Local Properties

1 Cauchy's integral formula

We begin with a simple lemma:

Lemma 1. *Let $a \in \mathbb{C}$ be fixed and let A be a set of finite measure. Then $\frac{1}{z-a}$ is integrable on A.*

Proof. Let $B := \{z \in A : |z-a| \geqslant 1\}$. Then

$$\int_B \frac{dm(z)}{|z-a|} \leqslant \int_B dm(z) \leqslant m(A) < \infty.$$

On the other hand,

$$\int_{A\setminus B} \frac{dm(z)}{|z-a|} \leqslant \int_{|z-a|\leqslant 1} \frac{dm(z)}{|z-a|} = \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{r} = 2\pi.$$

□

We now prove the Cauchy–Green formula. An important consequence of the Cauchy–Green formula is Cauchy's integral theorem.

Theorem 2 (Cauchy–Green formula). *Let $U \subset \mathbb{C}$ be a bounded domain with piece-wise regular boundary given positive orientation. Let f be a $\mathbb{R}$-differentiable function on some neighbourhood of $\overline{U}$ and $\overline{\partial} f$ be continuous on $\overline{U}$. Then for each $z \in U$, we have:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(w)}{w-z} dw - \frac{1}{\pi} \int_U \frac{\overline{\partial} f(w)}{w-z} dm(w)$$

Proof. THe proof is an application of Green's theorem on the form $\rho = \frac{f(w)}{w-z} dw$. However, this form has a singularity at the point $w = z$. To deal with this issue, we delete from U a small disk $\overline{D}(z,\varepsilon)$ where ε is chosen so small that the closed disk is fully-contained on U and set $U_\varepsilon := U \setminus \overline{D}(z,\varepsilon)$. Observe that U_ε is also domain with piece-wise regular boundary and if we orient $C(z,\varepsilon)$ with the clockwise orientation, then ∂U_ε would have partial orientation.

Write ρ as $\frac{f(w)}{w-z}dx + i\frac{f(w)}{w-z}dy$, then rho is of the form $Pdx + Qdy$ where $P = \frac{f(w)}{w-z}$ and $Q = \frac{f(w)}{w-z}$ and thus

$$Q_x - P_y = \frac{\partial}{\partial x}\left(i\frac{f(w)}{w-z}\right) - \frac{\partial}{\partial y}\left(\frac{f(w)}{w-z}\right) = 2i\frac{\partial f}{\partial \overline{w}} = \frac{1}{2i}\frac{\partial f}{\partial \overline{w}}\frac{1}{w-z}.$$

□

The RHS of the above is continuous function on $\overline{U}_\varepsilon$. Thus we can apply Green's theorem to conclude that

$$\int_{\partial U_\varepsilon} \rho = 2i \int_{U_\varepsilon} \frac{\partial f}{\partial \overline{w}} \frac{1}{w-z} dm(w).$$

Now,

$$\begin{aligned} \int_{\partial U_\varepsilon} rho &= \int_{\partial U} + \int_{C(z,\varepsilon)} \rho \\ &= \int_{\partial U} \frac{f(w)}{w-z} dw - i \int_0^{2\pi} f(z+\varepsilon^{i\theta}) d\theta, \end{aligned}$$

and this converges to $\int_{\partial U} \frac{f(w)}{w-z} - 2\pi i f(z)$ as $\varepsilon \to 0$.

The function $w \mapsto \frac{1}{w-z}$ is integrable on U and therefore so is $\frac{\partial f}{\partial \overline{w}} \frac{1}{w-z}$. An application of the dominated convergence theorem gives that

$$\int_{U_\varepsilon} \frac{\partial f}{\partial \overline{w}} \frac{1}{w-z} \to \int_U \frac{\partial f}{\partial \overline{w}} \frac{1}{w-z}$$

which proves the result.

Corollary 3. *Let $U \subset \mathbb{C}$ be a bounded domain with piece-wise regular boundary given positive orientation. Let f be holomorphic in a neighbourhood of $\overline{U}$. Then for each $z \in U$:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(w)}{w-z} dm(w).$$

2 Holomorphic functions are analytic

We now come to the proof of one of the central facts of the subject. Complex-analytic and holomorphic are equivalent concepts.

Theorem 4. *Let $f \in H(D(a,R))$ and let*

$$c_n := \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(w)}{(w-z)^{n+1}} dw, \quad r < R.$$

Then the c_n's are independent of r and the power series $\sum c_n (z-a)^n$ converges to $f(z)$ for $z \in D(a,R)$.

Proof. For $r < R$ and $|z-a| < r$, Cauchy's integral formula gives:

$$f(z) = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(w)}{w-z} dw.$$

We will now expand the Cauchy kernel $\frac{1}{z-w}$ as a power series about the point a. This is easily done as we did for proving that rational functions are analytic

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{\left(\frac{w-a}{z-a} - 1\right)(z-a)} = -\sum \frac{(w-a)^n}{(z-a)^{n+1}}.$$

Now fix z with $|z-a| < r$. Then $|f(w)| \cdot \frac{|z-a|^n}{|w-a|^{n+1}} \leqslant |f(w)||\zeta|^n$ where $|\zeta| < 1$ (independent of w) and therefore by the Weirstrass M-test, it follows that $\sum f(w)\frac{(z-a)^n}{(w-a)^{n+1}}$ converges uniformly on $C(a,r)$ to $\frac{f(w)}{w-z}$. Thus, we can interchange the integral and summation in the following:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{C(a,r)} \sum f(w)\frac{(z-a)^n}{(w-a)^{n+1}} = \sum \int_{C(a,r)} f(w)\frac{(z-a)^n}{(w-a)^{n+1}}dw \end{aligned}$$

This shows that the power series $\sum c_n(z-a)^n$ converges to $f(z)$ on $D(a,r)$. Any two power series development of f around the point a must be equal. This shows that the coefficients c_n are independent of r. By taking $r \to R$, it follows that the radius of convergence of $\sum c_n(z-a)^n$ is at least R. □

This proves that holomorphic functions are complex-analytic and hence infinitely differentiable in both the real and complex sense. Another consequence of the above theorem is the following extension of Cauchy's integral formula.

Corollary 5. *Let $f \in H(D(a,R))$ then for each $0 < r < R$, we have*

$$f^{(n)}(a) = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(w)}{(w-a)^{n+1}}.$$

Proof. This follows immediately from Taylor's theorem and comparing coefficients. □

The following theorem can be viewed as a converse of Cauchy's theorem.

Theorem 6 (Morera). *Let f be a continuous function on the open set U. Then $f \in H(U)$ iff $\int_{\partial\triangle} f(z)dz = 0$ for any closed $\triangle \subset U$.*

Proof. One implication follows from Cauchy's theorem. On the other hand, if it is true that $\int_{\partial\triangle} f(z)dz = 0$ for any triangle, then f locally has anti-derivatives. This means that f is holomorphic. □

3 The principle of analytic continuation

We will now study some of the aspects of holomorphic functions that arise out of complex-analyticity. One main distinguishing feature of analytic functions as opposed to $\mathcal{C}^\infty$ functions is that analytic functions are more rigid. This is made precise in this section. We first define the order of a holomorphic function at a point.

Lemma 7. *Let $U \subset \mathbb{C}$ be a domain, $a \in U$ and let $f \in H(U)$. Then the following conditions are all equivalent:*

(i) $f^{(n)}(a) = 0 \ \forall n$.
(ii) $f(z) = 0$ *in a neighbourhood of* a.
(iii) $f \equiv 0$ *on* U.

Proof. Obviously $(c) \implies (b), (b) \implies a$. It suffices to prove that $(a) \implies (c)$. To do this we use a connectedness argument. Let

$$A := \{z \in U : f(z) = 0 \text{ in some neighbourhood of } z\}.$$

Clearly, A is an open set. Let $z_m \in A$ be such that $z_m \to z \in U$. Then $f^{(n)}(z_m) = 0$ by the definition of A and consequently by passing to limits, $f^{(n)}(z) = 0$. But this means that the Taylor expansion of f centred at 0 has all coefficients 0 and this means that f vanishes on some disk around z and hence $z \in A$. This proves that A is closed in U and consequently $A = U$ proving that $(a) \implies (c)$. □

Definition 8. Let $U \subset \mathbb{C}$ be a domain and $f \in H(U)$. Suppose $f \not\equiv 0$ and $f(a) = 0$ $a \in U$. Then we can find a smallest $m > 0$ such that $f^{(m)} \neq 0$. This m is called the *order of* f *at* a. We also say that f vanishes to order m at the point a.

If f vanishes to order m at the point a then it follows that the Taylor series of f around the point a is of the form

$$f(z) = c_m(z-a)^m + \ldots, \; c_m \neq 0.$$

Thus, we can write

$$f(z) = (z-a)^m g(z),$$

in a neighbourhood of a, where $g(z) = c_m + c_{m+1}(z-a) + \ldots$. If for some z, the series above converges then so does the series for f as $(z-a)^n$ is an analytic function with radius of convergence ∞. Conversely, if $f(z)$ converges for some z then so does the series for $g(z)$. This shows that the function g is analytic and therefore holomorphic on the disk of convergence $D(a, R)$. Moreover, shrinking this disk, we may assume that $D(a, R) \subset U$ and that g does not take the value 0 in the disk $D(a, R)$. Now, define

$$Z(f) := \{z \in U : f(z) = 0\},$$

If $f \not\equiv 0$ then the above shows that around any point $a \in Z(f)$, we can find a disk around which $f(z) = (z-a)^n g(z)$ and g has no zeroes in $D(a, R)$. This proves that on $D(a, R)$ the function f takes the value 0 only at a. Consequently we have proved the

Theorem 9. *Let* f *be holomorphic and not identically* 0 *on a domain* U. *Then* $Z(f)$ *is a closed and discrete subset of* U. *Consequently, on any compact subset of* U, f *has only finitely many zeroes and only countably many zeroes on* U.

We can summarize everything we have established above with the following theorem.

Theorem 10 (Principle of analytic continuation). *Let* f, g *be holomorphic on a domain* U. *Then* $f \equiv g$ *iff any one of the following equivalent conditions holds:*

1. *For some point* $a \in U$, $f^{(n)}(a) = g^{(n)}(a), n = 0.1, \ldots$.
2. *For some indiscrete set* $A \subset U$, $f|_A = g|_A$.
3. *There is an open set* $V \subset U$ *such that* $F|_U = G|_V$.

We now give an application of the above results.

Definition 11. A domain $U \subset \mathbb{C}$ is said to be *symmetric about the real-axis* if $\bar{z} \in U$ whenever $z \in U$.

Theorem 12 (Schwarz reflection principle). *Let* $U \subset \mathbb{C}$ *be symmetric about the real-axis and* $f \in H(U)$ *and* f *takes real values on* $\mathbb{R} \cap U$*. Then* $f(\bar{z}) = \overline{f(z)}$*.*

Proof. Let $g(z) = \overline{f(\bar{z})}$. It is easy to see that g is also holomorphic on U. But on $\mathbb{R} \cap U$, g agrees with $\bar{f}$ and consequently by the principle of analytic continuation $g \equiv f$. □

4 The open mapping theorem

Let $f : U \to \mathbb{C}$ be holomorphic and nowhere vanishing. We have shown that f has a continuous branch of the logarithm iff f'/f has an anti-derivative. We have also proved that on any star-shaped domain, any holomorphic function has an antiderivative. But from the results above, f'/f is a holomorphic function and consequently we get:

Proposition 13. *On a convex open set, any nowhere vanishing holomorphic function admits a continuous branch of the logarithm and therefore a continuous branch of the argument. This implies that the function also admits a branch of the* n*-th root function. Every holomorphic function on a domain locally admits a continuous branch of logarithm, argument and* n*-th root.*

Now we will study the local behaviour of holomorphic functions. Suppose $f(a) = b$ and the function $f(a) - b$ vanishes to order m at a. This means that

$$f(z) - b = (z - a)^m g(z),$$

in a disk $D := D(a, R)$ and g has no zeroes in D. From the previous proposition, we can find a function $h \in H(D)$ such that $h^m = g$. This means that

$$f(z) = ((z - a)h(z))^m := f_1(z)^m. \tag{4.1}$$

We see that the function $f_1(z)$ satisfies $f_1(a) = 0$ and $f_1'(a) = h(a) \neq 0$. By the inverse function theorem, it follows that f_1 is a local-homeomorphism. We can also see this directly. By translating and scaling, we may assume that $a = 0$ and $g_1'(0) = 1$. Thus the power-series expnasion of f_1 is of the form

$$z + f_2(z),$$

where $f_2'(0) = 0$. Fix $0 < \varepsilon < 1$, we can find a $r < R$ such that for $z \in D(0, r), |f_2'(z)| < \varepsilon$. For $z, w \in D(0, r)$, by the fundamental theorem of complex calculus, we see that

$$|f_2(w) - f_2(z)| \leqslant \varepsilon |z - w|.$$

The above inequality shows that

$$(1 - \varepsilon)|z - w| \leqslant |f_1(z) - f_1(w)| \leqslant (1 + \varepsilon)|z - w|, \quad z, w \in D(0, r).$$

This shows that the map f_1 is injective on $D(0, r)$. We will now show that every ζ with $|\zeta|$ sufficiently small has a preimage in $D(0, r)$. If $f_1(z) = \zeta$ then z is a fixed point of $\zeta - f_2(z)$.

We define a sequence as follows: let $z_0 = 0$ and $z_{n+1} = \zeta - f_2(z_n)$. Now, $|z_{n+1} - z_n| = |f_2(z_n) - f_2(z_{n-1})| \leqslant \varepsilon|z_n - z - n - 1|$ and we get

$$|z_{n+1} - z_n| \leqslant \varepsilon^n |z_1 - z_0| = \varepsilon^n |\zeta|.$$

This shows that the sequence (z_n) is Cauchy and therefore converges Now,

$$|z_n| \leqslant \sum_{i=1}^{n} |z_i - z_{i-1}| \leqslant |\zeta| \frac{1}{1-\varepsilon}.$$

If $|\zeta| leq r(1-\varepsilon)$, it follows that we can find $z \in D(0, r)$ such that $f_1(z) = \zeta$. This proves that f_1 is a local-homeomorphism of some neighbourhood V of 0 onto a disk $D(0, s)$. The inverse map f^{-1} is automatically a holomorphic map by previous results. From (4.1), it follows that the image $f(V)$ is exactly $D(b, s^m)$. This proves the

Theorem 14 (Local mapping theorem). *Let f be holomorphic in neighbourhood of the point $a \in \mathbb{C}$ and suppose that $f(a) = b$ with multiplicity m. Then we can find a neighbourhood V of a and a disk $D(b, \delta)$ such that $f(v) = D(b, \delta)$ and furthermore each point $\zeta \in D(b, \delta) \setminus \{b\}$ has exactly m-preimages under f in V.*

Remark 15. f is locally injective at a iff $f'(a) \neq 0$. Only one implication is true for real variable functions and this implication follows from the inverse function theorem. Note that the function e^z has non-vanishing derivative at each point of $\mathbb{C}$ but is not injective globally.

Observe from the proof that $\delta = s^m$ and

Theorem 16 (The open mapping theorem). *Let $u \subset \mathbb{C}$ be a domain and let $f \in H(U)$ be non-constant. Then f is an open map.*

Proof. Let $V \subset U$ be open. Then f has finite multiplicity at each point of V. From the previous theorem it follows that $f(V)$ contains some disk $D(b, \delta)$. □

5 Maximum principle and applications

Theorem 17 (Maximum principle). *If f is holomorphic on a domain U and is non-constant then $|f|$ cannot have a local maximum on U.*

Proof. If f attains a local maximum at $a \in U$ then f cannot be an open map near a. □

Corollary 18. *Let $U \subset \mathbb{C}$ be a bounded domain and $f \in \mathcal{C}(U) \cap H(U)$. Then the maximum of $|f|$ on $\overline{U}$ is attained on ∂U.*

Proof. If the maximum is attained at $a \in U$ then a is local maximum of $|f|$. □

If $f \in H(U)$ and has no zeroes then $1/f \in H(U)$ and applying the maximum principle for $1/f$, we conclude that if $m \leqslant |f| \leqslant M$ on ∂U then $m \leqslant |f| leq M$ on $\overline{U}$.

Using Cauchy's integral formula, we can get nice estimates on the derivatives of a holomorphic function.

Theorem 19 (Cauchy's inequalities). *Let* $f \in H(\overline{D}(a,R))$ *and* $|f(z) \leqslant M$ *whenever* $z \in C(a,R)$ *then*

$$|f^{(n)}(a)| \leqslant M\frac{n!}{R^n}, \quad n = 0, 1, \ldots$$

Proof. By Cauchy's integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(a+Re^{i\theta})}{R^{n+1}e^{i(n+1)\theta}} iRe^{i\theta}d\theta = \frac{n!}{2\pi R^n}\int_0^{2\pi} f(a+Re^{i\theta})e^{-in\theta}d\theta,$$

from which the result follows immediately. □

Theorem 20 (Liouville). *An entire function that is bounded is constant.*

Proof. Apply Cauchy inequalities on larger and larger disks. □

Theorem 21 (The fundamental theorem of algebra). *Let* $P(z) = a_0 + a_1 z + \cdots + a_n z_n$ *be a polynomial whose degree* $n \geqslant 1$*. Then* P *has a zero in* $\mathbb{C}$*.*

Proof. Consider $f(z) = 1/P(z)$. If P has no zeroes then f is well-defined and holomorphic on $\mathbb{C}$. But

$$P(z) = z^n\left(a_n + \frac{a_{n-1}}{z}) + \cdots + \frac{a_0}{z^n}\right),$$

and it is clear that as $|z| \to \infty, |P(z)| \to \infty$. This means that the function f is bounded and thus by Liouville's theorem is constant which is absurd as the degree of P is higher than 1. □

Now, we shall see an interesting consequence of the maximum principle that has a number of interesting applications.

Theorem 22 (Schwarz lemma). *Let* $\mathbb{D}$ *be the open unit disk in the complex plane* $\mathbb{C}$ *centered at the origin and let* $f : \mathbb{D} \to \mathbb{D}$ *be a holomorphic map such that* $f(0) = 0$*. Then,* $|f(z)| \leqslant |z|\ \forall z \in \mathbb{D}$ *and* $|f'(0)| \leqslant 1$*.*

Moreover, if $|f(z)| = |z|$ *for some non-zero* z *or* $|f'(0)| = 1$*, then* $f(z) = az$ *for some* $a \in \mathbb{C}$ *with* $|a| = 1$*.*

Proof. Consider the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Then $g \in H(D)$ as g is rational with nowhere zero denominator on $\mathbb{D}\setminus\{0\}$ and $f'(0)$ exists. On the closed disk $\overline{D}(0,r)$, the maximum principle implies that we can find an $z_r \in C(0,R)$ such that $|g(z)| \leqslant |g(z_r)|$ in $\overline{D}(0,r)$. This means that $|g(z)| \leqslant \frac{|f(z_r)|}{|z_r|} \leqslant \frac{1}{r}$ on $\overline{D}(0,r)$. Taking $r \to 1$ shows that $|g(z)| \leqslant 1$ and this delivers the first part of the result.

Moreover, if $|f(z)| = |z|$ or $|f'(0)| = 1$ then $|g(z)| = 1$ for some $z \in \mathbb{D}$ and as $|g(z)| leq 1$ on $\mathbb{D}$, the maximum principle shows that $g \equiv 1$ on $\mathbb{D}$. □

Let $U \subset \mathbb{C}$ be a domain. We define

$$\text{Aut}(U) := \{f : U \to U : f \in H(U), f \text{ is bijective}\}.$$

If $f \in \mathrm{Aut}(U)$ then by the local mapping theorem it follows that f' does not vanish at any point of U. Consequently, $f^{-1} \in \mathrm{Aut}(U)$ as well. From this, it is easy to show that $\mathrm{Aut}(U)$ is group under composition of functions. Note also that the group $\mathrm{Aut}(U)$ acts naturally on the domain U by $(g, z) \mapsto g(u)$.

Theorem 23. *The automorphism group of the unit disk is the set of linear fractional transformations of the form*

$$\phi_{a,\alpha} := e^{i\alpha}\frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D}, \alpha \in \mathbb{R}.$$

Proof. Let $z = e^{i\theta}$. Then

$$\phi_{a,\alpha}(e^{i\theta}) = e^{i\alpha}\frac{e^{i\theta}-a}{1-\overline{a}e^{-i\theta}} = e^{i(\alpha-\theta)}\frac{e^{i\theta}-a}{e^{-i\theta}-\overline{a}},$$

which proves that $|\phi_{a,\alpha}(e^{i\theta})| = 1$. But $\phi_{a,\alpha}(a) = 0$ and consequently $\phi_{a,\alpha}$ maps the unit disk onto itself. This proves that $\phi_{a,\alpha} \in \mathrm{Aut}(\mathbb{D})$.

Now let $f \in \mathrm{Aut}(\mathbb{D})$ and let $f(0) = a$. Then $g := \phi_{a,0} \circ f$ satisfies $g(0) = 0$. Therefore $|g(z)| \leqslant |z|$. It is clear that $|g'(0)| = 1$ and consequently by Schwarz lemma $g(z) = e^{i\theta}z$ is just a rotation and hence

$$\phi_{a,0} \circ f(z) = e^{i\theta}z$$

which proves that $f(z) = \phi_{a,0}^{-1}(e^{i\theta}z)$. But an easy computation shows that $\phi_{a,0}^{-1} = \phi_{-a,0}$ and thus

$$f(z) = \frac{e^{i\theta}z + a}{1+\overline{a}e^{i\theta}z} = \phi_{-e^{i\theta}a, e^{i\theta}}.$$

□