of the point we choose on it. So, choose a point P_1 on C_1 and a point P_2 on C_2. Now, the bent of C_1 at P_1 is certainly stiffer or more than that at P_2 on C_2. Thus the curvature of a circle may be taken as something that varies inversely with the radius. We also use the term *radius of curvature* for the reciprocal of the curvature.

To compare the curvatures of these circles at those points, let us take another pair of points, say Q_1 near P_1 on C_1 and Q_2 near P_2 on C_2. The angle between the tangents at P_1 and Q_1 is more than the angle between the tangents to C_2 at P_2 and Q_2. It shows that the rate of change of the angle between a tangent and the x-axis with respect to the length of the curve is faster when the corresponding angles between the tangents is more. We thus define the curvature as follows.

Curvature of a curve at a point P on the curve is the rate of change in direction with respect to the arc length of the curve at P. Writing κ for the curvature, our definition yields the formula

$$\kappa = \left|\frac{d\psi}{ds}\right|$$

where s is the arc length and ψ is the angle between the tangent to the curve at that point and the positive x-axis. Further, the **radius of curvature**, denoted by ρ is given by

$$\rho = \frac{1}{\kappa} = \left|\frac{ds}{d\psi}\right|.$$

We now derive the formulas for calculating curvature in cartesian coordinates.

As we know,

$$\tan\psi = \frac{dy}{dx}, \quad ds = \sqrt{dx^2 + dy^2}, \quad \frac{dx}{ds} = \cos\psi.$$

Then $\sec^2\psi \, \dfrac{d\psi}{dx} = \dfrac{d^2y}{dx^2}$. Consequently,

$$\kappa = \left|\frac{d\psi}{ds}\right| = \left|\frac{d\psi}{dx}\,\frac{dx}{ds}\right| = \left|\frac{d^2y}{dx^2}\cos^3 x\right| = \left|y''\,[1+(y')^2]^{-3/2}\right|.$$

$$\rho = \frac{1}{\kappa} = \left|\frac{[1+(y')^2]^{3/2}}{y''}\right|.$$

In these formulas, we should take care that at the point where the curvature or radius of curvature are computed, the quantities in the denominator are not 0.

Similarly, in polar coordinates, the formula for the radius of curvature is given by

$$\rho = \frac{1}{\kappa} = \left|\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right]\right|.$$

Notice that the curvature of a straight line is 0 at every point since ψ, the angle between any tangent to it (tangent is the straight line itself) and the x-axis never changes.

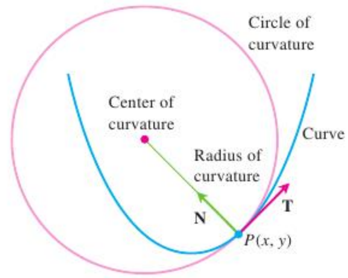
Example 2.14. Find the radius of curvature for a circle of radius a, say, centred at the origin.

In polar coordinates, the circle is given by $r = a$. We have

$$\frac{dr}{d\theta} = 0, \; \frac{d^2r}{d\theta^2} = 0.$$

$$\rho = \frac{1}{\kappa} = \left|[a^2]^{3/2}[a^2]^{-1}\right| = \left|\frac{a^3}{a^2}\right| = a.$$

If C is a curve having radius of curvature ρ at a point P on the curve, then imagine a circle of radius ρ touching the curve at P and lying on the convex side of the curve.



Then the centre of such a circle , now called an **osculating circle** is called the **centre of curvature** of the curve at that point. Clearly, the centre of curvature of a circle is the centre of the circle, whatever point P on the circle we choose.

If (a, b) is the centre of curvature of a curve $y = y(x)$ at the point (x, y), then

$$a = x - \rho \sin\psi = x - \frac{y'[1 + (y')^2]}{y''}, \quad b = y + \rho\cos\psi = y + \frac{1 + y'}{y''}.$$

Example 2.15. Find the radius and centre of curvature for the parabola $y^2 = 4ax$.

Let (x, y) be any point on the parabola $y^2 = 4ax$. at P, we have

$$\frac{dy}{dx} = \frac{2a}{y}, \quad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}.$$

$$\rho = \Big[1 + \frac{4a^2}{y^3}\Big]^{3/2}\Big(-\frac{4a^2}{y^3}\Big)^{-1} = \frac{2(x+a)^{3/2}}{\sqrt{a}}.$$

If (a, b) is the centre of curvature at (x, y), then

$$a = x - \frac{y'[1 + (y')^2]}{y''} = x + \frac{4a^2 - y^2}{2a}, \quad b = y + \frac{1 + y'}{y''} = y - \frac{y^3 + 2ay}{4a}.$$

2.6 Review Problems

Problem 1: What is the derivative of $f(x) = \sqrt{|x|}$?

For $x > 0$, $\sqrt{|x|} = \sqrt{x}$. So, $f'(x) = \frac{1}{2\sqrt{x}}$.
For $x < 0$, $\sqrt{|x|} = \sqrt{-x}$. So, $f'(x) = \frac{-1}{2\sqrt{-x}}$.

For the derivative of $f(x)$ at $x = 0$, $\dfrac{f(h) - f(0)}{h} = \begin{cases} 1/\sqrt{h} & \text{if } h > 0 \\ 1/\sqrt{-h} & \text{if } h < 0. \end{cases}$

So, $\displaystyle\lim_{h\to 0+} \frac{f(h) - f(0)}{h} = \infty$ and $\displaystyle\lim_{h\to 0-} \frac{f(h) - f(0)}{h} = -\infty$.

Therefore, the derivative of $f(x)$ at $x = 0$ does not exist. Even the left hand and right hand derivatives do not exist at $x = 0$.

Problem 2: Find $f'(x)$ if $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0. \end{cases}$

If $x > 0$, then $f'(x) = 2x$. If $x < 0$, then $f'(x) = -2x$. For $x = 0$,

$$\lim_{h\to 0+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0+} \frac{h^2}{h} = 0. \quad \lim_{h\to 0-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0-} \frac{-h^2}{h} = 0.$$

Thus $f'(0) = 0$. That is, $f'(x) = 2|x|$.

Problem 3: Find the derivative of $y = f(x)$ where x and y are given parametrically:
(a) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le \pi$. Upper semi-circle $y = \sqrt{a^2 - x^2}$.
(b) $x = a\cos t,\ y = b\sin t,\ 0 \le t \le \pi$. Upper part of an ellipse $y = b\sqrt{1 - x^2/a^2}$.
(c) $x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi$. The cycloid.
(d) $x = a\cos^3 t,\ y = a\sin^3 t,\ 0 \le t \le 2\pi$. The asteroid $x^{2/3} + y^{2/3} = a^{2/3}$.

If a function $y = f(x)$ is given parametrically by $x = \phi(t),\ y = \psi(t)$, then assuming that $\phi'(t) \neq 0$, we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\psi'(t)}{\phi'(t)}.$$

(a) $\dfrac{dy}{dx} = \dfrac{d(a\sin t)}{dt} \Big/ \dfrac{d(a\cos t)}{dt} = \dfrac{a\cos t}{-a\sin t} = -\dfrac{x}{y}.$

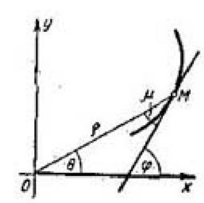
(b) $\dfrac{dy}{dx} = \dfrac{d(b\sin t)}{dt} \Big/ \dfrac{d(a\cos t)}{dt} = \dfrac{b\cos t}{-a\sin t} = -\dfrac{b^2}{a^2}\dfrac{a\cos t}{b\sin t} = \dfrac{b^2 x}{a^2 y}.$

(c) $\dfrac{dy}{dx} = \dfrac{d(a(t-\sin t))}{dt} \Big/ \dfrac{d(a(1-\cos t))}{dt} = \dfrac{a(1-\cos t)}{\sin t} = \cot\left(\dfrac{t}{2}\right).$

(d) $\dfrac{dy}{dx} = \dfrac{d(a\sin^3 t)}{dt} \Big/ \dfrac{d(a\cos^3 t)}{dt} = \dfrac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\dfrac{\sin t}{\cos t} = -\left(\dfrac{y}{x}\right)^{1/3}.$

Problem 4: If a curve is given in polar coordinates, show that the derivative of the radius vector with respect to the polar angle is equal to the radius vector multiplied by the cotangent of the angle between the radius vector and the tangent to the curve at the given point.

In polar coordinates, $x = \rho\cos\theta$, $y = \rho\sin\theta$, where the curve is given by $\rho = \rho(\theta)$. Let ϕ be the angle formed by the tangent to the curve at some point $M(\rho, \theta)$ with the positive x-axis. Then



$$\tan\phi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\rho'(\theta)\sin\theta + \rho\cos\theta}{\rho'(\theta)\cos\theta - \rho\sin\theta}.$$

Let μ be the angle between the radius vector ρ and the tangent at that point M. Then $\mu = \phi - \theta$, and then

$$\tan\mu = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} = \frac{(\rho'\sin\theta + \rho\cos\theta)\cos\theta - (\rho'\cos\theta - \rho\sin\theta)\sin\theta}{(\rho'\cos\theta - \rho\sin\theta)\cos\theta + (\rho'\sin\theta + \rho\cos\theta)\sin\theta} = \frac{\rho}{\rho'}.$$

This gives $\rho'(\theta)\theta = \rho(\theta)\cot\mu$.

Problem 5: Consider the function $f(x) = 1 - x^{2/3}$ defined on the interval $[-1, 1]$. Its derivative is $f'(x) = -\frac{2}{3}x^{-1/3}$. Notice that $f(-1) = f(1) = 0$ but $f'(x) \neq 0$ on $[-1, 1]$. Why?

Reason: Rolle's theorem is not applicable because, at $x = 0$, $f(x)$ is not differentiable.
Also, notice that at $x = 0$, $f(x)$ attains its maximum value 1.

Problem 6: Find the maximum and minimum values of $(1 - x^2)^{-1}$ in $(0, 1)$.

Let $f(x) = (1 - x^2)^{-1}$. Now, $f'(x) = \dfrac{2x}{(1 - x^2)^2}$. It is 0 only for $x = 0$.

When x is close to 1, $f(x)$ becomes arbitrarily large. So, $f(x)$ has no maximum in $(0, 1)$.
When $x < 0$, $f'(x) < 0$ and when $x > 0$, $f'(x) > 0$. That is, at $x = 0$, $f'(x)$ changes sign from $-$ to $+$. Therefore, $f(0) = 1$ is a minimum value of $f(x)$. This is the only minimum value since $f'(x)$ is never 0 for any other point.

Problem 7: Examine $f(x) = 2\sin x + \cos 2x$ for maxima/minima for $x \in \mathbb{R}$.

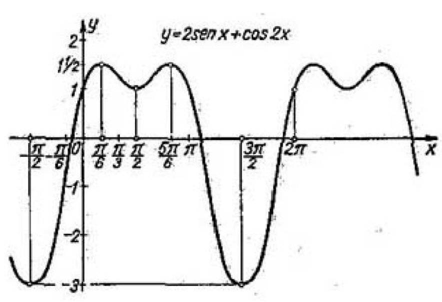
Since $f(x)$ has period 2π, we consider it on $(0, 2\pi)$. Now,

$$f'(x) = 2\cos x - 2\sin 2x = 2(\cos x - 2\sin x \cos x) = 2\cos x(1 - 2\sin x).$$

The critical points are $a = \pi/6,\ b = \pi/2,\ c = 5\pi/6,\ d = 3\pi/2$.

$$f''(x) = -2\sin x - 4\cos 2x.$$

$f''(\pi/6) = -2(1/2) - 4(1/2) = -3 < 0$. Hence $f(x)$ has a local maximum at $x = \pi/6$ and this maximum value is $f(\pi/6) = 3/2$.
$f''(\pi/2) = -2 \cdot 1 + 4 \cdot 1 = 2 > 0$. Thus $f(x)$ has a local minimum at $x = \pi/2$ and this minimum value is $f(\pi/2) = 1$.
$f''(5\pi/6) = -2(1/2) - 4(1/2) = -3 < 0$. So, $f(x)$ has a local maximum at $x = 5\pi/6)$ and this maximum value is $f(5\pi/6) = 3/2$.
$f''(3\pi/2) = -2(-1) - 4(-1) = 6 > 0$. Then $f(x)$ has a local minimum at $x = 3\pi/2$ and this minimum value is $f(3\pi/2) = -3$.
What about the points $x = 0$ and $x = 2\pi$? Due to periodicity we need to consider $x = 0$ only.
$f'(0) = 2 \neq 0$. Hence $x = 0$ is not a point of maxima/minima.
In summary, the points of local maxima are $2n\pi + \pi/6$ and $2n\pi + 5\pi/6$ for $n \in \mathbb{Z}$.
The points of local minima are $2n\pi + \pi/2$ and $2n\pi + 3\pi/2$ for $n \in \mathbb{Z}$.
The absolute maximum value of $f(x)$ is $3/2$ and the absolute minimum value of $f(x)$ is -3.



Problem 8: Determine the points of inflection of $y = (x - 1)^{1/3}$.

$$y' = \frac{1}{3}(x - 1)^{-2/3},\ y'' = -\frac{2}{9}(x - 1)^{-5/3}.$$

The second derivative does not vanish anywhere. But at $x = 1$, y'' does not exist. There is a possibility of a point of inflection at $x = 1$.

For $x < 1,\ y'' > 0$. So, the curve is concave up in a left neighborhood of $x = 1$.
For $x > 1,\ y'' < 0$. So, the curve is concave down in a right neighborhood of $x = 1$.
Therefore, $x = 1$ is a point of inflection.
Notice that at $x = 1$ the curve has a vertical tangent since $y'(x) \to \pm\infty$ as $x \to 1$.

Problem 9: Investigate the curve $x(t) = \dfrac{3at}{1+t^3},\ y(t) = \dfrac{3at^2}{1+t^3}$ for a give $a > 0$ for its decreasing/increasing behavior. Sketch the curve.

Both the functions are defined for all values of t except for $t = -1$. We see that

$$\begin{aligned}\lim_{t\to -1-} x(t) &= \lim_{t\to -1-} \frac{3at}{1+t^3} = \infty, \quad \lim_{t\to -1+} x(t) = -\infty \\ \lim_{t\to -1-} y(t) &= \lim_{t\to -1-} \frac{3at^2}{1+t^3} = -\infty, \quad \lim_{t\to -1+} y(t) = \infty\end{aligned}$$

$$x(0) = y(0) = 0,\ \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0,\ \lim_{t\to -\infty} x(t) = 0,\ \lim_{t\to -\infty} y(t) = 0.$$

Also, $\dfrac{dx}{dt} = \dfrac{6a(1/2 - t^3)}{(1+t^3)^2}, \quad \dfrac{dy}{dt} = \dfrac{3at(2-t^3)}{(1+t^3)^2}; \dfrac{dy}{dx} = \dfrac{t(2-t^3)}{1-2t^3}$. For the parameter t, we have the four critical values (when $dx/dt = 0 = dy/dt$):

$$t_1 = -1,\ t_2 = 0,\ t_3 = 2^{-1/3},\ t_4 = 2^{1/3}.$$

At $t = 0, x = 0, y = 0$, we have $\frac{dy}{dx} = 0$ and when $t \to \infty, x = 0, y = 0$, we have $\frac{dy}{dx} \to \infty$.
Thus the curve cuts the origin twice:

with the tangent parallel to x-axis and with the tangent parallel to y-axis.

Also, when $t = 2^{-1/3}, x = a4^{1/3}, y = a2^{1/3}$, $\frac{dy}{dx} \to \infty$.
At this point the tangent to the curve is vertical.
When $t = 2^{1/3}, x = a2^{1/3}, y = a4^{1/3}$, $\frac{dy}{dt} = 0$.
At this point the curve has a horizontal tangent.
The information obtained from the derivative are as follows:

Range of t	Range of x	Range of y	sign of dy/dx	variation in $y = f(x)$
$-\infty < t < -1$	$0 < x < \infty$	$0 > y > -\infty$	$-$	decreases
$-1 < t < 0$	$-\infty < x < 0$	$\infty > y > 0$	$-$	decreases
$0 < t < 2^{1/3}$	$0 < x < a4^{1/3}$	$0 < y < a2^{1/3}$	$+$	increases
$2^{-1/3} < t < 2^{1/3}$	$a4^{1/3} > x > a2^{1/3}$	$a2^{1/3} < y < a4^{1/3}$	$-$	decreases
$2^{1/3} < t < \infty$	$a2^{1/3} > x > 0$	$a4^{1/3} > y > 0$	$+$	increases

$$\lim_{x\to\infty} \frac{y}{x} = \lim_{t\to 1-} \frac{3at^2(1+t^3)}{3at(1+t^3)} = -1.$$

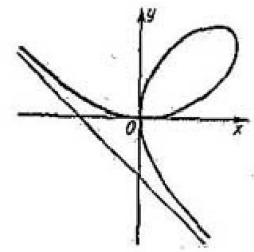
$$\lim_{x\to\infty} (y - (-1)x) = \lim_{t\to 1-} \frac{3at(t+1)}{1+t^3} = -a.$$

Hence the straight line $y = -x - a$ is an asymptote to a branch of the curve as $x \to \infty$. Similarly,

$$\lim_{x\to-\infty} \frac{y}{x} = -1, \quad \lim_{x\to-\infty} (y - (-1)x) = -a.$$

Therefore, the straight line $y = -x - a$ is also an asymptote to a branch of the curve as $x \to -\infty$.

Now, we can have a rough sketch of the curve, called Descartes' Folium.



Problem 10: Let $f : (a, b) \to \mathbb{R};\ c \in (a, b)$. Suppose that $|f|$ is differentiable at $x = c$ and f is continuous at $x = c$. Show that f is differentiable at $x = c$.

Notice that f may not be continuous but $|f|$ can be differentiable. For example, the function that takes value -1 for all rational numbers and takes value 1 for all irrational numbers.

If $f(c) > 0$, then $f(x)$ remains positive in a neighborhood of $x = c$. That is, in that neighborhood, $f(x) = |f(x)|$. Then f is differentiable at $x = c$.

Similarly, if $f(c) < 0$, then in a neighborhood of $x = c$, $f(x) = -|f(x)|$ and then $f(x)$ is differentiable at $x = c$.

If $f(c) = 0$, then $x = c$ is a local minimum point of $|f(x)|$. So, $|f|'(c) = 0$. This means

$$0 = \lim_{h\to 0} \frac{|f(c+h)| - |f(c)|}{h} = \lim_{h\to 0} \frac{|f(c+h)|}{h} \Rightarrow f'(c) = 0.$$

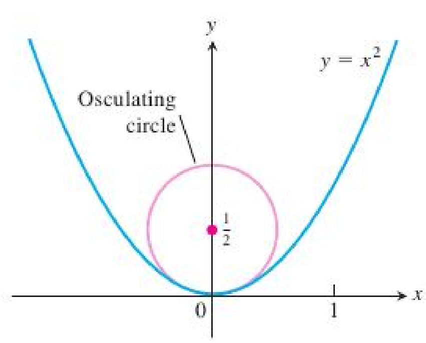
Problem 11: Find the radius and centre of curvature for $y = x^2$ at $(0, 0)$.

Here, $y' = 2x,\ y'' = 2$. That is, at the origin, $y' = 0, y'' = 2$. Thus

$$\rho = \left|(y'')^{-1}[1 + (y')^2]^{3/2}\right| = (1/2)^{-1}(1) = \frac{1}{2}$$

The centre of curvature at the origin is (a, b), where

$$a = x - y'[1 + (y')^2](y'')^{-1} = 0, \quad b = y + (1 + y')(y'')^{-1} = \frac{1}{2}.$$

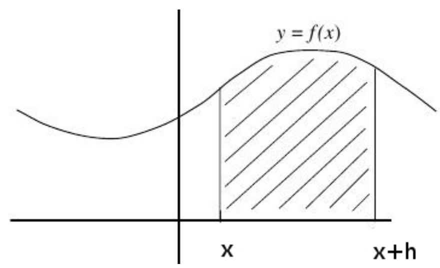


Chapter 3

Integration

3.1 The definite integral

Integration is a reverse process of integration. Look at the figure:



Imagine h is very small. Then $f(x)$ can be approximated by the shaded area divided by h. Write the area as $F(x)$, say, starting from some point $x = a$ on the left. Then the shaded area is $F(x+h) - F(x)$. Consequently,

$$f(x) \approx \frac{F(x+h) - F(x)}{h}.$$

Or, in the limit, we have $f(x) = F'(x)$. This means:

If $F(x)$ denotes the area of the region bounded by $y = f(x)$ and the x-axis, starting from some point $x = a$ up to the point $x = c$, then $f(c) = F'(c)$. That is, the function which may serve as an inverse process of differentiation is this area. How do we find this area?

We can approximate the area by dividing the x-interval, the domain of $f(x)$ into smaller parts, and then take the sum of all smaller rectangles of width as the small part on x-axis and the height as the function value at some point inside the smaller interval.

That is, suppose that $f : [a, b] \to \mathbb{R}$. Divide $[a, b]$ into smaller sub-intervals by choosing the break points as $a = x_0 < x_1 < \ldots < x_n = b$.

The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a **partition of** $[a, b]$.

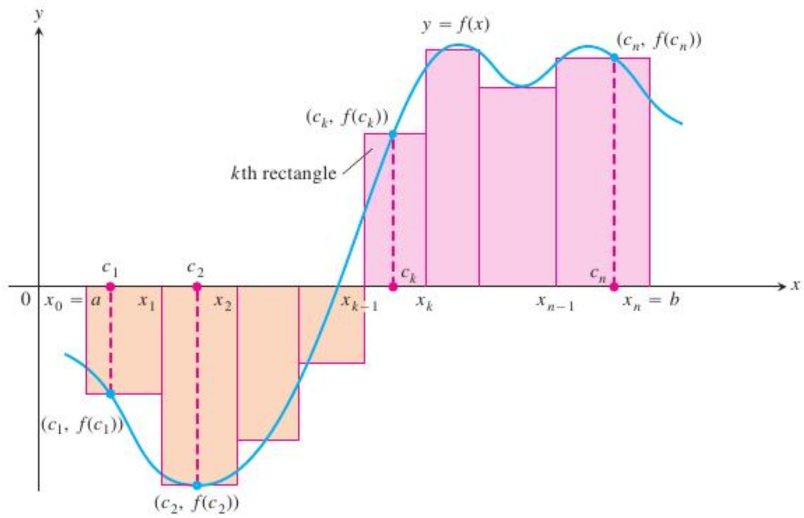
Now P divides $[a, b]$ into n sub-intervals: $[x_0, x_1], \cdots, [x_{n-1}, x_n]$. Here, the kth sub-interval is $[x_{k-1}, x_k]$. The area under the curve $y = f(x)$ raised over the kth sub-interval is approximated by $f(c_k)(x_k - x_{k-1})$ for some choice of the point $c_k \in [x_{k-1}, x_k]$.

Write the choice points (also called **sample points**) as a set $C = \{c_1, \ldots, c_n\}$.

Then the **Riemann sum**

$$S(f, P, C) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

is an approximation to the whole area raised over $[a, b]$ and lying between the curve $y = f(x)$ and the x-axis.



If the sub-intervals become smaller and smaller, we expect better and better approximation of the area by the Riemann sum. This hints at a limit process. That is, by taking the **norm** of the partition as $\|P\| = \max_k (x_k - x_{k-1})$, we would say that when the norm of the partition approaches 0, the Riemann sum would approach the required area. Thus, we define the area of the region bounded by the lines $x = a,\ x = b,\ y = 0$, and $y = f(x)$ as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

provided that this limit exists. Notice that we can choose c_k from each sub-interval in many ways. We can choose n in many ways. Thus the limit of he Riemann sum must exist for all such choices, and all those limits must be equal. If the limit of all such Riemann sums exist, and are equal, we call this limit as the integral of $f(x)$ from a to b. Our heuristics says that the integral is nothing but the area of the bounded region we just considered. In fact, when an area of such a region is required, we simply compute this integral. We write the integral as in the following:

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} S(f, P, C) = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

We say that the **integral exists** if for all such choices of P and of c_k, the right hand side limit exists. In that case, the integral can be computed by choosing a particular type of P, or even a particular way of choosing these c_k from the sub-intervals. The notation demands that the limit, when exists, must not depend on the particular choice points in C. Since the above limit looks different from $\lim_{x \to c} f(x)$, we must state it correctly. It means the following:

There exists $\ell \in \mathbb{R}$ so that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|P\| < \delta$, then for all choices of C, $|S(f, P, C) - \ell| < \epsilon$.

Of course, we will not use this definition always. The key to its application is the following basic fact, which we accept without proof:

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $\int_a^b f(x)\, dx$ exists.*

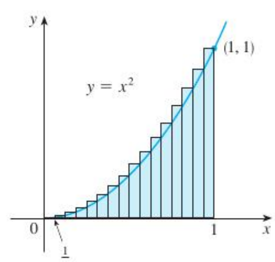
Since the integral is the area bounded by the curve and the x-axis between the two vertical lines that are obtained from the limits of integration, a prerequisite for a function to be integrable is that it must be bounded. Due to Theorem 1.4, a continuous function on a closed bounded interval is bounded.

Example 3.1. Calculate the area of the region below the curve $y = x^2$ above x-axis and between the lines $x = 0$ and $x = 1$.

That is, we compute $\int_0^1 x^2\, dx$. By Theorem 3.1, this integral exists. We thus choose a particular type of partition, say, **uniform partition**

$$P:\ 0 = x_0 = \frac{1}{n} < x_1 = \frac{2}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = \frac{n}{n} = 1.$$

We also choose the sample points $C:\ c_1 = x_0 = \frac{1}{n}, \cdots c_n = x_{n-1} = \frac{n-1}{n}$, the left end-points of the partition sub-intervals. That is,



The Riemann sum is

$$S(f, P, C) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

Notice that as $\|P\| \to 0$, we have $n \to \infty$. Then

$$\int_0^1 x^2\, dx = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{1}{3}.$$

Example 3.2. Show that $\displaystyle\int_0^1 \frac{1}{x}\, dx$ does not exist.

The function $1/x$ is not defined at 0. As such $\int_0^1 (1/x)$ is not defined.
Suppose we redefine the function by taking its value at 0 as 0. That is, we ask for $\int_0^1 f(x)dx$, where $f(0) = 0$ and $f(x) = 1/x$ for $x \in (0, 1]$. Here, $f(x)$ is an unbounded function since $\displaystyle\lim_{x\to 0+} (1/x) = \infty$. Therefore, $f(x)$ is not integrable.

Now, you see that an integrable function must be bounded in its domain of definition. If $f(x)$ becomes unbounded, then the limit of the Riemann sum will not exist. Even though a function is bounded, it need not be integrable.

Example 3.3. Show that $\int_0^1 f(x)dx$ does not exist, where $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Again, choose a uniform partition of $[0,1]$ as $P: \ 0 < \frac{1}{n} < \cdots < \frac{n}{n} = 1$. Let us choose two separate sets of sample points

$$C_1: \ x_1 < x_2 < \cdots < x_n, \quad C_2: \ t_1 < t_2 < \cdots < t_n,$$

where each x_k is a rational number in the k-th subinterval and t_k is an irrational number in the same k-th subinterval. The Riemann sums are the following:

$$S(P, f, C_1) = \sum_{k=1}^{n} \frac{f(x_k)}{n} = 0, \quad S(P, f, C_2) = \sum_{k=1}^{n} \frac{f(t_k)}{n} = \sum_{k=1}^{n} \frac{1}{n}.$$

The limits of the first Riemann sum as $n \to \infty$ is 0. The limit of the second Riemann sum if exists, will remain positive. Therefore, $f(x)$ is not integrable. In fact, limit of the second Riemann sum does not exist.

Form the definition of the definite integral we observe an interesting fact. Suppose we take any partition P of the interval $[a, b]$ on which the function $f(x)$ is defined, say, into n subintervals. Assume that $f(x)$ is bounded. Choose three different sets of sample points. Choose x_k as such a point that $f(x_k)$ is minimum in that k-th subinterval. Take t_k be any point in the subinterval. Finally, choose y_k as a point where $f(y_k)$ is maximum in the subinterval. Call C_1 as the set of sample points x_k, C_2 for the t_ks and C_3 for the y_ks. The Riemann sums will satisfy the following:

$$S(P, f, C_1) \le S(P, f, C_2) \le S(P, f, C_3).$$

If the function $f(x)$ is integrable, then as $\|P\| \to 0$, all of them will approach the limit $\int_a^b f(x)dx$. It is also clear that if $\lim_{\|P\| \to 0} S(P, f, C_1) = S(P, f, C_3)$, then $f(x)$ is integrable. The sum $S(P, f, C_1)$ is called the **lower sum** and $S(P, f, C_3)$ is called the **upper sum**; their limits if exist, are respectively called the **lower integral** and the **upper integral**. Integrability is equivalent to the statement that "as the norm of a partition approaches 0, the difference between the lower and upper sums approach 0". Sometimes this characterization helps in proving that a given function is integrable. We will not use this in usual circumstances due to Theorem 3.1. In fact, Theorem 3.1 is proved using this technique!

Since $\int_a^b f(x)\,dx$ is a number, it is called a **definite integral**.

Example 3.4. Evaluate the definite integral $\int_0^1 \cos x\,dx$.

Since $\cos x$ is continuous on $[0,1]$, the integral exists. To evaluate it, choose a partition

$$P: 0 = x_0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = 1.$$

Consider the sub-interval $[x_{k-1}, x_k]$. Since $(\sin x)' = \cos x$. By MVT, there exists $c_k \in [x_{k-1}, x_k]$ such that

$$\sin x_k - \sin x_{k-1} = (\cos c_k)(x_k - x_{k-1}).$$

Choose these c_ks as the sample points in the sub-intervals. Then

$$\begin{aligned} \int_0^1 \cos x\, dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \cos c_k (x_k - x_{k-1}) = \lim_{n\to\infty} \sum_{k=1}^{n} [\sin x_k - \sin x_{k-1}] \\ &= \lim_{n\to\infty} (\sin x_n - \sin x_0) = \lim_{n\to\infty} (\sin 1 - \sin 0) = \sin 1. \end{aligned}$$

The following properties of the definite integral are obvious from the definition.

Theorem 3.2. (Properties of Definite Integral)

1. *Let $f(x)$ have domain $[a, b]$. Let $c \in (a, b)$. Then $f(x)$ is integrable on $[a, b]$ iff $f(x)$ is integrable on both $[a, c]$ and $[c, b]$. In this case,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2. *Let $f(x)$ and $g(x)$ be integrable on $[a, b]$. Then $(f + g)(x)$ is integrable on $[a, b]$ and*

$$\int_a^b (f+g)(x)dx = \int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

3. *Let $f(x)$ be integrable on $[a, b]$. Let $c \in \mathbb{R}$. Then $(cf)(x)$ is integrable on $[a.b]$ and*

$$\int_a^b (cf)(x)\, dx = \int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx.$$

4. *Let $f(x)$ and $g(x)$ be integrable on $[a, b]$. If for each $x \in [a, b],\ f(x) \le g(x)$, then*

$$\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx.$$

5. *Let $f(x)$ be integrable on $[a, b]$. If $m \le f(x) \le M$ for all $x \in [a, b]$, then*

$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a).$$

6. *(*Average Value Theorem*) Let $f(x)$ be continuous on $[a, b]$. Then there exists $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

7. *Let $f(x)$ be continuous on $[a, b]$. If $f(x)$ has the same sign on $[a, b]$ and $\int_a^b f(x)\, dx = 0$, then $f(x)$ is the zero function, i.e., $f(x) = 0$ for each $x \in [a, b]$.*

Notice that (6) follows from (5) and the intermediate value theorem for continuous functions. The value $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. is called the **average value of the function f(x)** and (6) says that a continuous function achieves its average value. It also follows from (6) that if $\displaystyle\int_a^b f(x)dx = 0$, then $f(x)$ has a zero in $[a, b]$.

Convention: $\displaystyle\int_a^b f(x)\,dx$ makes sense when $a < b$. We extend the integral even when $a \not< b$ by the following:

If $a = b$, then we take $\displaystyle\int_a^b f(x)\,dx = 0$.

If $a > b$, then we take $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Notice that this convention goes well with the properties listed in Theorem 3.2. Moreover, we also extend Theorem 3.2(1) for any real number c; even when c is outside the interval (a, b) by

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

In all these extensions, we assume that the definite integrals exist.

3.2 The Indefinite Integral

If $F'(x) = f(x)$, then we say that the function $F(x)$ is an **antiderivative** of $f(x)$. Example 3.4 shows that if we know an antiderivative of a function, then we can readily get its integral.

Theorem 3.3. (Fundamental Theorem of Calculus $-$ 1) *Let $f(x)$ be continuous on $[a, b]$. Let $F(x)$ be an antiderivative of $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.*

Proof: Since $f(x)$ is continuous on $[a, b]$, the definite integral exists. To evaluate it, choose a partition

$$P : a = x_0 < x_1 = a + \frac{b-a}{n} < \cdots < x_{n-1} = a + \frac{(n-1)(b-a)}{n} < x_n = b.$$

Consider the sub-interval $[x_{k-1}, x_k]$. Since $F'(x) = f(x)$, By MVT, there exists $c_k \in [x_{k-1}, x_k]$ such that $F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$. Choose these c_ks as the sample points in the sub-intervals. Then

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(c_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^n [F(x_k) - F(x_{k-1})] \\ &= \lim_{n\to\infty}(F(x_n) - F(x_0)) = F(b) - F(a).\end{aligned}$$ □

We may remember the conclusion of this theorem as

$$\int_a^b g'(x)\,dx = g(x)\Big|_a^b = g(b) - g(a).$$

FTC-1 makes evaluation of definite integrals easy, which you are familiar with. For example,

$$\int_{-\pi/4}^{0} \sec x \tan x \, dx = \int_{-\pi/4}^{0} (\sec x)' \, dx = \sec x \Big|_{-\pi/4}^{0} = \sec 0 - \sec(-\pi/4) = 1 - \sqrt{2}.$$

Its use is limited, for, we require an antiderivative of the integrand in closed form.

FTC-1 considers taking the integral of the derivative of $f(x)$; it is $f(x) - f(a)$. What about taking the derivative of the integral? For this, we need the integral to be a function. Define $g(x) = \int_a^x f'(t)\, dt$ for each $x \in [a, b]$. We then show that $g'(x) = f(x)$.

Theorem 3.4. (Fundamental Theorem of Calculus $-$ 2) *Let $f(x)$ be continuous on $[a, b]$. Then $g(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover,*

$$g'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

Proof: Let $x \in (a, b)$ and let $h > 0$ such that $x + h \in [a, b]$. Now,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \left[\int_a^{x+h} f(t)\, dt - \int_a^{x} f(t)\, dt \right] = \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(c)$$

for some $c \in [x, x+h]$, by the Average Value theorem. We see that if $h \to 0$, then $c \to x$. Thus

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{c \to x} f(c) = f(x).$$

That is, $g(x)$ is differentiable and $g'(x) = f(x)$. Also, continuity of $g(x)$ follows from its differentiability. When x is either equal to a or b, we consider the appropriate one-sided limits:

$$\lim_{h \to 0+} \big[g(a+h) - g(a) \big] = \lim_{h \to 0+} \int_a^{a+h} f(x)\, dx = \lim_{h \to 0+} h f(d) = 0.$$

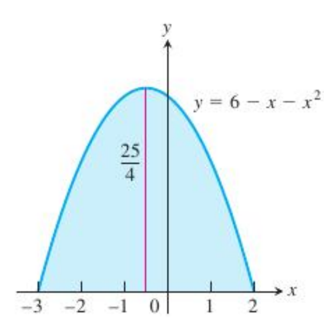
Here d is some point in $[a, a+h]$ which exists due to Theorem 3.2(5). This proves that $g(x)$ is continuous at $x = a$. Similar argument proves continuity of $g(x)$ at $x = b$. ☐

The two fundamental theorems say that integration and differentiation are inverse processes. The reason behind introducing the definite integral as the area is to obtain these two theorems. In fact, starting from "inverse processes", we have no way of relating the integral to the area. Using the Riemann integration, we get both! Moreover, the fundamental theorems say that the differential equation $y' = f(x)$ has a solution provided that $f(x)$ is continuous.

Example 3.5. Calculate the area bounded by x-axis and the parabola $y = 6 - x - x^2$.

We get the points of intersection of the curve with x-axis by setting $y = 0 = 6 - x - x^2 = (x+3)(2-x)$.
That is, $x = 2$ or $x = -3$.
The curve is non-negative on $[-3, 2]$.
Therefore, the required area is

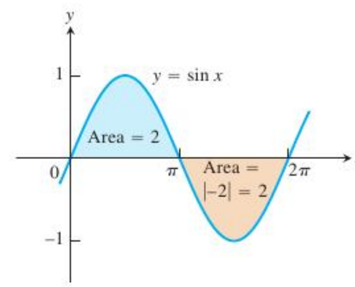
$$\int_{-3}^{2} (6 - x - x^2)\, dx = \frac{125}{6}.$$



Caution: The actual area is $\int_a^b |f(x)|\,dx$. The signed area is given by $\int_a^b f(x)\,dx$. If $f(x)$ is not of the same sign throughout $[a, b]$, then the integral gives the sum total of the signed areas only.

In Example 3.5, the function $f(x) = 6-x-x^2 \geq 0$ for each $x \in [-3, 2]$. Therefore, $|f(x)| = f(x)$. Hence the area is equal to the integral as computed there.

Example 3.6. Calculate the area of the region bounded by the x-axis, the lines $x = 0,\ x = 2\pi$, and the curve $y = f(x) = \sin x$.

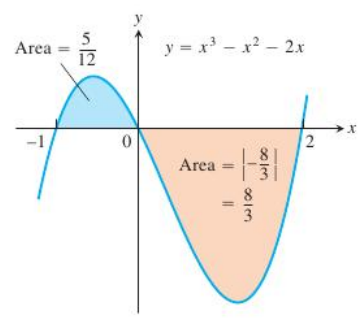


$\displaystyle\int_0^{2\pi} f(x)\,dx = -\cos x\Big|_0^{2\pi} = 0.$
This gives the signed area, where the portions below and above the x-axis get canceled.
The true area is given by

$\displaystyle\int_0^{2\pi} |f(x)|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} -\sin x\,dx$

$= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi} = 4.$

Example 3.7. Find the area of the region bounded by the lines $x = -1,\ x = 2,\ y = 0$, and the curve $y = x^3 - x^2 - 2x$.



$f(x) = x^3 - x^2 - 2x = x(x+1)(x-2).$

On $[-1, 0],\ f(x) \geq 0$. On $[0, 2],\ f(x) \leq 0$.

Thus the area of the given region is

$\displaystyle\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx - \int_0^2 (x^3 - x^2 - 2x)\,dx$

$= \dfrac{5}{12} - \dfrac{-8}{3} = \dfrac{37}{12}.$

Example 3.8. What is wrong with $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = -2$?

For each $x \in [-1, 1],\ \dfrac{1}{x^2} \geq 0$. So, $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$, if exists, must be non-negative. The problem is $\dfrac{1}{x^2}$ is not integrable on $[-1, 1]$, since it is not defined at $x = 0$. Moreover, it becomes unbounded as x approaches 0. Therefore, $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ does not exist.

Due to the Fundamental theorems, we introduce the **indefinite integral**:

$$\int f(x)dx = \int_a^x f(t)\,dt \quad \text{for any } a \in \mathbb{R}$$

whenever the definite integral makes sense. Notice that if $\int f(x)dx = g(x)$, then $g'(x) = f(x)$. Conversely, if $g'(x) = f(x)$, then

$$\int f(x)dx = \int_a^x f(t)\,dt = \int_a^x g'(t)dt = g(x) - g(a).$$

We write this by taking $g(a)$ as an arbitrary constant C; and we call it as a **constant of integration**. That is,

$$g'(x) = f(x) \quad \text{if and only if} \quad \int f(x)\,dx = g(x) + C.$$

The differentiation of usual functions give rise to their integration. For example, $(x^2)' = 2x$. Therefore, $\int (2x)dx = x^2 + C$

Below, we list integrals of some functions which we use often.

Common functions and their integrals

$f(x)$	$\int f(x)\,dx$
0	C
x^r	$\dfrac{x^{r+1}}{r+1} + C$ for $r \neq -1$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x + C$
$\text{cosec}\, x \cot x$	$-\text{cosec}\, x + C$
$\text{cosec}\,^2 x$	$-\cot x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$ for $\lvert x\rvert < 1$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + C$
$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1} x$ for $\lvert x\rvert > 1$

3.3 Substitution and Integration by Parts

The chain rule for differentiation: $(g(f(x))' = g(f(x))f'(x)$ is translated to integration as in the following.

Theorem 3.5. (Substitution)

1. *Let $u = g(x)$ be a differentiable function whose range is an interval I. Let $f(x)$ be continuous on I. Then*
$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

2. *Let $u = g(x)$ be a continuously differentiable function on $[a, b]$ whose range is an interval I. Let $f(x)$ be continuous on I. Then*
$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof: Since $f(x)$ is continuous, $F(x) = \int f(x)\,dx$ is an antiderivative of $f(x)$. That is, $F'(x) = f(x)$. By Chain rule for differentiation,

$$F(g(x))' = F'(g(x))g'(x) = f(g(x))g'(x).$$

Then $\displaystyle\int f(g(x))g'(x)\,dx = \int F(g(x))'dx = F(g(x)) + C$

$$= F(u) + C = \int F'(u)\,du + C = \int f(u)\,du.$$

This proves (1). Similarly, for (2) we see that

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big|_{x=a}^{x=b} = F(g(b)) - F(g(a)) = F(u)\Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)\,du. \qquad \square$$

When $u = g(x),\ du = g'(x)du$. Then the substitution rule looks like

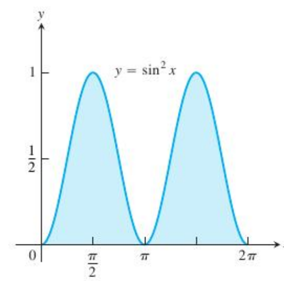
$$\int f(u)u'dx = \int f(u)du.$$

Similar comments apply for the definite integral, where we take care of the limits of integration in terms of u.

Example 3.9. $\displaystyle\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\frac{1}{2}\,d(2x) = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

Therefore, $\displaystyle\int_0^{2\pi} \sin^2 x dx = \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{2\pi} = \pi.$



Example 3.10. Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

Put $u = x^3 + 1$. Then $du = 3x^2\,dx;\ \sqrt{x^3+1} = \sqrt{u}$.
When $x = -1,\ u = 0$. When $x = 1,\ u = 2$. Thus

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^2 = \frac{4\sqrt{2}}{3}.$$

Sometimes for integration, you may require to use breaking a rational function into simple form using decomposition into partial fractions.

Example 3.11. Evaluate $\displaystyle\int \frac{10\,dx}{(x+1)(x^2+9)^2}.$

Here, we try to determine constants A, B, C, D and E from

$$\frac{10}{(x+1)(x^2+9)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} + \frac{Dx+E}{(x^2+9)^2}.$$

By simplifying and comparing, we obtain: $A = 1/10,\ B = -1/10,\ C = 1/10,\ D = -1$ and $E = 1$. Therefore,

$$\int \frac{10\,dx}{(x+1)(x^2+9)^2} = \int \frac{dx}{10(x+1)} - \int \frac{xdx}{10(x^2+9)} + \int \frac{dx}{10(x^2+9)} - \int \frac{xdx}{(x^2+9)^2} + \int \frac{dx}{(x^2+9)^2}.$$

For the second and fourth integrals use the substitution $u = x^2 + 9$ and for the last integral, use the substitution $x = 3\tan t$ and integrate to obtain:

$$\begin{aligned}\int \frac{10\,dx}{(x+1)(x^2+9)^2} &= \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} \\ &\quad + \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C \\ &= \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{7}{135}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x+9}{18(x^2+9)} + C.\end{aligned}$$

Example 3.12. Evaluate $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\,dt$.

Substitute $u = x^4$. Using FTC and the Chain rule,

$$\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \frac{d}{du}\int_1^{u} \sec t\,dt \times \frac{du}{dx} = \sec u \times \frac{du}{dx} = 4x^3\sec(x^4).$$

Recall that a function $f(x)$ is called **even** if $f(-x) = f(x)$. In that case,

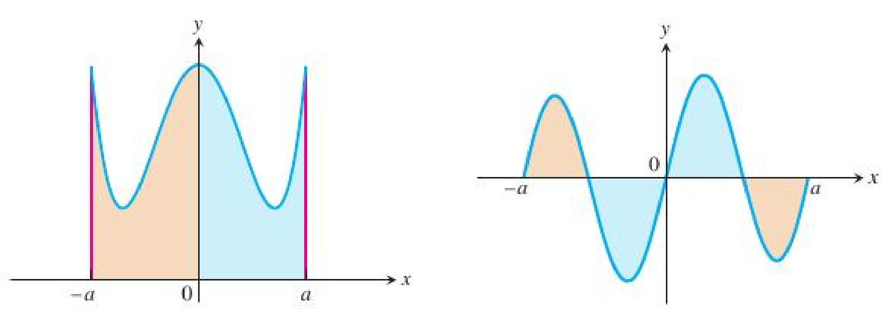
$$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-x)d(-x) = \int_0^a f(-x)dx = \int_0^a f(x)dx.$$

And when $f(x)$ is an **odd** function, i.e., when $f(-x) = -f(x)$, we see that

$$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-x)d(-x) = \int_0^a f(-x)dx = -\int_0^a f(x)dx.$$

In summary, let $f(x)$ be a continuous function on $[-a, a]$. Then

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd.} \end{cases}$$



The Fundamental theorems are used to translate the product rule for differentiation to integration. Recall that the product rule for differentiation says that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Write $h(x) = g'(x)$. Then $g(x) = \int h(x)\,dx$, and we get

$$f(x)h(x) = (f(x)g(x))' - f'(x)\int h(x)\,dx.$$

Integrate both the sides to get

$$\int f(x)h(x)\,dx = f(x)\int h(x)\,dx - \int \left[f'(x)\int h(x)\,dx\right]\,dx + C.$$

We remember it as follows (Read F as first and S as second):

Integral of $F \times S = F\times$ integral of $S -$ integral of (derivative of $F \times$ integral of S).

This is called the **integration by parts** formula. We also write it in the form

$$\int u\,dv = uv - \int v\,du + C.$$

Example 3.13. Find $\int x\cos x\,dx$.

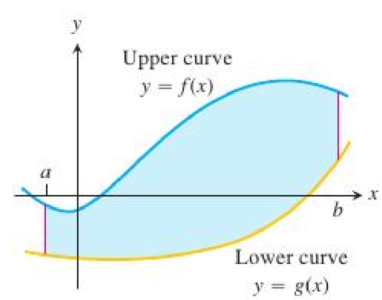
Integrating by parts, we have

$$\int x\cos x\,dx = x\int \cos x\,dx - \int\left[(x')\int \cos x\,dx\right]dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Since definite integral uses area as its definition, it is fairly straightforward to calculate the areas bounded by curves. Of course, we use the fundamental theorems for this purpose.

Let $f(x)$ $g(x)$ be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. The area between the curves $y = f(x)$ and $y = g(x)$ is

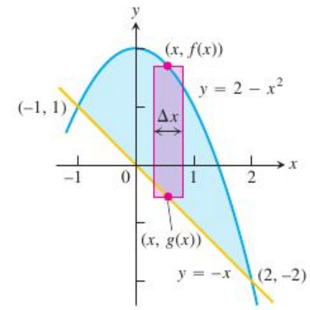
$$\int_a^b [f(x) - g(x)]\,dx.$$



In applying this result, you must consider where the curves intersect. It may turn out that inside the interval $[a, b]$ the curves intersect more than once. In that case, all the integrals are to be computed separately since there will be a sign change in $f(x) - g(x)$.

Example 3.14. Find the area of the region enclosed by the line $y = -x$ and the parabola $y = 2 - x^2$.

The curves here intersect to give a bounded region. The points of intersection are obtained from $-x = 2 - x^2$. They are $x = -1$ and $x = 2$. The line $y = -x$ lies below the parabola. Therefore, the area between them is

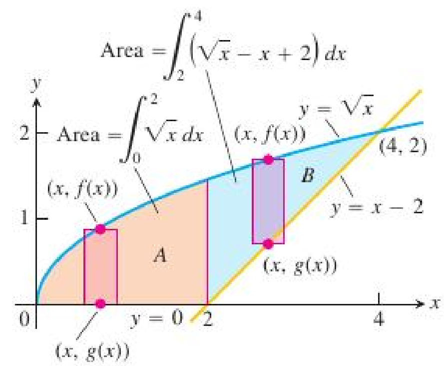


$$\int_{-1}^{2} [2 - x^2 - (-x)]\, dx = \Big[2x + \frac{x^2}{2} - \frac{x^3}{3}\Big]_{-1}^{2} +$$

$$= 4 - \frac{4}{2} - \frac{8}{3} - \big(-2 + \frac{1}{2} + \frac{1}{3}\big) = \frac{10}{3} - \frac{-7}{6} = \frac{9}{2}.$$

Example 3.15. Find the area of the region in the first octant bounded by the lines $y = 0,\ y = x-2$, and the parabola $x = y^2$.

The parabola $y = \sqrt{x}$ is the upper curve. But there are two lower lines. Their break point is at $x = 2$. We divide the region into two parts by drawing the vertical line $x = 2$. Then we find the areas of the regions marked A and B separately and take their sum. The required area is



$$\int_0^2 [\sqrt{x} - 0]\, dx + \int_2^4 [\sqrt{x} - (x-2)]\, dx$$

$$= \Big[\frac{2}{3}x^{3/2}\Big]_0^2 + \Big[\frac{2}{3}x^{3/2} - \frac{(x-2)^2}{2}\Big]_2^4$$

$$= \frac{2}{3}\,2^{3/2} + \frac{2}{3}\,4^{3/2} - \frac{2^2}{2} - \frac{2}{3}\,2^{3/2} = \frac{16}{3} - 2 = \frac{10}{3}.$$

Alternate Method: If you integrate with respect to y, then a single integral gives the result. Viewed from y-axis, the upper curve is the line $x = y + 2$ and the lower curve is $x = y^2$. The limits of integration are $y = 0$ to $y = 2$. Therefore, the required area is

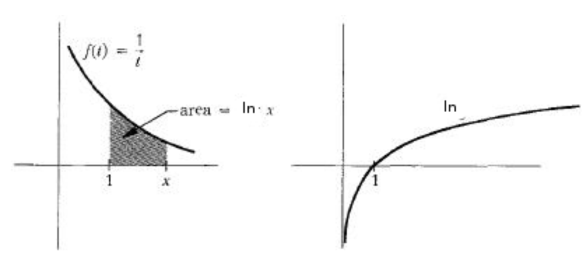
$$\int_0^2 (y + 2 - y^2)\, dy = \Big[\frac{y^2}{2} + 2y - \frac{y^3}{3}\Big]_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}.$$

3.4 Logarithm and Exponential

The function $f(t) = \dfrac{1}{t}$ is continuous on $(0, \infty)$. Therefore for any $x \in (0, \infty)$, the integral $\displaystyle\int_1^x \frac{1}{t}\, dt$ exists. Notice that when $x \in (0, 1)$, we look at the integral as $-\displaystyle\int_x^1 \frac{1}{t}\, dt$. So, we define its value as a the value of a function depending on x.

This is called the **natural logarithmic function** $\ln x$. That is,

$$\ln x = \int_1^x \frac{1}{t}\, dt \quad \text{for } x > 0.$$



We see that $\ln : (0, \infty) \to \mathbb{R}$. Since $\frac{1}{t} > 0$ for $t > 0$, $\ln$ is a strictly increasing function. Also, $\ln x > 0$ for $x > 1$, $\ln x < 0$ for $0 < x < 1$, and $\ln 1 = 0$. Moreover, $(\ln x)' = \frac{1}{x}$. Since $(\ln x)'' = -\frac{1}{x^2} < 0$, we conclude that the graph of $\ln x$ is concave down. Also, $\ln x \to \infty$ as $x \to \infty$ and $\ln x \to -\infty$ as $x \to 0+$. All but the limit properties are easy to see. We prove that $\lim\limits_{x\to\infty} \ln x = \infty$.

For this, suppose $1 \le k \le t \le m$. Then $1/t \ge 1/m$. Consequently,

$$\int_k^m \frac{1}{t}\, dt \ge \int_k^m \frac{1}{m}\, dt = \frac{1}{m}\int_k^m dt = \frac{m-k}{m}.$$

Let $n \in \mathbb{N}$. Let $x = 2^{2n}$. Using the above inequality, we have

$$\ln x = \int_1^x \frac{1}{t}\, dt = \int_1^{2^{2n}} \frac{1}{t} dt = \sum_{j=1}^{2n} \int_{2^{j-1}}^{2^j} \frac{1}{t}\, dt \ge \sum_{j=1}^{2n} \frac{2^j - 2^{j-1}}{2^j} = \sum_{j=1}^{2n} \frac{1}{2} = n.$$

That is, for each $n \in \mathbb{N}$, there exists $x > 1$ such that $\ln x \ge n$. This proves that $\lim\limits_{x\to\infty} \ln x = \infty$.

For the other limit, observe that by substituting $s = 1/t$, we have $ds = -(1/s^2)\, ds$. Then

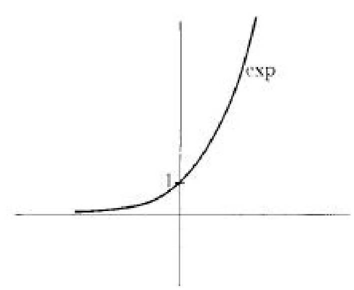
$$\ln \frac{1}{x} = \int_1^{1/x} \frac{1}{t}\, dt = \int_1^x s\, \frac{-1}{s^2}\, ds = -\int_1^x \frac{1}{s}\, ds = -\ln x.$$

Therefore,

$$\lim_{x\to 0+} \ln x = \lim_{y\to\infty} \ln \frac{1}{y} = -\lim_{y\to\infty} \ln y = -\infty.$$

Due to its strict increasing nature, the limit properties imply that $\ln : (0, \infty) \to \mathbb{R}$ is a one-one and onto function. Then $\ln x$ has an inverse function. We call this inverse function as the **natural exponential function** and denote it by $\exp$. That is,

$\exp : R \to (0, \infty); \quad y = \exp(x) \quad \text{iff} \quad x = \ln y.$



Hence $\exp(\ln x) = x$ for $x \in (0, \infty)$
and $\ln(\exp(x)) = x$ for $x \in \mathbb{R}$.

Differentiating the latter equation, we get

$\dfrac{1}{\exp(x)} \exp'(x) = 1$. That is, $\exp'(x) = \exp(x)$.

Similar to $\ln$, we have the limit properties:

$\lim\limits_{x\to\infty} \exp(x) = \infty$; and $\lim\limits_{x\to-\infty} \exp(x) = 0$.

Moreover, $\exp(x)$ is strictly increasing and is always positive.

Let $a, b > 0$. Consider $f(x) = \ln(xb) - \ln x$. Now, $f'(x) = \frac{b}{xb} - \frac{1}{x} = 0$. So, $f(x)$ is a constant. But $f(1) = \ln b - \ln 1 = \ln b$. Therefore, for each $x \in (0, \infty)$, $f(x) = \ln b$. In particular, $f(a) = \ln(ab) - \ln a = \ln b$. That is,

$$\ln(ab) = \ln a + \ln b, \quad \text{for all } a, b > 0.$$

Similarly, if $c, d \in \mathbb{R}$, let $a = \exp(c),\ b = \exp(d)$. Then $\ln a = c$ and $\ln b = d$. Now, the previous equation gives

$$\exp(c+d) = \exp(\ln a + \ln b) = \exp(\ln(ab)) = ab = \exp(c)\exp(d) \ \text{ for all } c, d \in \mathbb{R}.$$

Arbitrary powers of positive real numbers can then be defined using $\exp$ and $\ln$.

Let $a > 0$ We define the **exponential function** $a^x = \exp(x \ln a)$.

The **number** ***e*** is defined as the unique positive number satisfying $\ln e = 1$.

Then $e^x = \exp(x \ln e) = \exp(x)$. We dispense with the symbol $\exp(x)$ and write it henceforth as e^x. The exponent symbolism is consistent with the usual powers like $x^3 \times x^2 = x^{2+3}$, etc.

The definitions, the limit properties and the differentiation properties tell us that for $a > 0$,

$$\ln e = 1 = e^0, \quad e^{\ln x} = x, \quad \ln(e^x) = x, \quad a^x = e^{x \ln a},$$

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0+} \ln x = -\infty, \quad \lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to -\infty} e^x = 0,$$

$$(\ln x)' = \frac{1}{x}, \quad (e^x)' = e^x, \quad (a^x)' = (\ln a) a^x, \quad \int_1^e \frac{1}{t}\,dt = 1, \quad \int e^x\,dx = e^x.$$

Example 3.16. (a) $\displaystyle\lim_{h\to 0} \frac{\ln(1+xh)}{xh} = 1$ for $x \neq 0$.

As $h \to 0$, both $\ln(1+xh)$ and xh approach 0. By L' Hospital's rule, (with respect to h)

$$\lim_{h\to 0} \frac{\ln(1+xh)}{xh} = \lim_{h\to 0} \frac{x}{x(1+xh)} = 1.$$

(b) $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$

Using L' Hospital's rule again, we have $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} e^h = 1.$

(c) $\displaystyle\lim_{h\to 0} (1+xh)^{1/h} = e^x.$

If $x = 0$, then $(1+xh)^{1/h} = 1^{1/h} = e^{(1/h)\ln 1} = e^0 = 1$. For $x \neq 0$,

$$\lim_{h\to 0} \ln(1+xh)^{1/h} = \lim_{h\to 0} \frac{\ln(1+xh)}{h} = x \lim_{h\to 0} \frac{\ln(1+xh)}{xh} = x.$$

Since $\exp$ is a continuous function, we have

$$\lim_{h\to 0} (1+xh)^{1/h} = \exp\left(\lim_{h\to 0} \ln(1+xh)^{1/h}\right) = \exp(x) = e^x.$$

In particular, $e = \displaystyle\lim_{h\to 0} (1+h)^{1/h}$.

(d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^{1/k}} = 0$ for any $k \in \mathbb{N}$.

The limit is in $\frac{\infty}{\infty}$ form; apply L' Hospital's rule: $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^{1/k}} = \lim_{x\to\infty} \frac{k}{x\, x^{1/k-1}} = 0.$

(e) $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ for any $k \in \mathbb{N}$.

Again, the limit is in $\frac{\infty}{\infty}$ form. So, $\lim_{x\to\infty} \frac{x^k}{e^x} = \lim_{x\to\infty} \frac{kx^{k-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{k!}{e^x} = 0.$

(f) If $a > 1$, then $\lim_{x\to\infty} \frac{x^k}{a^x} = 0$ for any $k \in \mathbb{N}$.

For $a > 1$, $a^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{x^k}{a^x} = \lim_{x\to\infty} \frac{kx^{k-1}}{\ln(a)a^x} = \cdots = \lim_{x\to\infty} \frac{k!}{(\ln a)^k a^x} = 0.$$

Exercise: In proving that $\ln x \to \infty$ as $x \to \infty$, we have shown that $\ln(2^{2n}) \geq n$. Use this and the limit for e in (3) to prove that $2 \leq e \leq 4$.

The following properties of the exponential function can easily be proved:

$$\text{If } a > 1, \text{ then } \lim_{x\to\infty} a^x = \infty \text{ and } \lim_{x\to-\infty} a^x = 0.$$

$$\text{If } 0 < a < 1, \text{ then } \lim_{x\to\infty} a^x = 0 \text{ and } \lim_{x\to-\infty} a^x = \infty.$$

If $a \neq 1$, then the x-axis is a horizontal asymptote to the graph of $y = a^x$.

The hyperbolic functions are defined through the exponential and the logarithmic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), \quad \cosh^{-1} x = \ln(x + \sqrt{x^2-1}), \quad \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

Notice that $\cosh^{-1}$ has domain as $x \geq 1$ and $\tanh^{-1}$ has domain as $-1 < x < 1$.

The derivatives of these functions can be found out in the usual manner.

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x, \quad (\tanh x) = (\cosh x)^{-2}.$$

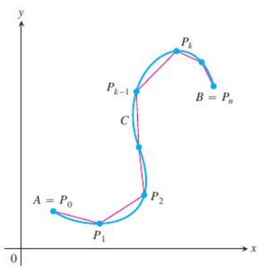
$$(\sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}}, \quad (\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}, \quad (\tanh^{-1} x)' = \frac{1}{1-x^2}.$$

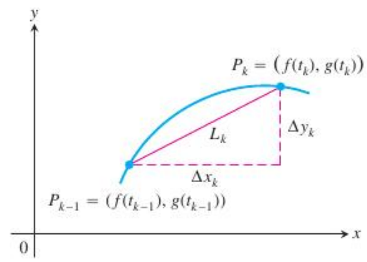
These formulas can be used for evaluating integrals as usual. For example,

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx = \sinh^{-1} x\Big|_0^1 = \sinh^{-1}(1) - \sinh^{-1}(0) = \ln(1+\sqrt{2}).$$

3.5 Lengths of Plane Curves

Let C be a curve given parametrically by $x = f(t), y = g(t),\ a \leq t \leq b$. We assume that both the functions $f(t)$ and $g(t)$ are continuously differentiable and the derivatives are not simultaneously zero on $[a, b]$. (Smoothness) We also assume that the curve is traversed exactly once as t increases from a to b.





Take a partition of $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of equal length Δt. and draw the secants on the curve by using the partition points. The length of the curve is defined as the limit of the sum of lengths of the secants as n approaches ∞. Now, the length of a single secant joining the points P_{k-1} to P_k is

$$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}.$$

By the MVT, there exist $a_k, b_k \in [t_{k-1}, t_k]$ such that

$$f(t_k) - f(t_{k-1}) = f'(a_k)\Delta t, \quad g(t_k) - g(t_{k-1}) = g'(b_k)\Delta t.$$

Then the sum of all the secants corresponding to the subintervals is

$$\sum_{k=1}^{n} \sqrt{[f'(a_k)]^2 + [g'(b_k)]^2}\,\Delta t.$$

Since the length of the curve is the limit of this sum, we have

$$\boxed{\text{Length of the curve} = L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt.}$$

If the curve is given as a function $y = f(x),\ a \le x \le b$, then take $x = t$ and $y = f(t)$ as its parameterization. We then have the length as

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_a^b \sqrt{1 + (y')^2}\,dx.}$$

Notice that this formula is applicable when $f'(x)$ is continuous on $[a, b]$.

Similarly, if a curve is given by a function $x = g(y),\ c \le y \le d$, where $g'(y)$ is continuous on $[c, d]$, the length of the curve is

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy = \int_c^d \sqrt{1 + (x')^2}\,dy.}$$

In the length formula, the integrand along with the differential is written as ds. Here, s stands for the length element on a given curve. We write $L = \int_a^b ds$ with limits a and b for the variable of integration, which may be $x,\ y$ or t. Here,

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

This view makes it possible to see the arc length as a function:

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

for the curve $y = f(x)$ where the curve is measured from the point $(a, f(a))$.

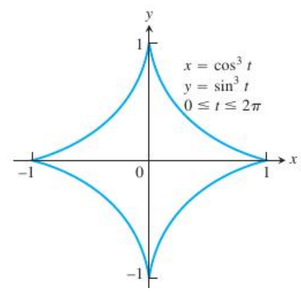
Example 3.17. Find the length of the curve $y = \dfrac{4\sqrt{2}}{3}x^{3/2} - 1$ traced for $0 \le x \le 1$.

Here, $y' = 2\sqrt{2}\,x^{1/2}$. The length of the curve is

$$L = \int_0^1 \sqrt{1 + (y')^2}\,dx = \int_0^1 \sqrt{1 + 8x}\,dx = \frac{2}{3}\cdot\frac{1}{8}(1+8x)^{3/2}\Big|_0^1 = \frac{13}{6}.$$

Example 3.18.

Find the length of the asteroid
$x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$.



Due to symmetry of the curve, its length is four times the arc traced when $0 \le t \le \pi/2$.

$(\cos^3 t)' = -3\cos^2 t \sin t, \quad (\sin^3 t)' = 3\sin^2 t \cos t.$

Therefore, the length of the asteroid is

$$L = 4\int_0^{\pi/2} \sqrt{[(\cos^3 t)']^2 + [(\sin^3 t)']^2}\,dt = 4\int_0^{\pi/2} 3\cos t \sin t\,dt = 6.$$

Example 3.19. Find the length of the arc of the curve $y = (x/2)^{2/3}$ for $0 \le x \le 2$.

Here, $y' = \dfrac{1}{3}\left(\dfrac{1}{x}\right)^{1/3}$. It is not defined at $x = 0$. We express the same curve as

$$x = g(y) = 2y^{3/2},\ 0 \le y \le 1.$$

Now, $x' = g'(y) = 3\sqrt{y}$, which is continuous on $[0, 1]$. The length L of the arc of the curve is

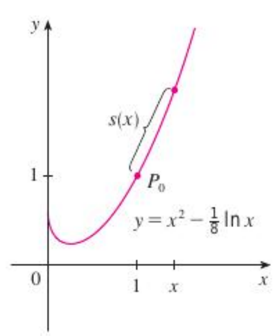
$$L = \int_0^1 \sqrt{1 + [g'(y)]^2}\,dy = \int_0^1 \sqrt{1 + 9y}\,dy = \frac{2}{27}(1+9y)^{3/2}\Big|_0^1 = \frac{2(10\sqrt{10} - 1)}{27}.$$

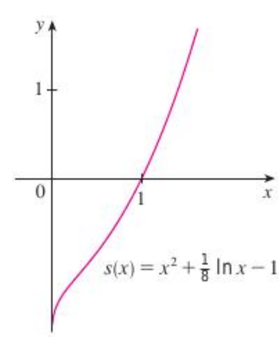
Example 3.20. Find the arc length function for the curve $y = x^2 - (\ln x)/8$, where the arc is measured from the point $(1, 1)$.

Write $f(x) = x^2 - (\ln x)/8$. Then $f'(x) = 2x - 1/(8x)$, and $\sqrt{1 + [f'(x)]^2} = 2x + 1/(8x)$. Then

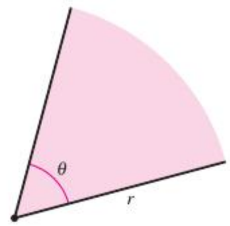
$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\,dt = \int_1^x \left(2x + \frac{1}{8x}\right)dt = x^2 + \frac{1}{8}\ln x - 1.$$

Notice that for $x < 1$, the values of $s(x)$ are negative, since s is measured from the point $(1, 1)$.

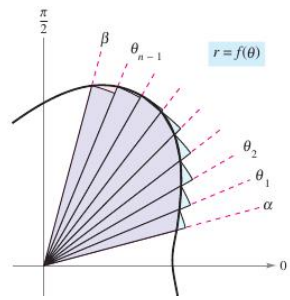




The calculation of area in **polar coordinates** is similar to the Cartesian coordinates; but it uses the basic area as a circular sector instead of a rectangle. Recall that a circular sector of angle θ while the circle has radius r is given by $\theta r^2/2$, where θ is measured in radians.



The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.



Suppose that a curve is given in polar coordinates by $r = f(\theta)$ for a continuous function $f(\theta)$, where $\alpha \le \theta \le \beta$. Divide the interval $[\alpha, \beta]$ into n sub-intervals $[\theta_{k-1}, \theta_k]$, say of equal length $\Delta\theta = \theta_k - \theta_{k-1}$. Approximate the area of the sector of the region bounded by the curve $r = f(\theta)$ and the rays $\theta = \theta_{k-1}$ and $\theta = \theta_k$ by a circular sector of the same angle $\Delta\theta$ and radius $f(r_k)$, where r_k is some point in the subinterval $[\theta_{k-1}, \theta_k]$. The approximate area of the sector is $[f(r_k)]^2\Delta\theta/2$. Form the Riemann sum as an approximation to the required area. Then take the limit as the norm of the partition, i.e., $\max(\theta_k - \theta_{k-1})$ approaches 0, that is, as $n \to \infty$. This gives the area as

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}[f(r_k)]^2\Delta\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta = \int_\alpha^\beta r^2\, d\theta.$$

Notice that our assumptions for the derivation of this formula are that the function $f(\theta)$ is continuous on the interval $[\alpha, \beta]$, where $\beta - \alpha \le 2\pi$, that is, when the curve is traversed only once. Moreover, we assume that $f(\theta)$ remains non-negative.

The arc-length of the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ can be obtained directly from the parametric form. We parameterize the curve in Cartesian coordinates as

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta, \quad \alpha \le \theta \le \beta.$$

Here, the parameter is θ. Then

$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta, \quad y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

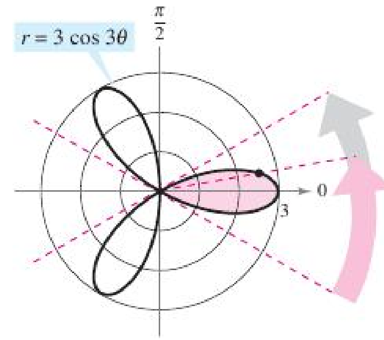
Consequently, $[(x'(\theta)]^2 + [y'(\theta)]^2 = [f(\theta)]^2 + [f'(\theta)]^2$. Therefore, the required length of the arc of the curve is

$$L = \int_\alpha^\beta \sqrt{[(x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{[(f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

Again, our assumption for this formula is that $f'(\theta)$ is continuous on the interval $[\alpha, \beta]$. In summary, for $r = r(\theta), \ \ \alpha \le \theta \le \beta$, the area of the sector and the arc length of the curve are

$$\boxed{\text{Area } = \int_\alpha^\beta r^2\, d\theta, \quad \text{Length } = \int_\alpha^\beta \sqrt{r^2 + (r')^2}\, d\theta}$$

Example 3.21. Find the area of one petal of the rose curve $r = 3\cos(3\theta)$.

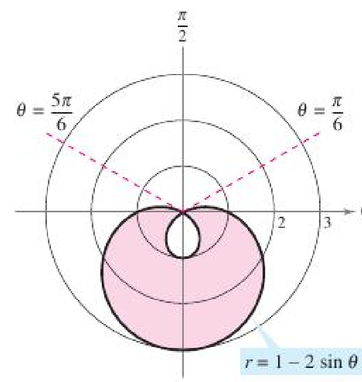


One petal of the rose curve is traversed when θ varies from $-\pi/6$ to $\pi/6$. Thus, the area enclosed by it is

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} [3\cos(3\theta)]^2\, d\theta$$

$$= \frac{9}{4}\int_{-\pi/6}^{\pi/6} (1+\cos(6\theta))\, d\theta = \frac{3\pi}{4}.$$

Example 3.22. Find the area of the region lying between the inner and the outer loops of the limacon $r = 1 - 2\sin\theta$.



The inner loop is traced when θ varies from $\pi/6$ to $5\pi/6$. The area of the region enclosed by the inner loop is

$$A_1 = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1-2\sin\theta)^2\, d\theta$$

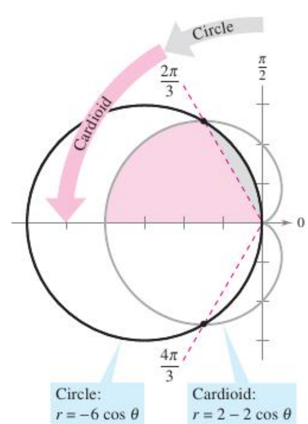
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [3 - 4\sin\theta - 2\cos(2\theta)]\, d\theta = \pi - \frac{3\sqrt{3}}{2}.$$

The outer loop is traced when θ varies from $5\pi/6$ to $13\pi/6$. Thus the area enclosed by the outer loop is

$$A_2 = \frac{1}{2}\int_{5\pi/6}^{13\pi/6} (1-2\sin\theta)^2\, d\theta = 2\pi + \frac{3\sqrt{3}}{2}.$$

Therefore, the required area is $A_2 - A_1 = \pi + 3\sqrt{3}$.

Example 3.23. Find the area of the region common to the interiors of the circle $r = -6\cos\theta$ and the cardioid $r = 2 - 2\cos\theta$.



Both the curves are symmetric about the x-axis. We work with the upper-half plane.

The gray shaded region lies between the circle and the radial line $\theta = 2\pi/3$. The circle has the coordinate $(0, \pi/2)$ at the pole. Thus this region has the area

$$\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos^2\theta)\, d\theta.$$

The region shaded pink is bounded by the radial lines $\theta = 2\pi/3$ and $\theta = \pi$. Thus it has the area

$$\frac{1}{2}\int_{3\pi/6}^{\pi} (2-2\cos\theta)^2\, d\theta.$$

Therefore, the required area is twice the sum of the above areas:

$$A = \int_{\pi/2}^{2\pi/3} (-6\cos^2\theta)\,d\theta + \int_{3\pi/6}^{\pi} (2-2\cos\theta)^2\,d\theta = 5\pi.$$

Example 3.24. Find the length of the arc of the cardioid $r = 2 - 2\cos\theta$ for $0 \le \theta \le 2\pi$.

The length is given by

$$\begin{aligned} L &= \int_0^{2\pi} [(f(\theta))^2 + (f'(\theta))^2]^{1/2}\,d\theta = \int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2}\,d\theta \\ &= \int_0^{2\pi} 2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta = \int_0^{2\pi} 4\sin(\theta/2)\,d\theta = 16. \end{aligned}$$

3.6 Review Problems

Problem 1: Evaluate the limit of $x^{-1}\ln(1+e^x)$ as $x \to \pm\infty$.

$$\lim_{x\to\infty} \frac{\ln(1+e^x)}{x} = \lim_{x\to\infty} \frac{e^x}{(1+e^x)} = \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1.$$

by using L' Hospital's rule and $\lim_{x\to\infty} e^x = \infty$. Similarly,

$$\lim_{x\to-\infty} \frac{\ln(1+e^x)}{x} = \lim_{x\to-\infty} \frac{e^x}{(1+e^x)} = 1.$$

Problem 2: Evaluate $\int \sqrt{a^2-x^2}\,dx$.

$$\int \sqrt{a^2-x^2}\,dx = \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx = a^2\sin^{-1}\frac{x}{a} - \int x\,\frac{x\,dx}{\sqrt{a^2-x^2}}.$$

Integrating the second integral by parts,

$$\int x\,\frac{x\,dx}{\sqrt{a^2-x^2}} = -x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2}\,dx.$$

Putting this result on the earlier equation and simplifying, we get

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C.$$

Problem 3: Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+3x-4}}$.

$$\begin{aligned} \frac{1}{\sqrt{x^2+3x-4}} &= \frac{1}{\sqrt{x+4}\,\sqrt{x-1}} = \frac{\sqrt{x+4}+\sqrt{x+1}}{\big(\sqrt{x+4}+\sqrt{x-1}\big)\sqrt{x+4}\,\sqrt{x-1}} \\ &= \frac{2}{\sqrt{x+4}+\sqrt{x-1}}\Big(\frac{1}{2\sqrt{x+4}} + \frac{1}{2\sqrt{x-1}}\Big) = \big[2\,\ln\big(\sqrt{x+4}+\sqrt{x-1}\,\big)\big]'. \end{aligned}$$

Therefore,

$$\int \frac{dx}{\sqrt{x^2+3x-4}} = 2\ln\left(\sqrt{x+4}+\sqrt{x-1}\right)+C.$$

Problem 4: Evaluate $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin x \sin^{n-1} x\,dx \\ &= -\sin^{n-1} x \cos x\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\,dx \\ &= (n-1)\int_0^{\pi/2} \sin^{n-2} x(1-\sin^2 x)\,dx = (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

That is, $I_n = \dfrac{n-1}{n} I_{n-2}$. Using this as an iteration formula, we have

$$I_{2m} = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{3}{4}\cdot\frac{1}{2}I_0, \quad I_{2m+1} = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{4}{5}\cdot\frac{2}{3}I_1.$$

Moreover, $I_0 = \int_0^{\pi/2} dx = \pi/2$ and $I_1 = \int_0^{\pi/2} \sin x\,dx = 1$. This gives a value for I_n depending on whether n is even or odd.

Problem 5: Compute the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}$.

A plane parallel to yz-plane at a distance x from the origin on the x-axis has equation:

$$\frac{y^2}{\left[b\sqrt{1-\frac{x^2}{a^2}}\right]^2}+\frac{z^2}{\left[c\sqrt{1-\frac{x^2}{a^2}}\right]^2}=1.$$

Its area is $A(x)=\pi\, bc\left(1-\dfrac{x^2}{a^2}\right)$. Thus the required volume is

$$V=\pi\, bc\int_{-1}^{a}\left(1-\frac{x^2}{a^2}\right)dx=\pi\, bc\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^{a}=\frac{4}{3}\pi abc.$$

Problem 6:

Find the length of the arc of the cycloid $x(t)=a(t+\sin t)$, $y(t)=a(1-\cos t)$ for $0\le t\le\theta$.

The required length is given by

$$L(\theta)=\int_0^\theta\left[(x'(t))^2+(y'(t))^2\right]dt=\int_0^\theta\sqrt{2a^2(1+\cos t)}\,dt=4a\sin\frac{\theta}{2}.$$

Problem 7: Find the length of the arc of the parabola $y^2=4ax$ cut by the line $3y=8x$.

The line cuts the parabola at the points $(0,0)$ and $(9a/16,3a/2)$. If $y^2=4ax$, then $2yy'=4a$ gives $(y')^2=(2a/y)^2=4a^2/(4ax)=a/x$. The arc length is

$$L=\int_0^{9a/16}\sqrt{1+(y')^2}\,dx=\int_0^{9a/16}\sqrt{1+a/x}\,dx.$$

With $x=a\tan^2\theta$, $dx=2a\tan\theta\sec^2\theta\,d\theta$, $0\le\theta\le\tan^{-1}(3/4)$,

$$L=\int_0^{\tan^{-1}(3/4)}\sec^3\theta\,d\theta=\Big[a\sec\theta\tan\theta+a\ln|\sec\theta+\tan\theta|\Big]_0^{\tan^{-1}(3/4)}=a\Big(\frac{15}{16}+\ln 2\Big).$$

Problem 8: Find the area of the region enclosed by the loop of Descartes' Folium $x^3+y^3=3axy$.

We cannot express this equation in the form $y=f(x)$. Changing to polar coordinates with $x=r\cos\theta,\ y=r\sin\theta$, the equation reduces to

$$r=f(\theta)=\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}.$$

The loop is traced when θ varies from 0 to $\pi/2$. Thus the required area is

$$A=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}\,d\theta=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta.$$

Substituting $t=\tan\theta$, we have

$$A=\frac{9a^2}{2}\int_0^\infty\frac{t^2\,dt}{1+t^3}=\frac{3a^2}{2}\Big[\frac{-1}{1+t^3}\Big]_0^\infty=\frac{3a^2}{2}.$$

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Index