Outline of solution to assignment 1 question 6

Fill in the details of the steps given below to complete the proof. If someone submits a different (correct) proof, they get bonus points.

1. Without loss of generality, assume that \( U \) is connected. Let \( g : U \to \mathbb{C} \) be a continuous branch of the square root function. Then we have

\[
g(z)^2 = z.
\]

Use this to show that \( g \in H(U) \) and that \( g'(z) = \frac{1}{2g(z)} \). You might freely use the chain rule, product rule, etc. Your are just proving the familiar fact that \( \frac{\partial \sqrt{x}}{\partial x} = \frac{1}{2\sqrt{x}} \) so don’t worry too much about being rigorous.

2. In this step, we will show that if \( \gamma : [0, 1] \to U \) is any closed path then \( \text{Ind}(\gamma, 0) = 0 \). Suppose not, then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} = k \in \mathbb{Z}, k \neq 0.
\]

Note that \( g(\gamma(z)) \) is also a closed path. Hence,

\[
\text{Ind}(g(\gamma), 0) = \frac{1}{2\pi i} \int_{g(\gamma)} \frac{dz}{z} = \frac{1}{2\pi i} \int_0^1 \frac{g'(\gamma(t))\gamma'(t)}{g(\gamma(t))} dt = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{2\gamma(t)} dt = k/2,
\]

which shows that the winding number of \( \gamma \) with respect to 0 is an even number. But any closed curve in the plane with non-zero winding number around the origin contains in its image a simple closed curve with winding number one around the origin, and this leads to a contradiction.

3. Fix \( z_0 \in U \) and define \( f(z_0) \) to be some arbitrary value in \( \text{arg} \, z_0 \). For any \( z \in U \), let \( \gamma : [0, 1] \to U \) be a path from \( z_0 \) to \( z \) and let \( \phi \) be the branch of \( \text{arg} \) along \( \gamma \) that satisfies \( \phi(z_0) = f(z_0) \). Set \( f(z) = \phi(z) \). Prove that \( f \) is well-defined and that it is the required continuous branch of \( \text{arg} \, z \) on \( U \).