

# Pi at School

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**Abstract:** In this paper, an attempt has been made to define  $\pi$  by using the greatest lower bound and the least upper bound properties of the real number system. Using this definition, the usual formulas for the perimeter and area of a circle are proved. Though the formulas are well-known, their proofs are not found in text books. This paper tries to fill that gap.

**Keywords:** Circle, Perimeter, Area,  $\pi$

## 1 Introduction

For the first time when a child sees a circular region, she thinks it should have an area since it is a closed figure with a very regular smooth boundary. She is then told that a circle does have an area and a perimeter whose formulas are, respectively,  $\pi r^2$  and  $2\pi r$ . The number  $\pi$  is not rational but can be approximated well, for all practical purposes, to  $22/7$ . She continues with her usual trust in her teachers and guardians, for good. It is natural for a child to raise doubts and her teachers have the responsibility of clearing these. In this case, the doubts are the following:

1. Why does the same number  $\pi$  work for both the perimeter and the area?
2. Why does the same  $\pi$  work for all circles?
3. And what is  $\pi$ ?

The teachers do their best in giving the explanation that it has been found experimentally by taking bigger and bigger circles that the ratio of the perimeter of a circle to its diameter is nearly a constant. Further, they say that this fact will be used to prove the area formula later using techniques from higher mathematics. In smaller circles, there is more error in the measurements and that is evened out with larger and larger radii. We see that the explanation is far from answering the genuine questions of the child. It answers an altogether different question. Further, there are deeper questions unasked.

Length of a straight line is well defined since our measurement tools are straight. As a circle is not straight, its circumference has a length is questionable. A rectangle has an area

is acceptable since measurement of an area uses squares. But whether a circle has an area is again questionable. Our aim, in this paper, is to clear these doubts in a manner that a school student can understand. The only requirement is the concept of a greatest lower bound of a set of real numbers. For generality, we also consider the approach of least upper bounds and finally show that both the approaches lead to the same  $\pi$ .

The concepts are simple enough. Let  $A$  be a non-empty subset of the set of real numbers  $\mathbb{R}$ . A real number  $\alpha$  is a lower bound of  $A$  if  $\alpha \leq x$  for each  $x \in A$ . A real number  $\beta$  is an upper bound of  $A$  if  $\beta \geq x$  for each  $x \in A$ . A real number  $\gamma$  is a greatest lower bound of  $A$  if  $\gamma$  is a lower bound of  $A$  and no number greater than  $\gamma$  is a lower bound of  $A$ . Similarly, a real number  $\delta$  is a least upper bound of  $A$  if  $\delta$  is an upper bound of  $A$  and no number less than  $\delta$  is an upper bound of  $A$ . The two properties of  $\mathbb{R}$  are:

*Greatest Lower Bound Property:*

Each non-empty subset of  $\mathbb{R}$  having a lower bound has a greatest lower bound.

*Least Upper Bound Property:*

Each non-empty subset of  $\mathbb{R}$  having an upper bound has a least upper bound.

In fact, the two properties of  $\mathbb{R}$  are equivalent. Moreover, whenever they exist, the greatest lower bound and the least upper bound are unique. Our approach is similar to defining  $\sqrt{2}$  as the greatest lower bound of the set of all positive rational numbers whose squares are greater than 2, and also as the least upper bound of the set of all positive rational numbers whose squares are less than 2.

In Section 2, we take the simpler approach of using the greatest lower bounds for defining the perimeter and area of a circle and the number  $\pi$ . Next, we derive the well known formulas for the perimeter and area of a circle. In Section 3, we redefine  $\pi$  taking the polygons inscribed in a circle, and prove that both the approaches lead to the same  $\pi$ . Section 4 concludes the paper.

## 2 Simpler $\pi$

We use the words *circumference* for the path itself and *perimeter* for the length of the circumference. Since the sides of a polygon are straight line segments, the circumference of a polygon has length. Since squares and rectangles have well defined areas, parallelograms and triangles also have areas. Again, polygons can be triangularized, and thus have areas. We use the perimeters and areas of polygons to define the perimeter and area of a circle.

Let  $C$  be a circle of radius  $r > 0$ . Denote by  $\ell(P)$  the perimeter and by  $\mathbb{k}(P)$  the area of any polygon  $P$ . Consider the sets

$$\mathcal{P}(C) = \{\ell(P) : P \text{ is a polygon circumscribing } C\},$$

$$\mathcal{A}(C) = \{\mathbb{k}(P) : P \text{ is a polygon circumscribing } C\}.$$

Both  $\mathcal{P}(C)$  and  $\mathcal{A}(C)$  are non-empty and have zero as a lower bound. By the greatest lower bound property of  $\mathbb{R}$ , there exist greatest lower bounds of the sets  $\mathcal{P}(C)$  and  $\mathcal{A}(C)$ .

**Definition 2.1.** *Let  $C$  be a circle of radius  $r > 0$ . The perimeter of  $C$  is the greatest lower bound of  $\mathcal{P}(C)$ . The area of  $C$  is the greatest lower bound of  $\mathcal{A}(C)$ .*

Due to Definition 2.1, the perimeter and area of any circle are real numbers. Notice that we neglect the units. First of all, we want to show that the perimeter and area of a circle do not depend on the centre.

**Theorem 2.1.** *All circles with radii of the same length have equal perimeters and equal areas.*

*Proof.* Let  $C, C'$  be circles with centres  $O, O'$ , each of radius  $r > 0$ . Let  $P$  be a polygon that circumscribes  $C$ . Suppose  $P$  touches  $C$  at the points  $A_1, A_2, \dots, A_n$ . Construct the radii  $OA_1, OA_2, \dots, OA_n$ . Mark a point on the circle  $C'$  as  $B_1$ . Construct the radius  $O'B_1$ . Construct the radii  $O'B_2, O'B_3, \dots, O'B_n$  so that  $\angle B_1O'B_2 = \angle A_1OA_2, \dots, \angle B_{n-1}O'B_n = \angle A_{n-1}OA_n$ , where the points  $B_1, B_2, \dots, B_n$  are on the circumference of  $C'$ . Draw tangents to  $C'$  at these points. This gives rise to a polygon  $P'$  that circumscribes  $C'$  and is congruent to  $P$ . Then,  $\ell(P) = \ell(P')$  and  $\mathbb{k}(P) = \mathbb{k}(P')$ .

Conversely, given any polygon  $P'$  circumscribing  $C'$ , we can construct a congruent polygon  $P$  that circumscribes  $C$ , in a similar way. Here also,  $\ell(P') = \ell(P)$  and  $\mathbb{k}(P') = \mathbb{k}(P)$ .

Therefore,  $\mathcal{P}(C) = \mathcal{P}(C')$  and  $\mathcal{A}(C) = \mathcal{A}(C')$ . Taking greatest lower bounds, we see that perimeters of  $C, C'$  are equal, and so are the areas of  $C$  and  $C'$ .  $\square$

Thus it makes sense to talk of *the* perimeter and *the* area of a circle of radius one (a unit circle) no matter wherever in the plane it is drawn.

**Theorem 2.2.** *The perimeter of a unit circle is twice its area.*

*Proof.* Let  $O$  be the centre of a unit circle  $S$ . Let  $P$  be a polygon that circumscribes  $S$ . Consider two consecutive vertices of  $P$ , say,  $A$  and  $B$ . Suppose the side  $AB$  touches  $S$  at  $D$ . See Figure 1. Since  $OD$  is perpendicular to  $AB$  and  $OD = 1$ ,  $AB$  equals twice the area of the triangle  $OAB$ . Summing over all such triangles  $OAB$ , we see that  $2\mathbb{k}(P) = \ell(P)$ . Therefore,

$$\mathcal{P}(C) = \{2\mathbb{k}(P) : P \text{ is a polygon circumscribing } C\}.$$

By taking greatest lower bounds of these sets, we see that the perimeter of  $S$  is twice its area.  $\square$

**Definition 2.2.** *The real number  $\pi$  is the area of a unit circle.*

Due to Theorem 2.2,  $\pi$  is also equal to half the perimeter of a unit circle. We next derive the well used formulas for the perimeter and area of a circle of any radius.

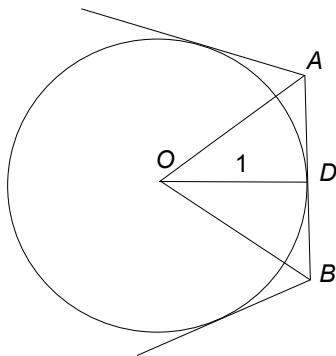


Figure 1: Figure for Theorem 2.2

**Theorem 2.3.** *Let  $C$  be a circle of radius  $r > 0$ . Then the perimeter of  $C$  is  $2\pi r$  and the area of  $C$  is  $\pi r^2$ .*

*Proof.* Let  $C$  be a circle with centre  $O$  and radius  $r > 0$ . Due to Theorem 2.1, we take a unit circle  $S$  having centre as  $O$ . Let  $P$  be a polygon that circumscribes  $C$ , touching  $C$  at  $A_1, A_2, \dots, A_n$ . Construct the radii  $OA_1, OA_2, \dots, OA_n$ . Let these radii intersect  $S$  at the points  $B_1, B_2, \dots, B_n$ . Construct tangents to  $S$  at the points  $B_1, B_2, \dots, B_n$ . These tangents are parallel to the sides of the polygon  $P$  and they make a polygon  $Q$  that circumscribes  $S$ . Moreover,  $Q$  is similar to  $P$ . Since the radii  $OA_i$  and  $OB_i$  are in the ratio  $r : 1$ , we have  $\ell(P) = r\ell(Q)$  and  $\mathbb{k}(P) = r^2\mathbb{k}(Q)$ .

Conversely, starting with any polygon  $Q$  that circumscribes  $S$ , a polygon  $P$  that circumscribes  $C$  can also be constructed so that  $\ell(P) = r\ell(Q)$  and  $\mathbb{k}(P) = r^2\mathbb{k}(Q)$ . It follows that

$$\mathcal{P}(C) = \{r\ell(Q) : Q \text{ is a polygon circumscribing } S\},$$

$$\mathcal{A}(C) = \{r^2\mathbb{k}(Q) : Q \text{ is a polygon circumscribing } S\}.$$

Taking greatest lower bounds, we see that perimeter of  $C$  is  $2\pi r$  and area of  $C$  is  $\pi r^2$ .  $\square$

### 3 Redefining $\pi$

That is enough for  $\pi$  at school. However, the approach lacks generality. For example, the length of a curve does not require rectifiability; only bounded variation suffices. This means, for circles, instead of defining perimeter and area via circumscribing polygons, we must try inscribed polygons. This approach requires some basic results from geometry, which we want to clear off now.

**Proposition 3.1.** *If the equal sides  $AB$  and  $AC$  of an isosceles triangle  $ABC$  are extended to  $AD$  and  $AE$ , then  $DE \geq BC$ .*

*Proof.* If the extension is trivial, i.e., when the points  $D$  and  $E$  coincide with  $B$  and  $C$ , respectively, then  $DE = BC$ . Otherwise, join  $DE$ . Without loss of generality, suppose  $BD < CE$ . Draw a line  $DF$  parallel to  $BC$ , and let it intersect  $CE$  at  $F$ . See Figure 2.

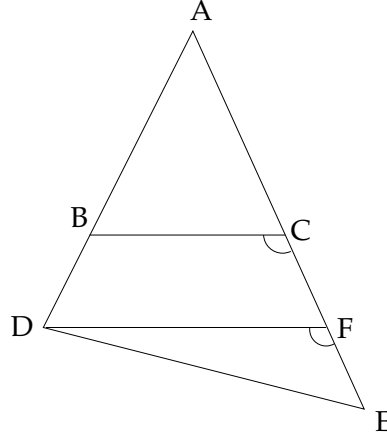


Figure 2: Figure for Proposition 3.1

The angle  $\angle DFE = \angle BCF = \angle ABC + \angle BAC > \frac{1}{2}(\angle ABC + \angle ACB) + \frac{1}{2}\angle BAC$ , a right angle. Hence  $\angle DFE$  is the biggest angle in the  $\triangle DEF$ . Its opposite side  $DE$  is therefore biggest of the three. That is,  $DE > DF$ . Since  $ADF$  is also an isosceles triangle and  $AD > AB$ , we have  $DF > BC$ . To sum up,  $DE > DF > BC$ .  $\square$

**Proposition 3.2.** *Any polygon inscribed in a circle has smaller perimeter and smaller area than that of the equilateral triangle that circumscribes the circle.*

*Proof.* Let  $C$  be a circle,  $P$  a polygon inscribed in  $C$ , and  $T$  an equilateral triangle that circumscribes  $C$ . Draw radii of the circle  $C$  joining the centre of  $C$  and the vertices of  $P$ . Extend these radii to intersect the boundary of  $T$ . For notational convenience, take two consecutive vertices of  $P$  as  $A$  and  $B$ . Extend the line segments  $OA$  and  $OB$  to meet the sides of  $T$  at  $D$  and  $E$ , respectively. This includes the degenerate cases that  $D$  and  $A$  coincide, and/or  $E$  and  $B$  coincide. There are two cases to consider:

- (a) The points  $D$  and  $E$  are on the same side of  $T$ .
- (b) The points  $D$  and  $E$  are on the two adjacent sides of  $T$ .

In the case (a),  $\triangle OAB$  is an isosceles triangle and the sides are extended to the points  $D$  and  $E$ . By Proposition 3.1,  $DE \geq AB$ . See Figure 3.

In the case (b), let the vertex of the triangle  $T$  common to both the adjacent sides be  $F$ . In the triangle  $DFE$ ,  $DF + FE > DE$ . But  $DE \geq AB$ , by Proposition 3.1. Thus, corresponding to the radii  $OA$  and  $OB$ , we see that the side  $AB$  of the polygon  $P$  is less than or equal to the lengths of the corresponding line segments on  $T$ .

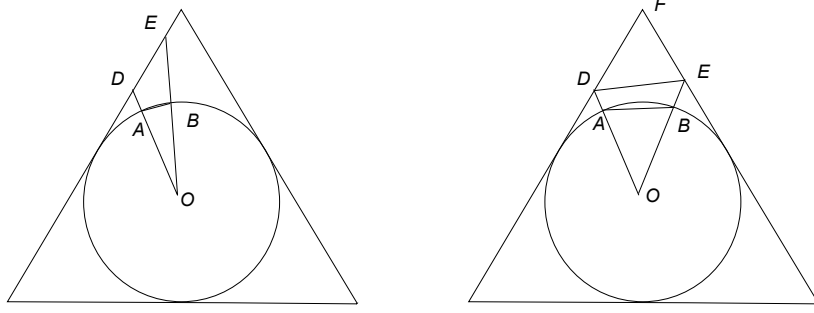


Figure 3: Figure for Proposition 3.2

Considering all the radii of the circle as drawn earlier, and taking sum over all pairs of consecutive radii forming the sides of the polygon  $P$ , we find that the perimeter of  $P$  is less than or equal to the perimeter of the triangle  $T$ . This proves (1). Proof of (2) is trivial since  $P$  is contained in  $T$ .  $\square$

Let  $C$  be a circle of radius  $r > 0$ . Consider the sets

$$\mathcal{L}(C) = \{\ell(P) : P \text{ is a polygon inscribed in } C\}$$

$$\mathcal{K}(C) = \{\mathbb{k}(P) : P \text{ is a polygon inscribed in } C\}$$

Due to Proposition 3.2,  $\ell(P) \leq 6\sqrt{3}r$ , the perimeter of  $T$ , and  $\mathbb{k}(P) \leq 3\sqrt{3}r^2$ , the area of  $T$ , where  $T$  is an equilateral triangle circumscribing  $C$ . That is, both the sets  $\mathcal{L}(C)$  and  $\mathcal{K}(C)$  have upper bounds. By the least upper bound property of  $\mathbb{R}$ , there exist least upper bounds of the sets  $\mathcal{L}(C)$  and  $\mathcal{K}(C)$ .

**Definition 3.1.** Let  $C$  be a circle of radius  $r > 0$ . Perimeter of  $C$ , denoted by  $\ell(C)$ , is the least upper bound of  $\mathcal{L}(C)$ . Area of  $C$  is the least upper bound of  $\mathcal{K}(C)$ , and is denoted by  $\mathbb{k}(C)$ .

Due to Definition 3.1, the perimeter and area of any circle are real numbers. Once again, Theorem 2.1 holds true even with Definition 3.1.

**Theorem 3.3.** If radii of two circles  $C, C'$  are equal, then  $\ell(C) = \ell(C')$  and  $\mathbb{k}(C) = \mathbb{k}(C')$ .

*Proof.* Proof is similar to that of Theorem 2.1.  $\square$

In particular, all unit circles have the same perimeter and the same area. Further, the connection of area to perimeter of a unit circle as stated in Theorem 2.2 also holds.

**Theorem 3.4.** Let  $S$  be a unit circle. Then  $\ell(S) = 2\mathbb{k}(S)$ .

*Proof.* Let  $S$  be a unit circle with centre at  $O$ . Consider a polygon  $P$  inscribed in  $S$ . Join the centre  $O$  to the vertices of  $P$  forming radii of length one each. Let  $A, B$  be two consecutive vertices of  $P$ . Draw a perpendicular  $OD$  to  $AB$ ,  $D$  being a point on  $AB$ . See Figure 4.

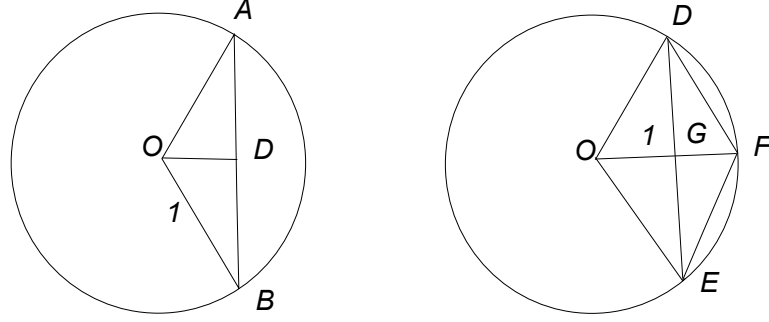


Figure 4: Figure for Theorem 3.4

Since  $OD < OB = 1$ , we find that  $2\mathbb{k}(\triangle OAB) = AB \cdot OD < AB$ . Summing over all such  $\triangle OAB$  in  $P$ , we obtain:  $2\mathbb{k}(P) < \ell(P)$ . Since for each inscribing polygon  $P$ , we have  $\ell(P) \leq \ell(S)$ , this yields the inequality  $2\mathbb{k}(S) \leq \ell(S)$ .

For the other inequality, consider any polygon  $P$  inscribed in  $S$ , a unit circle with centre  $O$ . Join  $O$  to the vertices of  $P$  forming radii of length one each, as earlier. Construct angle bisectors of each pair of consecutive radii forming new radii. Consider all the new points so obtained on  $S$  along with the vertices of  $P$ . They form a new polygon  $P'$  inscribed in  $S$ .

Let  $D, E$  be two consecutive vertices of  $P$ . Let  $F$  be the vertex of  $P'$  mid-way between  $D$  and  $E$  on  $S$ . The points  $D, F, E$  are now three consecutive vertices of  $P'$ . See Figure 4. Let  $G$  be the point of intersection of  $DE$  and  $OF$ . Now,  $\angle OGD$  is a right angle. Since  $OG < OF = 1$ , we see that  $DE = 2DG = 2DG \cdot OF = 2\mathbb{k}(OCDE)$ . Summing over all the radii of  $P$ , we obtain:  $\ell(P) = 2\mathbb{k}(P') \leq 2\mathbb{k}(S)$ . It then follows that  $\ell(S) \leq 2\mathbb{k}(S)$ .  $\square$

We use a unit circle for defining  $\pi$  as earlier. Due to the new definition of perimeter and area of a circle, we use the symbol  $\pi_u$ .

**Definition 3.2.** *The real number  $\pi_u$  is equal to  $\mathbb{k}(S)$ , where  $S$  is a unit circle.*

Due to Theorem 3.4,  $\pi_u$  is also equal to half of  $\ell(S)$  for a unit circle  $S$ . We next derive the well used formulas for the perimeter and the area of a circle of any radius.

**Theorem 3.5.** *Let  $C$  be a circle of radius  $r > 0$ . Then  $\ell(C) = 2\pi_u r$  and  $\mathbb{k}(C) = \pi_u r^2$ .*

*Proof.* Let  $C$  be a circle of radius  $r > 0$  with centre  $O$ . Due to Theorem 3.3, take a unit circle  $S$  with centre  $O$ . Let  $P$  be a polygon inscribed in  $S$ . Construct the radii joining  $O$  to the vertices of  $P$ . Let  $Q$  be the polygon inscribed in  $C$  formed by the points of intersection of these radii (extended, if necessary) with  $C$ .

For notational convenience, Let  $A, B$  be two consecutive vertices of  $P$  and let  $A', B'$  be the corresponding vertices of  $Q$ . Since triangles  $OAB$  and  $OA'B'$  are similar,  $A'B' = r \times AB$  and  $\mathbb{k}(\triangle ODE) = r^2 \mathbb{k}(\triangle OAB)$ . Summing over all such triangles we find that  $\ell(P) = r \ell(Q)$  and  $\mathbb{k}(P) = r^2 \mathbb{k}(Q)$ .

Conversely, starting from any polygon  $Q$  inscribed in  $C$ , a similar construction shows that there exists a polygon  $P$  inscribed in  $S$  such that  $r \ell(P) = \ell(Q)$  and  $r^2 \mathbb{k}(P) = \mathbb{k}(Q)$ .

Therefore,

$$\mathcal{L}(C) = \{r \ell(P) : P \text{ is a polygon inscribed in } S\},$$

$$\mathcal{K}(C) = \{r^2 \mathbb{k}(P) : P \text{ is a polygon inscribed in } S\}.$$

Taking least upper bounds, we see that  $\ell(C) = r \ell(S)$  and  $\mathbb{k}(C) = r^2 \mathbb{k}(S)$ . Due to Definition 3.2,  $\ell(C) = 2\pi_u r$  and  $\mathbb{k}(C) = \pi_u r^2$ .  $\square$

Thus, we see that a more general approach for defining  $\pi$  using the inscribed circles can also be taken. Whatever be the approach, by defining the radian unit for measuring angles as “in a unit circle, an arc length of  $\theta$  subtends an angle of  $\theta$  radian at the centre” the arc length and area of a sector with an angle of  $\theta$  radian at the centre of a circle of radius  $r$  turn out to be  $r\theta$  and  $r^2\theta/2$ , respectively. The radian measure of the angle contained in a circle is thus  $2\pi$  (or,  $2\pi_u$ ). The order formulas of trigonometric ratios for small arguments can then be derived.

**Proposition 3.6.** For  $0 < \theta < 1/2$ ,  $\tan \theta - \sin \theta < 2\theta^3$ .

*Proof.* Let  $A, B$  be two points on the unit circle  $S$  that subtend an angle  $\theta$  at the centre  $O$  of  $S$ . Let the tangent at  $A$  meet the extended line  $OB$  at  $C$ . Draw  $AD$  perpendicular to  $OB$ . See Figure 5. The arc  $AB = \theta$ . Write  $p = AD$ . Then  $0 < p < \theta$ ,  $\tan \theta = p/\sqrt{1-p^2}$ , and  $\sin \theta = p$ .

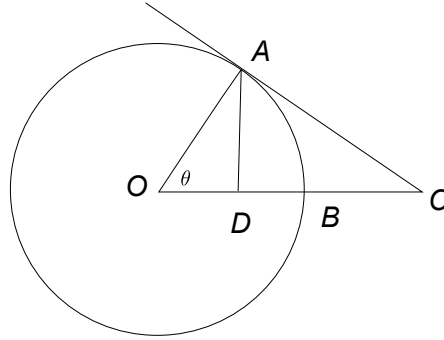


Figure 5: Figure for Proposition 3.6

Now,

$$\tan \theta - \sin \theta < 2\theta^3$$

$$\text{if } 1/\sqrt{1-p^2} < 1 + 2p^2$$

$$\text{if } 1 < (1 + 2p^2)^2(1 - p^2) = 1 + 4p^2 + 4p^4 - p^2(1 + 4p^2 + 4p^4)$$

$$\text{if } 0 < 4 + 4p^2 - 1 - 4p^2 - 4p^4$$

$$\text{if } 4p^4 < 3.$$

This is the case since  $0 < p < 1/2$ .  $\square$



**Theorem 3.7.**  $\pi = \pi_u$ .

*Proof.* Since angles can be bisected repeatedly, we restrict to the angles  $\pi/2^k$ , and thereby, to regular polygons with number of sides  $2^k$ . Let  $S$  be a unit circle. Let  $P$  be a regular polygon of  $2^n$  sides that circumscribes  $S$ . If  $A_i$  and  $A_{i+1}$  are two consecutive points where  $P$  touches  $S$ , then draw a radius  $OB_i$  bisecting the angle  $A_iOA_{i+1}$ . The points  $A_1, B_1, A_2, B_2, \dots, A_{2^n}, B_{2^n}$  form the regular polygon  $Q$  of  $2^{n+1}$  sides inscribed in  $S$ . We estimate the difference between the areas of  $P$  and  $Q$ .

See Figure 5, where  $A$  and  $B$  are two consecutive vertices of  $Q$ . We observe that

$$\begin{aligned} \mathbb{k}(P) - \mathbb{k}(Q) &= 2^{n+1} \times (\mathbb{k}(\triangle AOC) - \mathbb{k}(\triangle AOB)) \\ &= 2^{n+1}(\tan \theta - \sin \theta)/2, \text{ where } \theta = \pi/2^{n+1} \\ &< \pi^3/2^{2n+2} \end{aligned}$$

Since given  $\epsilon > 0$ , a suitable  $n$  can be chosen so that  $\pi/2^{2n+2} < \epsilon$ , the last estimate thus says that given any  $\epsilon > 0$ , there exist numbers  $\alpha = \mathbb{k}(P) \in \mathcal{K}(S)$  and  $\beta = \mathbb{k}(Q) \in \mathcal{A}(S)$  such that  $\alpha - \beta < \epsilon$ . Thus the greatest lower bound of  $\mathcal{K}(S)$  coincides with the least upper bound of  $\mathcal{A}(S)$ . Therefore,  $\pi = \pi_u$ .  $\square$

## 4 Conclusions

In this paper, we have introduced four approaches to defining  $\pi$ . The approaches show, as a side result, that the same  $\pi$  works for perimeter as well as area of a circle giving rise to the well known formulas as  $2\pi r$  and  $\pi r^2$ , respectively. Though, for school children, the greatest lower bound approach using the circumscribing polygons of Section 2 is sufficient, we also took the more general approach of inscribed polygons. The fact that both the approaches lead to the same  $\pi$  uses the estimate  $\tan \theta - \sin \theta < 2\theta^3$  for small  $\theta$ . These results, in turn, depend on the sector formulas that in a unit circle, the area of a sector with an angle  $\theta$  is  $\theta/2$  and the length of the arc that subtends an angle  $\theta$  at the centre is  $\theta$ . We ask the reader to prove these results rigorously following a similar approach.