Proper holomorphic mappings of balanced domains

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What are proper mappings?

Let $X$, $Y$ be topological spaces, and let $f : X \to Y$ be a continuous map. We say that $f$ is proper if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact.
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5. Let $D_1$ and $D_2$ be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and let $f : D_1 \to D_2$ be continuous. Then $f$ is proper iff for all sequences $\{x_n\} \subset D_1$ that has no limit point in $D_1$, the sequence $\{f(x_n)\}$ has no limit point in $D_2$. 
Some major results

The result I shall present today stems from the following major result of Alexander:

**Result (Alexander, 1977)**

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Slightly before the work of Alexander, Pinchuk, in 1973-74, had established the following:

Result (Pinchuk, 1973)

Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded strictly pseudoconvex domain, and let $f : D \to D$ be a proper holomorphic mapping. If $f$ extends to a $C^1$ mapping on $D$, then $f$ is an automorphism of $D$. 
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If $D \subset \mathbb{C}^n, n > 1,$ is a bounded weakly pseudoconvex domain with smooth real-analytic boundary, then any proper holomorphic self mapping of $D$ is an automorphism.
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Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded domain with smooth real-analytic boundary. Then any proper self-map of $D$ that extends smoothly to $\partial D$ must be an automorphism.
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Note that in the above result, pseudoconvexity of $D$ is not assumed.
Newer results

Recent results related to Alexander’s theorem have focussed on domains that need not possess a real-analytic boundary. Two important results in this regard are:

**Result (Berteloot, 1998)**

Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded complete Reinhardt domain with $C^2$-smooth boundary. Then every proper holomorphic self-map of $D$ is an automorphism.

**Result (Coupet, Pan and Sukhov, 1999)**

Let $D \subset \mathbb{C}^2$ be a smoothly bounded balanced pseudoconvex domain of finite type. Then every proper holomorphic self mapping of $D$ is an automorphism.
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A note on finite type domains

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**Lemma**

Let \( D \subset \mathbb{C}^n \) be a smoothly bounded pseudoconvex domain of finite type. Then any holomorphic map \( \phi : \mathbb{D} \to \overline{D} \) such that \( \phi(0) = p \) is constant.
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4. The notion of D’Angelo finite has been extensively used in the literature.
Motivation for a key conjecture

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- it is pseudoconvex and of finite type,
- it is a bounded symmetric domain and hence homogeneous,
- it is balanced and convex, etc.

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Key question

What role, if any, did the various attributes of $\mathbb{B}^n$ listed above play in the phenomenon exhibited in Alexander’s theorem?
The central conjecture

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Let $D \subset \mathbb{C}^n$, $n > 1$, be a bounded domain with $C^2$-smooth boundary. Then every proper holomorphic self-mapping of $D$ is an automorphism.
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The above conjecture is nowhere close to being settled even with the additional hypotheses of pseudoconvexity and finiteness of type. However, significant progress has been made when the domain in question admits some symmetries. A case in point is the result of Coupet, Pan and Sukhov.
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Theorem (J., Math. Z., 2015)

Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded balanced domain of finite type. Assume that $\Omega$ has a smooth defining function that is plurisubharmonic in $\Omega$. Then every proper holomorphic self-map $F : \Omega \to \Omega$ is an automorphism.
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The special feature of the above theorem is that it is the first result in the literature for domains in $\mathbb{C}^n$, $n > 2$, with boundary not necessarily real-analytic or strictly pseudoconvex where the automorphism group need not be “large”.
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A $C^2$-smooth function $h : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{C}^n$, is said to be plurisubharmonic if

$$
\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \bar{z}_k}(p) v_j \bar{v}_k \geq 0 \quad \forall p \in \Omega, \forall (v_1, \ldots, v_n) \in \mathbb{C}^n
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Examples

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2. Complex ellipsoids also satisfy all the hypotheses of our theorem.

3. The domain given by the defining function $|z_1|^2 + |z_2|^2|z_2z_3|^2 − 1$ is an example of a balanced but not Reinhardt domain that satisfies all our hypotheses.
A structure theorem for the branch locus

The following structure result for the branch locus plays a key role in our result.
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**Theorem (Structure Theorem)**

Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded balanced pseudoconvex domain of finite type. Let $F : \Omega \to \Omega$ be a proper holomorphic mapping, and assume that the branch locus

$$V_F := \{ z \in \Omega : \text{Jac}_{\mathbb{C}} F(z) = 0 \} \neq \emptyset.$$ 

Let $X$ be an irreducible component of $V_F$. Then for each $z \in X$, the set $(\mathbb{C} \cdot z) \cap \Omega$ is contained in $X$. 
Bell’s result

Bell has obtained several results on proper holomorphic mappings of circular domains. The following lemma is a straight-forward consequence of one of his main results.
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**Lemma (Bell)**

Let $f : D_1 \to D_2$ be a proper holomorphic map between bounded balanced domains. Assume that the intersection of every complex line passing through 0 with $\partial D_1$ is a circle. Then $f$ extends holomorphically to a neighbourhood of $\overline{D}_1$. 
Some dynamics

To complete the proof of the main theorem, we need some basic facts from dynamics. The following theorem is the starting point of iteration theory.
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Result

Let $X$ be a taut manifold, and $f \in \text{Hol}(X, X)$. Then either the sequence $\{f^k\}$ of iterates of $f$ is compactly divergent, or there exists a complex submanifold $M$ of $X$ and a holomorphic retraction $\rho : X \to M$ (i.e., $\rho^2 = \rho$) such that every limit point $h \in \text{Hol}(X, X)$ of $\{f^k\}$ is of the form $h = \gamma \circ \rho$, where $\gamma$ is an automorphism of $M$. Moreover,

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1. even $\rho$ is a limit point of the sequence $\{f^k\}$,
2. $f(M) \subset M$, and $f|_M$ is an automorphism of $M$;
3. The set of limit points of the iterates of $f$ is a compact abelian group; in fact it is isomorphic to the closed subgroup of $\text{Aut}M$ generated by $f|_M$. 
Some dynamics

**Definition**

With the notation as in the above theorem, we say that $f$ is *non-recurrent* if the sequence $\{f^k\}$ of iterates of $f$ is compactly divergent. Otherwise, we say that $f$ is *recurrent*, and we call the map $\rho$ the *limit retraction*, and the manifold $M$ the *limit manifold*. 

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**Result (Opshtein)**

Let $D \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex domain which admits a defining function that is plurisubharmonic. Let $f : D \to D$ be a proper holomorphic self-map that is recurrent. Then the limit manifold of $f$ is necessarily of dimension higher than 1.
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A classical result of Cartan

The behaviour of the iterates of a holomorphic self-map of a taut manifold $X$ depends on whether $f$ has a fixed point or not. The following theorem known as the Cartan-Carathéodory theorem gives a quantitative description of the behaviour of the differential $f'$ at a fixed point of $f$. 

Result

Let $X$ be a taut complex manifold, and let $f \in \text{Hol}(X, X)$ have some fixed point $z_0 \in X$. Then

1. the spectrum of $f'(z_0)$ is contained in $D$;
2. $|\text{Jac}_C(f)(z_0)| = 1$ if and only if $f$ is an automorphism;
3. $T_{z_0}X$ admits a $f'(z_0)$-invariant splitting $T_{z_0}X = L_N \oplus L_U$ such that the spectrum of $f'(z_0)|_{L_N}$ is contained in $D$, the spectrum of $f'(z_0)|_{L_U}$ is contained in $\partial D$ and $f'(z_0)|_{L_U}$ is diagonalizable.
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The role of finite type in our proof

On the boundary of any smoothly bounded pseudoconvex domain $D$, we can define an upper semi-continuous function, $\tau : \partial D \to \mathbb{Z}_+ \cup \{0\}$ which, loosely speaking, measures the “type” of a boundary point.
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2. The function was introduced by Bedford and Bell and has been used in several Alexander-type results. The key properties that is of relevance to our result are:

   ▶ If $f : D \to D$ is a proper holomorphic map that extends smoothly to $D$, then $\tau(p) \geq \tau(f(p))$, where the inequality is strict iff $p$ is a branch point of $f$.

   ▶ If $D$ is of finite type, then $\tau$ is a bounded function.

Another crucial property used is the analytical-disk lemma that was stated previously.
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- Using the machinery of complex geodesics we show that $\mathcal{M}$ is actually the intersection of some complex subspace with $\Omega$. This step uses the analytical-disk lemma.
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Connections with Cartan’s theorem for biholomorphisms

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Result (Bell)

Let $\Omega_1$ and $\Omega_2$ be bounded domains in $\mathbb{C}^n$ and let $\{f_i : \Omega_1 \to \Omega_2\}$ be a sequence of proper holomorphic mappings all of which have multiplicity $\leq m$. Suppose $f_i$ converge uniformly on compacts to a map $f : \Omega \to \mathbb{C}^n$. Then the following conditions are equivalent:

1. $f$ is a proper holomorphic mapping.
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Let $\Omega_1$ and $\Omega_2$ be bounded strictly pseudoconvex domains in $\mathbb{C}^n$, $n > 1$.

Then a sequence of proper holomorphic mappings $\{f_i: \Omega_1 \to \Omega_2\}$ cannot converge to a holomorphic map $f: \Omega_1 \to \Omega_2$ that is not proper.

What is the relationship between the above conjecture and Opshtein's result?

Well, Opshtein's result follows if the words "strictly pseudoconvex" are replaced by the words "weakly pseudoconvex with plurisubharmonic defining function".
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Jaikrishnan Janardhanan
Proper holomorphic mappings
16th November
Proof Sketch

Proving the key conjecture for a general class of domains

We outline a general recipe for proving the key conjecture for a class of domains $\mathcal{D}$.

1. Establish an analogue of Wong’s conjecture for the class $\mathcal{D}$.

2. Use the properties enjoyed by a domain in $\Omega \in \mathcal{D}$ to show that the iterate of any branched proper holomorphic self-map of $\Omega$ must converge to a non-proper holomorphic map of $\Omega$ to itself.

3. The conclusion of Step 2. is in conflict with the conclusion of Step 1. and this proves that any proper holomorphic self-map of $\Omega$ must be unbranched.

4. A result of Pinchuk shows that any unbranched proper holomorphic self-map of a bounded domain with smooth boundary is automatically an automorphism and we are done.
THANK YOU