Finiteness theorems for holomorphic mappings from products of hyperbolic Riemann surfaces

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Notation

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Some classical finiteness/non-existence theorems

1. Our objective is to study scenarios under which the cardinality of $O^*(X, Y)$ or $O_{\text{dom}}(X, Y)$ is finite.

Liouville's Theorem: If $D$ is a bounded planar domain then $O^*(\mathbb{C}, D) = \emptyset$.

Liouville's Theorem: $O^*(\mathbb{C}, \{0, 1\}) = \emptyset$.

Theorem of de Franchis: If $R$ and $S$ are compact Riemann surfaces both of genus higher than 2 then $O^*(R, S)$ is a finite set.

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**Result (Imayoshi, 1982)**

*Let $R$ be a Riemann surface of finite type and let $S$ be a Riemann surface of finite type $(g, n)$ with $2g - 2 + n > 0$. Then $O^\ast(R, S)$ is a finite set.*
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Let $R$ be a Riemann surface of finite type and let $S$ be a Riemann surface of finite type $(g, n)$ with $2g - 2 + n > 0$. Then $O^*_+(R, S)$ is a finite set.

A Riemann surface of finite type $(g, n)$ is a Riemann surface that is biholomorphic to a Riemann surface obtained by removing $n$ points from a compact Riemann surface of genus $g$. 
The role of hyperbolicity

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3. The integrated version of the above metric, also denoted \( \rho \), is called the Poincaré distance.

4. The Schwarz–Pick lemma can now be reinterpreted to say that holomorphic self-maps of the unit disk are distance decreasing under the Poincaré metric and distance.
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3. The crucial observation is that one cannot equip such a distance on non-hyperbolic Riemann surfaces. One way to see this is to observe that one can embed as large an analytic disk as one desires inside a non-hyperbolic Riemann surface.

4. Using the seminal work of Ahlfors on the Schwarz lemma, one can give illuminating and unified proofs of Liouville’s theorem and the theorems of Picard and several other related results like Schottky’s theorem and Montel’s theorem.
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$$d_X(f(x), f(y)) \leq \rho(x, y) \quad \forall x, y \in \mathbb{D}, f \in O(\mathbb{D}, X).$$
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5. It also follows trivially that there are no complex lines sitting inside a hyperbolic manifold.
Shiga’s theorem

The following result of Shiga is a higher-dimensional analogue of Imayoshi’s theorem.

**Result (Shiga, 2004)**

Let $X = \mathbb{B}^n / G$ be a complex hyperbolic manifold of divergence type and let $Y = \Omega / \Gamma$ be a geometrically finite $n$-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and $\Gamma$ is fixed-point-free discrete subgroup of $\text{Aut}(\Omega)$. Suppose $G$ is finitely generated and that $\Omega$ is complete with respect to the Kobayashi distance. Then $O_{\text{dom}}(X, Y)$ is a finite set. Furthermore, if the essential boundary dimension of $\Omega$ is zero, then $O_{\ast}(X, Y)$ is a finite set.
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We will explain the meaning of the terms geometrically finite and essential boundary dimension 0 in a later slide.
Our main result

Theorem (Divakaran and Jaikrishnan, 2017, IJM)

Let $X := X_1 \times \cdots \times X_n$ be a product of hyperbolic Riemann surfaces of finite type and let $Y = \Omega/\Gamma$ be an $m$-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and $\Gamma$ is fixed-point-free discrete subgroup of $\text{Aut}(\Omega)$.

1. If $N$ is a tautly embedded complex submanifold of $Y$ then $O_{\text{dom}}(X, N)$ is a finite set.
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2. If $Y$ is geometrically finite and $\Omega$ is complete hyperbolic then $O_{\text{dom}}(X, Y)$ is a finite set.

3. If in addition to the conditions in (2), the essential boundary dimension of $\Omega$ is zero, then $O_*(X, Y)$ is a finite set.
Our result extends the result of Imayoshi to product manifolds.

The proof also works *mutatis mutandis* to give a proof of Shiga’s result.

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However, the crux of our argument involves a (to our knowledge) new normal families argument that neatly clarifies the underlying role of (Kobayashi) hyperbolicity in the above results.
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Relationship to earlier results

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Tautness and normal families

- If $M$ and $N$ are two hyperbolic complex manifolds, then by the distance decreasing property the space $O(M, N)$ is an equicontinuous family. Is $O(M, N)$ relatively compact as a subspace of $C(M, N)$?
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**Definition**

A subset $F \subset C(M, N)$ is said to be a *normal family* if every sequence $\{f_n\} \subset F$ has either a subsequence that converges uniformly on compacts to a function in $C(M, N)$ or has a compactly divergent subsequence.

A complex manifold $N$ is said to be *taut* if for every complex manifold $M$ the set $O(M, N)$ is a normal family.

Let $N$ be a complex manifold and let $Y$ be a complex submanifold. We say that $Y$ is *tautly embedded* in $N$ if every sequence of holomorphic mappings $\{f_n : M \to Y\}$, where $M$ is any complex manifold, admits a subsequence that converges uniformly on compacts to a holomorphic map $f : M \to N$. 
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- Complete hyperbolic manifolds are taut.
Meaning of technical terms

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4. The essential boundary dimension of a bounded domain in $\mathbb{C}^n$ is roughly the maximal dimension of analytic sets sitting in $\partial D$. The unit ball in $\mathbb{C}^n$ and more generally strictly pseudoconvex domains have essential boundary dimension 0 whereas the polydisk has essential boundary dimension $n - 1$. 
A rigidity result

Theorem

Let $X := X_1 \times \cdots \times X_n$ be a product of hyperbolic Riemann surfaces of finite type and let $Y = \Omega/\Gamma$ be an $m$-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and $\Gamma$ is fixed-point-free discrete subgroup of $\text{Aut}(\Omega)$. Write $X$ as $\mathbb{D}^n/G$, where $G := \bigoplus_{i=1}^n G_i$ and $G_i$ is the Fuchsian group of divergence type such that $X_i = \mathbb{D}/G_i$. Suppose $\phi, \psi : X \to Y$ are holomorphic mappings such that we can find lifts $\widetilde{\phi}, \widetilde{\psi} : \mathbb{D}^n \to \Omega$ that induce the same homomorphism on $G$.

1. If $\phi$ (or $\psi$) is dominant, then $\phi = \psi$.
2. If $\phi$ (or $\psi$) is non-constant and $\Omega$ has essential boundary dimension zero, then $\phi = \psi$. 

Some remarks about the rigidity result

1. In the case when $\Omega \subset \mathbb{C}$ then the hypothesis about essential boundary dimension being 0 is vacuously satisfied. This version of the rigidity result was proved by Imayoshi. Our proof is a straightforward generalization of Imayoshi’s result.
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5. Our proof can be easily adapted to the situation where $X$ is a complex hyperbolic manifold of divergence type.
Proof sketch of our main result

We will now give a sketch of the proof of the first assertion our main result.

- Let \( \{f_k\} \subseteq O_{\text{dom}}(X, N) \) be a sequence of distinct dominant holomorphic mappings. As \( N \) is tautly embedded in \( Y \), we may assume that the sequence \( \{f_k\} \) converges in the compact-open topology to a map \( f : X \to Y \).
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- We can find a connected and compact set \( K \) such that finitely many closed loops contained in \( K \) based at a point \( x \in K \) generate \( \pi_1(X, x) \).
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- We can find a connected and compact set \( K \) such that finitely many closed loops contained in \( K \) based at a point \( x \in K \) generate \( \pi_1(X, x) \).
- We show that for suitably large \( k \), we can find lifts \( \tilde{f}_k \) and \( \tilde{f} \) that induce the same homomorphism on \( G \).
Proof sketch...

- We first choose $k$ suitably large so that $z_k := f_k(x)$ and $y := f(x)$ belong to an evenly covered coordinate ball in $Y$, say $U$. 
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- Choose $\tilde{f}$ and $\tilde{f}_k$ to be the lifts of $f$ and $f_k$, respectively, such that $\tilde{f}(\tilde{x}), \tilde{f}_k(\tilde{x}) \in \tilde{U}$.
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- Let $\chi$ and $\chi_k$ be the homomorphism induced by $\tilde{f}$ and $\tilde{f}_k$, respectively.
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- Choose \( \widetilde{f} \) and \( \widetilde{f}_k \) to be the lifts of \( f \) and \( f_k \), respectively, such that \( \widetilde{f}(\widetilde{x}), \widetilde{f}_k(\widetilde{x}) \in \widetilde{U} \).

- Let \( \chi \) and \( \chi_k \) be the homomorphism induced by \( \widetilde{f} \) and \( \widetilde{f}_k \), respectively.

- Each \( g \in G \) can be represented by a closed loop based at \( x \), say \( \gamma \). Then \( f \circ \gamma \) and \( f_k \circ \gamma \) are loops in \( Y \) based at \( y \) and \( z_k \), respectively.
Proof sketch...

- Let $\sigma := f \circ \gamma$ and $\sigma_k := \bar{\delta}_k \ast (f_k \circ \gamma) \ast \delta_k$ be two loops based at the point $y$, where $\delta_k$ is a curve lying in $U$ that connects $y$ to $z_k$. 

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Finiteness theorem for holomorphic mappings
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- Now $\sigma_k \to \sigma$ uniformly. A folklore result now shows that $\sigma$ and $\sigma_k$ are equivalent in $\pi_1(Y, y)$ for suitably large $k$. 
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- Now $\sigma_k \to \sigma$ uniformly. A folklore result now shows that $\sigma$ and $\sigma_k$ are equivalent in $\pi_1(Y, y)$ for suitably large $k$.
- Let $\tilde{\sigma}$ and $\tilde{\sigma}_k$ be the lifts of $\sigma$ and $\sigma_k$, respectively, that start at $\tilde{f}(\tilde{x})$. As $\sigma$ and $\sigma_k$ represent the same element in $\pi_1(Y, y)$, the endpoints of $\tilde{\sigma}$ and $\tilde{\sigma}_k$ must be the same and equal to $\chi(g) \left( \tilde{f}(\tilde{x}) \right)$. 
Conclusion of proof sketch

Since the quotient map is a homeomorphism from $\tilde{U}$ to $U$ and $\delta_k$ lies entirely in $U$, a lift of $\delta_k$ starting at $\tilde{f}(\tilde{x})$ ends in $\tilde{U}$. Similarly, a lift of $\tilde{\delta}_k$ that ends in $\chi(g)(\tilde{U})$ has to begin in $\chi(g)(\tilde{U})$. Thus a lift of $f_k \circ \gamma$ starting in $\tilde{U}$ (at $\delta_k(1)$) has to end in $\chi(g)(\tilde{U})$. Thus, $\chi_k(g)(\tilde{U}) \cap \chi(g)(\tilde{U}) \neq \emptyset$. Since $U$ is an evenly covered neighborhood, it follows that $\chi_k(g) = \chi(g)$. 
THANK YOU