ADVANCED ANALYSIS: SOME TOPICS

M.THAMBAN NAIR

Contents

Date: January - May 2020: Notes for the course MA 6110: Topics in Advanced Analysis.

.

1. **Positive Measure and Integration- A Review**

1.1. **Outer measure and measure.**

Definition 1. Let X be a set. A function $\mu^* : 2^X \to [0, \infty]$ is said to be an **outer measure** on X if

1. $\mu^*(\emptyset) = 0$;
2. $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$ (Monotonicity);
3. $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$(Countable sub-additivity). ♢

The following result can be easily verified.

Theorem 2. *Let μ^* be an outer measure on a set X. If for every pair of disjoint sets A, B in 2^X,*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B),$$

then for every countable mutually disjoint family $\{A_n : n \in \mathbb{N}\}$ in 2^X,

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

There are sets X and outer measures μ^* on X which does not satisfy

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B),$$

for every pair of disjoint sets A, B in 2^X This prompts us to look for some subclass of 2^X where the above condition is satisfied.

Definition 3. Let μ^* be an outer measure on a set X. A subset E of X is said to be μ^***-measurable** if for every $A \subseteq X$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. ♢

Let us denote the set of all μ^*-measurable subsets of X by $\mathcal{A}^*$. Note that $E \in \mathcal{A}^*$ if and only if

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall\, A \subseteq X.$$

Theorem 4. *Let μ^* be an outer measure on a set X. Then $\mathcal{A}^*$ has the following properties:*

1. $X \in \mathcal{A}^*$.
2. $A \in \mathcal{A}^* \Rightarrow A^c \in \mathcal{A}^*$.
3. $A_n \in \mathcal{A}^*$, $n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*$.

Further, for every countable mutually disjoint family $\{A_n : n \in \mathbb{N}\}$ in $\mathcal{A}$,

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$$

and for every $A \in \mathcal{A}^*$ with $\mu^*(A) = 0$, $B \subseteq A$ implies $B \in \mathcal{A}^*$.

A family of sets satisfying the three conditions as in the above theorem is called a *σ-algebra*. More precisely, we have the following definition.

Definition 5. A family $\mathcal{A}$ of subsets of X is called a **σ-algebra** on X if

(1) $X \in \mathcal{A}$,
(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
(3) $A_n \in \mathcal{A}$, $n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. ♢

Definition 6. Let $\mathcal{A}$ be a σ-algebra on X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a **measure** on $\mathcal{A}$ if $\mu(\emptyset) = 0$, and for every countable mutually disjoint family $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$,

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

The measure μ is called a **complete measure** if for every $A \in \mathcal{A}$ with $\mu(A) = 0$, $B \subseteq A$ implies $B \in \mathcal{A}$. ♢

Remark 7. If μ^* and $\mathcal{A}^*$ are as in Theorem 4, then the restriction of μ^* on $\mathcal{A}^*$ is a complete measure. ♢

1.2. **Caratheodory extension of measure on an algebra.**

Definition 8. A family $\mathcal{A}_0$ of subsets of X is called an **algebra** on X if

(1) $X \in \mathcal{A}$,
(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
(3) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$. ♢

Clearly, every α-algebra is an algebra.

Example 9. The family $\mathcal{A}_0$ of all finite unions of disjoint intervals of the forms $[a, b)$, $(-\infty, a)$,$[b, \infty)$ form an algebra. This $\mathcal{A}_0$ is not a σ-algebra. ♢

Definition 10. Let $\mathcal{A}_0$ be an algebra on X. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a **measure** on $\mathcal{A}_0$ if $\mu(\emptyset) = 0$, and for every countable mutually disjoint family $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_0 \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n). \qquad ♢$$

Theorem 11. *Let $\mathcal{A}_0$ be an algebra on X and μ_0 be a measure on $\mathcal{A}_0$. For $A \subseteq X$, let*

$$\mu^*(A) = \inf\{\sum_{n=1}^{\infty} \mu_0(A_n) : A \subseteq \bigcup_{n=1}^{\infty} A_n\}.$$

Then

(1) *μ^* is an outer measure,*

(2) $\mathcal{A}^* \supseteq \mathcal{A}_0$, *where* $\mathcal{A}^*$ *is the* σ*-algebra of all* μ^**-measurable sets,*
(3) $\tilde{\mu} := \mu^*|_{\mathcal{A}^*}$ *is an extension of* μ_0.

Exercise 12. *Let* $\mathcal{A}_0$, μ_0, $\mathcal{A}^*$ *and* μ^* *be as in Theorem 11. Show that if* $\mathcal{A}$ *is a* σ*-algebra on* X *such that* $\mathcal{A} \supseteq \mathcal{A}_0$ *and* $\mu^*|_{\mathcal{A}}$ *is a measure, then* $\mathcal{A} \subseteq \mathcal{A}^*$.

Exercise 13. Suppose $\mathcal{A}_0$ is a σ-algebra on X and μ_0 is a measure on $\mathcal{A}_0$. Is $\mathcal{A}^*$ the completion of $\mathcal{A}_0$?

1.3. Integration and integrability.

Definition 14. A function $\varphi : X \to \mathbb{R}$ is called a **simple** function if its range is a finite set. $\Diamond$

Theorem 15. *If* φ *is a simple function and if* $\alpha_1, \dots, \alpha_n$ *are the distinct values of* φ, *then* $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$, *where* $A_i := \{x \in X : \varphi(x) = \alpha_i\}$, $i = 1, \dots, n$.

Definition 16. For a simple function φ, the representation as in Theorem 15 is called its **canonical representation**. $\Diamond$

Definition 17. Let $\mathcal{A}$ be a σ-algebra on X. A simple function φ with canonical representation $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$ is said to be **measurable** with respect to $\mathcal{A}$ if $A_i \in \mathcal{A}$ for $i = 1, \dots, n$. $\Diamond$

Definition 18. Let $\mathcal{A}$ be a σ-algebra and μ be a measure on $\mathcal{A}$. Let φ be a simple measurable function with canonical representation $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$. Then the **integral** of φ is defined by

$$\int_X \varphi \, d\mu = \sum_{i=1}^n \alpha_i \mu(A_i). \qquad \Diamond$$

Theorem 19. *Given any function* $f : X \to [0, \infty]$, *there exists a monotonically increasing sequence* (φ_n) *of simple functions such that* $\varphi_n \to f$ *pointwise.*

Definition 20. Let $\mathcal{A}$ be a σ-algebra on X. A function $f : X \to [-\infty, \infty]$ is said to be **measurable** with respect to $\mathcal{A}$ if there exists a sequence (φ_n) of simple measurable functions such that $\varphi_n \to f$ pointwise. $\Diamond$

Theorem 21. *Let* $\mathcal{A}$ *be a* σ*-algebra on* X. *A function* $f : X \to [-\infty, \infty]$ *is measurable with respect to* $\mathcal{A}$ *if and only if for every open set* G *of* $\mathbb{R}$, $f^{-1}(G) \in \mathcal{A}$.

Theorem 22. *Given any measurable function* $f : X \to [0, \infty]$, *there exists a monotonically increasing sequence* (φ_n) *of simple measurable functions such that* $\varphi_n \to f$ *pointwise.*

Theorem 23. *Let $\mathcal{A}$ be a σ-algebra on X. If (φ_n) and (ψ_n) are monotonically increasing sequences of non-negative simple measurable functions such that*

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \psi_n(x) \quad \forall\, x \in X,$$

then

$$\lim_{n\to\infty} \int_X \varphi_n \, d\mu = \lim_{n\to\infty} \int_X \psi_n \, d\mu.$$

Definition 24. Let $\mathcal{A}$ be a σ-algebra on X and $f : X \to [0, \infty]$ be measurable with respect to $\mathcal{A}$. Let (φ_n) be a monotonically increasing sequence of simple measurable functions such that $\varphi_n \to f$ pointwise. Then the **integral** of f is defined by

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X \varphi_n \, d\mu. \qquad \Diamond$$

Theorem 25. (Monotone convergence theorem) *If (f_n) is an increasing sequence of non-negative measurable functions, then*

$$\int_X (\lim_{n\to\infty} f_n) d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

In the above theorem, note that $f := \lim\limits_{n\to\infty} f_n$ exists as a measurable function and $(\int_X f_n d\mu)$ converges in $[0, \infty]$.

Theorem 26. (Fatou's lemma) *If (f_n) is a sequence of non-negative measurable functions, then*

$$\int_X (\liminf_n f_n) d\mu \le \liminf_n \int_X f_n d\mu.$$

Definition 27. Let $\mathcal{A}$ be a σ-algebra on X and $f : X \to \mathbb{R}$ be measurable with respect to $\mathcal{A}$. If $\int_X f^+ \, d\mu < \infty$ or $\int_X f^- \, d\mu < \infty$, then

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu. \qquad \Diamond$$

Definition 28. Let $\mathcal{A}$ be a σ-algebra on X. A function $f : X \to \mathbb{C}$ is said to be **measurable** if its real and imaginary parts are measurable, and in that case

$$\int_X f \, d\mu := \int_X (\mathrm{Re}(f) \, d\mu + i \int_X \mathrm{Im}(f) \, d\mu,$$

provided $\int_X (\mathrm{Re}(f) \, d\mu$ and $\int_X \mathrm{Im}(f) \, d\mu$ are well-defined as per Definition 27.

Definition 29. Let $\mathcal{A}$ be a σ-algebra on X and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. A measurable function $f : X \to \mathbb{K}$ is said to be **integrable** if $\int_X |f| d\mu < \infty$.

Notation: The set of all $\mathbb{K}$-valued integrable functions is denoted by $\mathcal{L}(X, \mathcal{A}, \mu)$ or $\mathcal{L}(X)$ or $\mathcal{L}(\mu)$.

Theorem 30. *Let $\mathcal{A}$ be a σ-algebra on X. Then $\mathcal{L}(\mu)$ is a vector space over $\mathbb{K}$ and for every $f \in \mathcal{L}(\mu)$, $\int_X f\, d\mu$ is well-defined and*

$$\Big| \int_X f\, d\mu \Big| \le \int_X |f| d\mu,$$

and the map $f \mapsto \int_X |f| d\mu$ is a semi-norm on $\mathcal{L}(\mu)$.

Theorem 31. (Dominated convergence theorem) *If (f_n) is a sequence of $\mathbb{K}$-valued measurable functions. If (f_n) converges pointwise and $|f_n| \le |g|$ for all $n \in \mathbb{N}$ for some $g \in \mathcal{L}(\mu)$, then*

$$\int_X (\lim_{n\to\infty} f_n) d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

1.4. **The space $L^p(\mu)$.** For $0 < p < \infty$, let

$$\mathcal{L}^p(\mu) := \{f : f \text{ measurable such that } |f|^p \in \mathcal{L}(\mu)\}.$$

It can be seen that

- $\mathcal{L}^p(\mu)$ is a vector space over $\mathbb{K}$.

Theorem 32. *For $\mathbb{K}$-valued measurable function $f, g : X \to \mathbb{K}$ and $1 < p < \infty$,*

(1) $\int_X |fg|\, d\mu \le \Big(\int_X |f|^p d\mu \Big)^{\frac{1}{p}} \Big(\int_X |g|^q d\mu \Big)^{\frac{1}{q}}$, *where q is such that $\frac{1}{p} + \frac{1}{q} = 1$;*

(2) $\Big(\int_X |f+g|\, d\mu \Big)^{\frac{1}{p}} \le \Big(\int_X |f|^p d\mu \Big)^{\frac{1}{p}} + \Big(\int_X |g|^p d\mu \Big)^{\frac{1}{p}}$.

The inequalities in (1) and (2) in the above theorem are called **Hölder's inequality** and **Minkowski inequality** respectively.

Theorem 33. *For $1 \le p < \infty$, the map $f \mapsto \Big(\int_X |f|^p d\mu \Big)^{\frac{1}{p}}$ is a semi-norm on $\mathcal{L}^p(\mu)$.*

For $1 \le p < \infty$, let $\mathcal{Z}_p := \{f \in \mathcal{L}^p(\mu) : f = 0 \text{ a.e.}\}$.

Theorem 34. *For $1 \le p < \infty$, $\mathcal{Z}_p$ is a subspace of $\mathcal{L}^p(\mu)$, and*

$$[f] \mapsto \Big(\int_X |f|^p d\mu \Big)^{\frac{1}{p}}$$

is a norm on the quotient space $L^p(\mu) := \mathcal{L}^p(\mu)/\mathcal{Z}_p$.

For $f \in \mathcal{L}^p(\mu)$ we use the notation

$$\|f\|_p := \Big(\int_X |f|^p d\mu \Big)^{\frac{1}{p}},$$

and we shall write f in place of $[f]$.

Theorem 35. *For $1 \le p < \infty$, the normed linear space $L^p(\mu)$ is a Banach space.*

We use the following lemma.

Lemma 36. *Let (f_n) be a Cauchy sequence in $L^p(\mu)$. If (f_n) converges a.e. to a measurable function f, then $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$.*

Proof. Let f be a measureable function such that $f_n \to f$ a.e. Then for each $m \in \mathbb{N}$, $|f_n - f_m|^p \to |f - f_m|^p$ a.e. Hence, by Fatou's lemma

$$\int_X |f - f_m|^p d\mu = \int_X \liminf_n |f_n - f_m|^p d\mu \le \liminf_n \int_X |f_n - f_m|^p d\mu = \liminf_n \|f_n - f_m\|_p^p.$$

Since (f_n) is a Cauchy sequence in $L^p(\mu)$, it follows that (*How?*) $\|f - f_m\|_p \to 0$. □

Proof of Theorem 35. Let (f_n) be a Cauchy sequence in $L^p(\mu)$. By the above lemma, it is enough to show that (f_n) has a subsequence which converges a.e.

Since (f_n) is a Cauchy sequence in $L^p(\mu)$, there exists a sequence (n_k) in $\mathbb{N}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2^k} \quad \forall\, k \in \mathbb{N}.$$

Note that $f_{n_{k+1}} = f_{n_1} + \sum_{i=1}^k (f_{n_{i+1}} - f_{n_i})$. We show that $(f_{n_{k+1}})$ converges a.e. Let

$$g_k = \sum_{i=1}^k |f_{n_{i+1}} - f_{n_i}|, \quad g = \sum_{i=1}^\infty |f_{n_{i+1}} - f_{n_i}|.$$

Note that (g_k) increase to g. Hence, by MCT, $\int_X g_k^p d\mu \to \int_X g^p d\mu$. Since

$$\|g_k\|_p \le \sum_{i=1}^k \|f_{n_{i+1}} - f_{n_i}\|_p \le 1$$

we have $\|g\|_p \le 1$ so that g is finite almost everywhere. Thus, $(f_{n_{k+1}})$ converges a.e. □

Corollary 37. *Every Cauchy sequence in $L^p(\mu)$ has an almost everywhere convergent subsequence.*

It is to be observed that if $X := \mathbb{N}$ and μ is the counting measure on $2^{\mathbb{N}}$, then

$$L^p(\mu) = \mathcal{L}^p(\mu) = \ell^p := \{(\alpha_n) \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^\infty |\alpha_n|^p < \infty\},$$

and $(\alpha_n) \mapsto \left(\sum_{n=1}^\infty |\alpha_n|^p\right)^{\frac{1}{p}}$ is the corresponding norm. Also, if $X = \{1, \ldots, m\}$ and μ is the counting measure on 2^X, then $L^p(\mu) = \mathbb{K}^m$.

Let

$$\mathcal{L}^\infty(\mu) := \{f : f \text{ measurable and bounded a.e.}\}.$$

Thus,

$$f \in \mathcal{L}^\infty(\mu) \iff f : X \to \mathbb{K} \text{ is measurable and } \exists\, M_f > 0 \text{ such that } |f| \le M_f \text{ a.e.}$$

Theorem 38. *$\mathcal{L}^\infty(\mu)$ is a vector space and $\mathcal{Z} := \{f \in \mathcal{L}^\infty(\mu) : f = 0 \text{ a.e.}\}$ is a subspace of $\mathcal{L}^\infty(\mu)$.*

We denote $L^\infty(\mu) := \mathcal{L}^\infty(\mu)/\mathcal{Z}$, and for a measurable function $f : X \to \mathbb{K}$, we denote

$$\|f\|_\infty := \inf\{M > 0 : |f| \le M \text{ a.e.}\}.$$

Thus, $f \in \mathcal{L}^\infty(\mu) \iff \|f\|_\infty < \infty$.

We see that the map $f \mapsto \|f\|_\infty$ is a semi-norm on $\mathcal{L}^\infty(\mu)$ and it is a norm on $L^\infty(\mu)$.

Theorem 39. *$L^\infty(\mu)$ is a Banach space.*

Proof. For k, m, n, let

$$A_k = \{x : |f_k(x)| > \|f_k\|_\infty\}, \quad B_{m,n} = \{x : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}.$$

Then $\mu(A_k) = 0 = \mu(B_{m,n})$. Let $\Omega := [(\cup_k A_k) \cup (\cup_{m,n} B_{m,n})]^c$. Then we have

$$|f_k(x)| \le \|f_k\|_\infty, \quad |f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \quad \forall\, x \in \Omega.$$

Let $\tilde{f}_n = f_n|_\Omega$. From the above, we see that $(\tilde{f}_n)$ is a Cauchy sequence in $B(\Omega)$. Let $g \in B(\Omega)$ be such that $\|\tilde{f}_n - g\|_\infty \to 0$. We also have $f_n \to g$ pointwise on Ω. Taking $f = g$ on Ω and $f = 0$ on Ω^c, we have $f \in L^\infty(\Omega)$ and $\|f_n - f\|_\infty \to 0$. □

Theorem 40. *For measurable functions $f, g : X \to \mathbb{K}$,*

$$\int_X |fg| d\mu \le \|f\|_1 \|g\|_\infty.$$

- We observe that, for $p = 2$, the norm $f \mapsto \|f\|_2$ on $L^2(\mu)$ is induced by the inner product
$$\langle f, g\rangle = \int_X f\bar{g} d\mu, \quad f, g \in L^2(\mu).$$
Thus, $L^2(\mu)$ is a Hilbert space with respect to the above inner product.

We recall the Riesz representation theorem Functional Analysis:

Theorem 41. (Riesz representation theorem) *Corresponding to every continuous linear functional φ on a Hilbert space $\mathcal{H}$, there exists a unque $g \in \mathcal{H}$ such that*

$$\varphi(f) = \langle f, g\rangle_{\mathcal{H}} \quad \forall\, f \in \mathcal{H},$$

and we also have $\|g\|_2 = \|\varphi\|$.

- If λ and μ are measures on $(X, \mathcal{A})$, then we write $\lambda \le \mu$ whenever $\lambda(E) \le \mu(E)$ for every $E \in \mathcal{A}$.

Theorem 42. *Let λ and μ be finite measures on $(X, \mathcal{A})$ such that $\lambda \le \mu$. Then $L^2(\mu) \subseteq L^1(\lambda)$ and there exists a unique $g \in L^2(\mu)$ such that*

$$\int_X f d\lambda = \int_X f\bar{g} d\mu \quad \forall\, f \in L^2(\mu).$$

Proof. Since μ is a finite measure, $L^2(\mu) \subseteq L^1(\mu)$ and since $\lambda \leq \mu$, $L^1(\mu) \subseteq L^1(\lambda)$. Hence, for $f \in L^2(\mu)$, the integral $\int_X f d\lambda$ is well-defined, and the map $\varphi : L^2(\mu) \to \mathbb{K}$ defined by

$$\varphi(f) = \int_X f d\lambda, \quad f \in L^2(\mu),$$

is a continuous linear functional on $L^2(\lambda)$. Indeed, this map is linear and

$$\Big| \int_X f d\lambda \Big| \leq \int_X |f| d\lambda \leq \int_X |f| d\mu \leq \|f\|_2 \sqrt{\mu(X)} \quad \forall f \in L^2(\mu).$$

Hence, by Riesz representation theorem, there exists a unique $g \in L^2(\mu)$ such that

$$\int_X f d\lambda = \int_X f \bar{g} d\mu \quad \forall f \in L^2(\lambda).$$

□

Corollary 43. *Let λ and μ be finite measures on $(X, \mathcal{A})$ such that $\lambda \leq \mu$. Then there exists a unique $g \in L^2(\mu)$ such that*

$$\lambda(E) = \int_E g d\mu \quad \forall E \in \mathcal{A}.$$

2. Radon-Nykodym theorem

Let μ be a (positive) measure on a measurable space $(X, \mathcal{A})$ and f be a non-negative measurable function on $(X, \mathcal{A})$. Let

$$\nu(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{A}.$$

Then we know that ν is also a measure on $(X, \mathcal{A})$ and for every $E \in \mathcal{A}$,

$$\mu(E) = 0 \quad \Rightarrow \quad \nu(E) = 0.$$

Definition 44. Let μ_1 and μ_2 be positive measures on a measurable space $(X, \mathcal{A})$. Then μ_1 is said to be **absolutely continuous** with respect to μ_2, written as $\mu_1 << \mu_2$, if for every $E \in \mathcal{A}$,

$$\mu_2(E) = 0 \quad \Rightarrow \quad \mu_1(E) = 0.$$ ◊

Theorem 45. (Radon-Nykodym theorem (RNT)) *Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{A})$ such that $\nu << \mu$. Then there exists a unique a.e. $[\mu]$ non-negative measurable function f on $(X, \mathcal{A})$ such that*

$$\nu(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{A}.$$

A particular case of the theorem RNT is Corollary 43: If μ and ν are finite measures such that $\nu \leq \mu$, then by Corollary 43, there exists a unique $f \in L^2(\mu)$ such that

$$\lambda(E) = \int_E f d\mu \quad \forall\, E \in \mathcal{A}.$$

For the proof in the general setting, we shall make use of the following two lemmas:

Lemma 46. *Let μ be a finite measure on $(X, \mathcal{A})$. Let S be a closed subset of $\mathbb{K}$ and let $f \in L^1(\mu)$ be such that for every $E \in \mathcal{A}$ with $\mu(E) > 0$,*

$$\frac{1}{\mu(E)} \int_E f d\mu \in S.$$

Then $f(x) \in S \quad$ for a.a. $x \in X$.

Proof. Since S^c is an open subset of $\mathbb{K}$, it is a countable union of closed balls. Hence, it is enough to show that $\mu(\{x \in X : f(x) \in D\}) = 0$ for any closed ball $D \subseteq S^c$. Let $D = \{\alpha \in \mathbb{K} : |\alpha - \alpha_0| \leq r\}$ for some $\alpha_0 \in S^c$ with $D \subseteq S^c$ and $r > 0$ and let $E := \{x \in X : f(x) \in D\}$. Suppose $\mu(E) > 0$. Then,

$$\Big|\frac{1}{\mu(E)} \int_E f d\mu - \alpha_0\Big| \leq \frac{1}{\mu(E)} \int_E |f - \alpha_0| d\mu \leq r,$$

which is a contradiction to the hypothesis $\frac{1}{\mu(E)} \int_E f d\mu \in S$. □

Lemma 47. *Let μ be a σ-finite measure on $(X, \mathcal{A})$. Then there exists $w \in L^1(\mu)$ such that $0 < w < 1$ on X. In particular,*

$$\tilde{\mu}(E) := \int_E w d\mu, \quad E \in \mathcal{A},$$

defines a (finite) measure such that for $E \in \mathcal{A}$, $\mu(E) = 0$ iff $\tilde{\mu}(E) = 0$.

Proof. Since μ is σ-finite, there exists $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ such that $X = \cup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for every $n \in \mathbb{N}$. Let w be defined by

$$w(x) = \sum_{n=1}^{\infty} \frac{2^{-n}\chi_{E_n}}{1 + \mu(E_n)}, \quad x \in X.$$

Then we have $0 < w < 1$ and $\int_X w\, d\mu = \sum_{n=1}^{\infty} \frac{2^{-n}\mu(E_n)}{1+\mu(E_n)} \leq 1$ so that $w \in L^1(\mu)$ and $\tilde{\mu}(X) = \|w\|_1 \leq 1$. □

Remark 48. Clearly, if μ is a finite measure, then the conclusion in the above lemma is obvious: One may take $w(x) = 1/2$ for all $x \in X$. ◊

Proof of RNT (Theorem 45). Assume first that ν is a finite measure. Let $\tilde{\mu}$ be the measure as in Lemma 47 and let $\lambda = \nu + \tilde{\mu}$. Then λ is a finite measure and for every $f \in L^2(\lambda)$,

$$\int_X |f| d\nu \leq \int_X |f| d\lambda \leq \|f\|_{L^2(\lambda)} \sqrt{\lambda(X)}.$$

This shows that $f \mapsto \int_X f d\nu$ is a continuous linear functional on $L^2(\lambda)$. Hence, by the Riesz-representaition theorem, there exists a unique $g \in L^2(\lambda)$ such that

$$\int_X f d\nu = \int_X g f d\lambda \quad \forall\, f \in L^2(\lambda). \tag{$*$}$$

From this, taking $f = \chi_E$, we have

$$\int_E g d\lambda = \nu(E) \leq \lambda(E) \tag{$*$}$$

so that for every $E \in \mathcal{A}$ with $\lambda(E) > 0$,

$$0 \leq \frac{1}{\lambda(E)} \int_E g d\lambda \leq 1.$$

Hence, by Lemma 46, we have $0 \leq g \leq 1$ a.e.$[\lambda]$. Without loss of generality, assume that $0 \leq g \leq 1$ on X. Note that

$$\int_X f d\nu = \int_X g f d\lambda = \int_X g f d\nu + \int_X g f d\tilde{\mu}$$

so that

$$\int_X (1-g) f d\nu = \int_X g f d\tilde{\mu}.$$

hence, for any $E \in \mathcal{A}$, taking

$$f := (1 + g + \cdots + g^n)\chi_E,$$

we have

$$\int_E (1 - g^{n+1}) d\nu = \int_E g(1 + g + \cdots + g^n) d\tilde{\mu}.$$

Let

$$A = \{x \in X : 0 \leq g(x) < 1\}, \quad B = \{x \in X : 0 \leq g(x) = 1\}.$$

Since

$$\int_E (1 - g^{n+1}) d\nu = \int_{E\cap A} (1 - g^{n+1}) d\nu + \int_{E\cap B} (1 - g^{n+1}) d\nu = \int_{E\cap A} (1 - g^{n+1}) d\nu,$$

we have

$$\int_{E\cap A} (1 - g^{n+1}) d\nu = \int_E g(1 + g + \cdots + g^n) d\tilde{\mu} \quad \forall\, E \in \mathcal{A}.$$

Since $g^n \to 0$, by DCT,

$$\nu(E \cap A) = \int_E h d\mu,$$

where $h = wg(1 + g + g^2 + \cdots)$. This also implies that $h := \sum_{n=1}^{\infty} g^n w \in L^1(\mu)$. From $(*)$, taking $f = \chi_B$, we have

$$\nu(B) = \int_B g d\lambda = \int_B g d\nu + \int_B g d\tilde{\mu} = \nu(B) + \tilde{\mu}(B).$$

Hence, $\tilde{\mu}(B) = 0$ so that $\mu(B) = 0$ and since $\nu << \mu$, $\nu(B) = 0$. Therefore,

$$\nu(E) = \nu(E \cap A) = \int_E h \, d\mu.$$

Next, suppose that ν is σ-finite. Then there exists a disjoint family $\{X_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ such that $X = \cup_{n=1}^{\infty} X_n$ and $\nu(X_n) < \infty$ for all $n \in \mathbb{N}$. Let $E \in \mathcal{A}$. For each $n \in \mathbb{N}$, there exists $h_n \in L^1(X_n, \mathcal{A}_n, \mu)$ such that

$$\nu(X_n \cap E) = \int_{X_n \cap E} h_n \, d\mu \quad \forall \, n \in \mathbb{N},$$

where $\mathcal{A}_n$ is the restriction of $\mathcal{A}$ on X_n. Then we have

$$\nu(E) = \sum_{n=1}^{\infty} \nu(X_n \cap E) = \sum_{n=1}^{\infty} \int_{X_n \cap E} h_n \, d\mu.$$

Note that

$$\int_{X_n \cap E} h_n \, d\mu = \int_E \chi_{X_n \cap E} h_n \, d\mu.$$

Thus,

$$\nu(E) = \int_E h \, d\mu, \quad h := \sum_{n=1}^{\infty} \chi_{X_n \cap E} h_n.$$

It follows that $h \in L^1(\mu)$ and it is non-negative. The uniqueness of h can be verified easily. □

3. **Signed Measures**

3.1. **Motivation, Definition, and Examples.** Let $(X, \mathcal{A}, \mu)$ be a measure space.

(1) If f is a non-negative measurable function, then we know that
$$\nu(E) := \int_E f d\mu, \quad E \in \mathcal{A},$$
defines another measure on $(X, \mathcal{A})$.

(2) If f is an extended real measurable function, then
$$\nu^+(E) := \int_E f^+ d\mu, \quad \nu^-(E) := \int_E f^- d\mu \quad \text{for} \quad E \in \mathcal{A},$$
$$\tilde{\nu}(E) := \nu^+(E) + \nu^-(E) = \int_E |f| d\mu$$
define measures on $(X, \mathcal{A})$.

(3) If f is an extended real measurable function and if either $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < \infty$, then the integral $\int_E f d\mu$ is defined by
$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu, \quad E \in \mathcal{A}.$$

(4) Let ν^+ and ν^- be as in (2) above such that one of them is finite. Then
$$\nu(E) := \int_E f d\mu, \quad E \in \mathcal{A},$$
has the following properties:
   (a) $\nu(\emptyset) = 0$,
   (b) $\{+\infty, -\infty\} \not\subseteq \{\nu(E) : E \in \mathcal{A}\}$,
   (c) $\nu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \nu(E_n)$ for every disjoint family $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$.

Definition 49. Let Let $(X, \mathcal{A})$ be a measurable space. A set function $\nu : \mathcal{A} \to [-\infty, \infty]$ is called a **signed measure** if

(1) $\nu(\emptyset) = 0$,
(2) $\{+\infty, -\infty\} \not\subseteq \{\nu(E) : E \in \mathcal{A}\}$,
(3) $\nu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \nu(E_n)$ for every disjoint family $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$. ◊

Example 50. Let f be a real measurable function on a measure space $(X, \mathcal{A}, \mu)$ such that $\int_X f d\mu$ is well defined. Then
$$\nu(E) := \int_E f d\mu, \quad E \in \mathcal{A},$$
defines a signed measure, and $\nu = \nu^+ - \nu^-$. In particular, if $f \in L^1(\mu)$, then ν is a signed measure such that $|\nu(E)| < \infty$ for every $E \in \mathcal{A}$. ◊

Example 51. If μ_1 and μ_2 are measures on $(X, \mathcal{A})$ with atleast one of them is finite, then $\nu := \mu_1 - \mu_2$ is a signed measure. $\Diamond$

3.2. **Hahn decomposition and Jordan decomposition.** Is every signed measure ν is of the form $\mu_1 - \mu_2$ for some measures μ_1, μ_2?

Suppose (A, B) is a measurable decomposition of X, that is, $A, B \in \mathcal{A}$ such that

$$X = A \cup B, \quad A \cap B = \emptyset.$$

Note that, for every $E \in \mathcal{A}$, $\nu(E) = \nu(E \cap A) + \nu(E \cap B)$. Suppose the above decomposition is such that

(1) $\nu(E \cap A) \geq 0$ for every $E \in \mathcal{A}$, and
(2) $\nu(E \cap A) \leq 0$ for every $E \in \mathcal{A}$.

Then, defining

$$\nu^+(E) = \nu(E \cap A), \quad \nu^-(E) = -\nu(E \cap B),$$

we see that ν^+, ν^- and $\nu^+ + \nu^-$ are measures and $\nu = \nu^+ - \nu^-$. Thus, the answer to our above posed question is in the affirmative if there exists a measurable decomposition (A, B) of X satisfying (1) and (2) above.

There does exist such a measurable decomposition!

Definition 52. A measurable decomposition (A, B) of X is said to be a **Hahn-decomposition** of ν if

$$\nu(E \cap A) \geq 0 \quad \text{and} \quad \nu(E \cap B) \leq 0 \quad \forall\, E \in \mathcal{A}. \qquad \Diamond$$

Theorem 53. (Hahn decomposition theorem HDT)) *Every signed measure has a Hahn-decomposition.*

Definition 54. Let ν be a signed measure on a measurable space $(X, \mathcal{A})$, and let $A \in \mathcal{A}$. Then A is said to be

(1) a **positive set** for ν if $\nu(E \cap A) \geq 0$ for every $E \in \mathcal{A}$,
(2) a **negative set** for ν if $\nu(E \cap A) \leq 0$ for every $E \in \mathcal{A}$,
(3) a **null set** for ν if it is positive and negative for ν. $\Diamond$

Exercise 55. *Let ν be a signed measure on a measurable space $(X, \mathcal{A})$, and let $A \in \mathcal{A}$. Show that*

(1) *A is a positive set for ν iff $\nu(E) \geq 0$ for every $E \in \mathcal{A}$ with $E \subseteq A$,*
(2) *A is a negative set for ν iff $\nu(E) \leq 0$ for every $E \in \mathcal{A}$ with $E \subseteq A$,*
(3) *A is a null set for ν iff $\nu(E) = 0$ for every $E \in \mathcal{A}$ with $E \subseteq A$.*

Theorem 56. *Let ν be a signed measure and (A, B) be a Hahn-decomposition of ν. Let*

$$\nu^+(E) = \nu(E \cap A), \quad \nu^-(E) = -\nu(E \cap B) \quad \forall E \in \mathcal{A}.$$

Then ν^+ *and* ν^- *are measures having the following properties:*

(1) $\nu = \nu^+ - \nu^-$,

(2) $\nu^+(E \cap B) = 0$ *and* $\nu^-(E \cap A) = 0$ *for all* $E \in \mathcal{A}$.

Exercise 57. *Show that a measurable decomposition* (A, B) *of* X *is said to be a Hahn-decomposition of* ν *if and only if* A *is positive and* B *is negative for* ν.

Definition 58. Signed measures ν_1 and ν_2 on a measurable space $(X, \mathcal{A})$ are said to be **mutually singular**, written as $\nu_1 \perp \nu_2$, if there exists a measurable decomposition (A, B) of X such that $\nu_1(E \cap B) = 0 = \nu_2(E \cap A)$ for all $E \in \mathcal{A}$. ♢

Exercise 59. *Show that signed measures* ν_1 *and* ν_2 *are mutually singular if and only if there exists a measurable decomposition* (A, B) *of* X *such that*

$$\nu_1(E) = \nu_1(E \cap A) \quad \text{and} \quad \nu_2(E) = \nu_2(E \cap B) \quad \forall\, E \in \mathcal{A}.$$

Exercise 60. *Show that measures* μ_1 *and* μ_2 *are mutually singular if and only if there exists a measurable decomposition* (A, B) *of* X *such that* $\mu_1(B) = 0 = \mu_2(A)$.

Definition 61. Let ν be a signed measure on a measurable space $(X, \mathcal{A})$. Then ν is said to be **concentrated** on a set $A \in \mathcal{A}$ if $\nu(E) = \nu(E \cap A)$ for all $E \in \mathcal{A}$. ♢

Exercise 62. *Show that* ν_1 *and* ν_2 *on a measurable space* $(X, \mathcal{A})$ *are mutually singular if and only if there exists a measurable decomposition* (A, B) *of* X *such that* ν_1 *is concentrated on* B *and* ν_2 *is concentrated on* A.

Theorem 63. *Let* ν *be a signed measure and* (A, B) *be a Hahn decomposition of* ν. *Let* ν^+ *and* ν^- *be as in theorem 56. Then*

(1) *A is positive for ν and B is negative for ν,*

(2) *ν^+ is concentrated on A and ν^- is concentrated on B.*

(3) $\nu^+ \perp \nu^-$.

Corollary 64. *Suppose* (A, B) *and* (A', B') *be a Hahn decompositions of a signed measure* ν. *Then* $A \triangle A'$ *and* $B \triangle B'$ *are null sets for* ν.

Proof. Let $E \subseteq A \triangle A'$. Recall that $A \triangle A' = (A \setminus A') \cup (A' \setminus A)$.

We write $E = E_1 \cup E_2$, where $E_1 = E \cap (A \setminus A')$, $E_2 = E \cap (A' \setminus A)$. Note that

$$\nu^+(E_1) = \nu(A' \cap E_1) = 0, \quad \nu^+(E_2) = \nu(A \cap E_2) = 0,$$

$$\nu^-(E_1) = -\nu(B \cap E_1) = 0, \quad \nu^-(E_2) = -\nu(B' \cap E_2) = 0.$$

Hence, $\nu^+(E) = 0 = \nu^-(E)$ so that $\nu(E) = 0$. Similarly, for any $F \subseteq B \triangle B'$, we obtain $\nu(F) = 0$. □

By the above corollary, we can say that Hahn decomposition is unique upto symmetric difference.

Definition 65. let ν be a singed measure. A pair (μ_1, μ_2) of measures is called a **Jordan decomposition** of ν if $\nu = \mu_1 - \mu_2$ and $\mu_1 \perp \mu_2$. $\Diamond$

Theorem 66. *Suppose (μ_1, μ_2) is a Jordan decomposition of a signed measure ν, and (A, B) is a corresponding measurable decomposition such that $\mu_1(B) = 0 = \mu_2(A)$. Then (A, B) is a Hahn decomposition of ν.*

Proof. Let $E \in \mathcal{A}$. Then we have

$$\nu(E \cap A) = \mu_1(E \cap A) - \mu_2(E \cap A) = \mu_1(E \cap A) = \mu_1(E),$$

$$\nu(E \cap B) = \mu_1(E \cap B) - \mu_2(E \cap B) = -\mu_2(E \cap B) = -\mu_2(B),$$

Thus, $\mu_1(E) = \nu(E \cap A)$ and $\mu_2(E) = -\nu(E \cap B)$ so that $\mu_1 = \nu^+$ and $\mu_2 = \nu^-$. $\square$

By Theorem 63, every signed measure has a Jordan decomposition. In fact, this decomposition is unique.

Theorem 67. (Jordan decomposition theorem (JDT)) *For every signed measure ν, there exists a unique pair (μ_1, μ_2) of measures such that*

$$\nu = \mu_1 - \mu_2, \quad \mu_1 \perp \mu_2.$$

3.3. **Proof of Hahn and Jordan decomposition theorems.**

Lemma 68. *Let ν be a signed measure on a measurable space $(X, \mathcal{A})$ and $E \in \mathcal{A}$ be such that $0 < \nu(E) < \infty$. Then there exists a positive set $A \subseteq E$ such that $\nu(A) > 0$.*

Proof. If E is a positive set, then we are done. So, assume that E is not a positive set. Let $n_1 \in \mathbb{N}$ be smallest positive integer such that there exists $E_1 \in \mathcal{A}$ with $E_1 \subseteq E$ and $\nu(E_1) < -\frac{1}{n_1}$. If $E \setminus E_1$ is positive, then we are done. Otherwise, let $n_2 \in \mathbb{N}$ be smallest positive integer such that there exists $E_2 \in \mathcal{A}$ with $E_2 \subseteq E \setminus E_1$ and $\nu(E_2) < -\frac{1}{n_2}$. Continue this to get $E_1, \ldots, E_k$ in $\mathcal{A}$ such that $n_j \in \mathbb{N}$ is smallest positive integer such that there exists $E_j \subseteq E \setminus (\cup_{i=1}^{j-1} E_i)$ and $\nu(E_j) < -\frac{1}{n_j}$ for $j = 1, \ldots, k$. This procedure will either terminate with some positive E_k, or else, we obtain $E_1, E_2, \ldots$ with the above properties. In the later case, let $A = E \setminus (\cup_{i=1}^{\infty} E_i)$. Then

$$\nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k).$$

Since $0 < \nu(E) < \infty$, it follows that $\sum_{k=1}^{\infty} \nu(E_k)$ converges so that $\sum_{k=1}^{\infty} \frac{1}{n_k}$ also converges. Since $\nu(E_k) < 0$, it also follows that $0 < \nu(E) < \nu(A)$.

Now suppose A is not positive. Then there exists $A_0 \in \mathcal{A}$ such that $A_0 \subseteq A$ and $\nu(A_0) < 0$. Since $n_k \to \infty$, we can find k large enough such that $\nu(A_0) < -\frac{1}{n_k - 1}$. But,

$$A_0 \subseteq A \subseteq E \setminus (\cup_{i=1}^{k-1} E_i).$$

This contradicts the definition of n_k and E_k. Thus, we have proved that A is a positive set. □

Proof of HDT (Theorem 53). Let ν be a signed measure. Without loss of generality, assume that

$$-\infty \leq \nu(E) < \infty \quad \forall\, E \in \mathcal{A}. \tag{$*$}$$

Let

$$\beta := \sup\{\nu(E) : E \text{ is positive}\}.$$

Since $\emptyset$ is positive and $\nu(\emptyset) = 0$, it follows that $\beta \geq 0$. Let (E_n) be as sequence of positive sets such that $\nu(E_n) \to \beta$, and let $A = \cup_{n=1}^{\infty} E_n$. It can be seen that A is positive. Since $E_n \subseteq A$ and since both E_n and A are positive, we obtain[1]

$$\nu(E_n) \leq \nu(A) \quad \forall\, n \in \mathbb{N}.$$

Hence,

$$\beta = \lim_{n\to\infty} \nu(E_n) \leq \nu(A) \leq \beta.$$

That is[2] $\beta = \nu(A)$. We now show that $B := X \setminus A$ is negative, so that (A, B) would be a Hahn decomposition.

Suppose B is not negative. Then there exists $E \subset B$ such that $\nu(E) > 0$. Then, by Lemma 68, there exists a positive set $E_0 \subseteq E$ such that $\nu(E_0) > 0$. Then, $\tilde{A} := A \cup E_0$ is a positive set satisfying

$$\nu(\tilde{A}) = \nu(A) + \nu(E_0) = \beta + \nu(E_0) > \beta,$$

since $0 \leq \beta < \infty$. This is a contradtion to the definition of α. Thus, we have proved that (A, B) is a Hahn decomposition of ν. □

Proof of JDT (Theorem 67). The existence of Jordan decomposition follows from the existence of Hahn decomposition $(A, .B)$. To show the uniqueness, let (μ_1, μ_2) be another Jordan decomposition of ν. Let (ν^+, ν^-) be the Jordan decomposition obtained from (A, B), and let (A', B') be a measurable decomposition of X such that ν is positive on A', negative on B' and $\mu_1(B') = 0 = \mu_2(A')$. We show that $\mu_1 = \nu^+$

[1]This follows from the fact that $A = E_n \cup (A \setminus E_n)$, where both E and $A \setminus E_n$ are positive.

[2]By assumption $(*)$, $0 \leq \beta < \infty$

and $\mu_2 = \nu^-$. Note that for every $E \in \mathcal{A}$,

$$\begin{aligned}
\nu^+(E) &= \nu(E \cap A) && \text{(by the definition of } \nu^+) \\
&= \nu(E \cap A \cap A') + \nu(E \cap A \cap B') && \text{(since} (A', B') \text{ is a decomposition of } X) \\
&= \nu(E \cap A \cap A') && \text{(since } A \text{ is positive and } B' \text{ is negative)} \\
&= \nu(E \cap A \cap A') + \nu(E \cap B \cap A') && \text{(since } A' \text{ is positive and } B \text{ is negative)} \\
&= \nu(E \cap A'). && \text{(since} (A, B) \text{ is a decomposition of } X) \\
&= \mu_1(E). && \text{(since} (A', B') \text{ is a Hahn decomposition)}
\end{aligned}$$

Similarly, it can be shown that $\nu^-(E) = \mu_2(E)$. □

4. **Positive Borel Measures**

4.1. **Motivation.** Some of the nice properties that Lebesgue measure m on $\mathbb{R}$ has are the following:

- For every compact $K \subseteq \mathbb{R}$ $m(K) < \infty$,
- For every Lebesgue measurable set $E \subseteq \mathbb{R}$,
$$m(E) = \inf\{m(V) : V \text{ open } E \subseteq V\},$$
- If E is Lebesgue measurable with $\mu(E) < \infty$ or if $E \subseteq \mathbb{R}$ is open, then
$$m(E) = \sup\{m(K) : K \text{ compact } K \subseteq E\}.$$
- If E is Lebesgue measurable and $m(E) = 0$, then $A \in \mathcal{M}$ for every $A \subseteq E$.

Now, suppose X is a topological space. One may ask:

> Does there exist a σ-algebra $\mathcal{A}$ on X containing all open sets and a measure μ on $\mathcal{A}$ such that it has the above type of nice properties?

We shall answer this question affirmatively if X is a *locally compact Hausdorff* space.

4.2. **Some topological preliminaries.**

Definition 69. A topological space X is said to be a **Hausdorff space** if for every distinct points $x, y \in X$, there exists disjoint open sets U and V such that $x \in U$ and $y \in V$. ♢

Definition 70. A topological space X is said to be a **locally compact space** if for every point $x \in X$, there exist an open set V and a compact set K such that $x \in V \subseteq K$. ♢

It can be easily seen that (Exercise) every closed subset of a compact set is compact. However, a compact subset need not be closed. However,

Theorem 71. *Every a compact subset of a Hausdorff space is closed.*

This follows from the following theorem.

Theorem 72. *Let X be a Hausdorff space, and K be compact in X. Then for each $x \notin K$ there exist disjoint open sets U and V such that $K \subseteq U$ and $x \in V$. In particular, K is closed.*

Proof. Suppose $x \notin K$. Then for each $y \in K$, there exist disjoint open sets G_y and H_y such that $y \in G_y$ and $x \in H_x$. Since $K \subseteq \cup_{y \in K} G_y$ and K is compact, there exists $y_1, \ldots, y_n$ in K such that $K \subseteq \cup_{i=1}^n G_{y_i}$. Now, $U := \cup_{i=1}^n G_{y_i}$ and $V := \cap_{i=1}^n H_{y_i}$ satisfy the requirements. The particular case is obvious, as V is an open set such that $x \in V \cap K = \emptyset$. □

The proof of the following corollary is left as an exercise.

Corollary 73. *Let X be a Hausdorff space. If K is compact and F is closed, then $F \cap K$ is compact.*

Corollary 74. *Let X be a Hausdorff topological space X. Then X is locally compact if and only if for every $x \in X$, there exists an open set V such that $x \in V$ and $\overline{V}$ is compact.*

Proof. Let $x \in X$. Suppose X is locally compact. Then we know that there exist an open set V and a compact set K such that $x \in V \subseteq K$. Since K compact and X is Hausdorff, K is closed. Hence, $\overline{V} \subseteq K$, and hence, being a closed subset of a compact set, $\overline{V}$ is compact. The converse follows from the definition. □

Theorem 75. (Finite intersection property) *Let X be a Hausdorff space and $\mathcal{K}$ is a family of compact sets in X such that $\cap_{K\in\mathcal{K}} K = \emptyset$. Then there exists compact sets $K_0, K_1, \ldots, K_n$ in $\mathcal{K}$ such that $\cap_{i=1}^n K_i = \emptyset$.*

Proof. Let $K_0 \in \mathcal{K}$. Then we have $K_0 \cap \left(\cap_{K\in\mathcal{K}-\{K_0\}} K\right) = \emptyset$. Hence,

$$K_0 \subseteq \left(\cap_{K\in\mathcal{K}-\{K_0\}} K\right)^c = \cup_{K\in\mathcal{K}-\{K_0\}} K^c.$$

Since K_0 is compact, there exists $K_1, \ldots, K_n$ in $\mathcal{K}$ such that $K_0 \subseteq \cup_{i=1}^n K_i^c$. Thus, $K_0 \cap (\cap_{i=1}^n K_i) = \emptyset$. □

Remark 76. Another way of writing Theorem 75 is the following: *Let X be a Hausdorff space and $\mathcal{K}$ is a family of compact sets in X. If $\cap_{K\in\mathcal{K}_0} K \neq \emptyset$ for every finite subfamily $\mathcal{K}_0 \subseteq \mathcal{K}$, then $\cap_{K\in\mathcal{K}} K \neq \emptyset$.* ♢

Theorem 77. *Let X be a locally compact Hausdorff space, K be compact in X and U be open in X such that $K \subseteq U$. Then there exists an open set V such that $\overline{V}$ compact and $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. By Corollary 74, for every $x \in X$, there exists an open set V_x containing x such that $\overline{V_x}$ is compact. Since $K \subseteq \cup_{x\in K} V_x$ and K is compact, there exist $x_1, \ldots, x_n$ in K such that $K \subseteq \cup_{i=1}^n V_i$, where $V_i := V_{x_i}$ for $i = 1, \ldots, n$. Taking $G = \cup_{i=1}^n V_i$ we have $\overline{G} \subseteq \cup_{i=1}^n \overline{V_i}$ where $\cup_{i=1}^n \overline{V_i}$ is compact so that $\overline{G}$ is also compact. Thus, $K \subseteq G \subseteq \overline{G} \subseteq X$, and hence, the theorem if $U = X$.

Next assume that $U \neq X$. Note that $K \cap U^c = \emptyset$. By Theorem 72, for every $x \notin K$, there exist disjoint open sets G_x and H_x such that $x \in G_x$ and $K \subseteq H_x$. Hence, $x \notin \overline{H_x}$. Now, let

$$W_x := U^c \cap \overline{G} \cap \overline{H_x}.$$

Note that W_x is a compact set. Also, we have $\cap_{x\in U^c} W_x = \emptyset$, as $y \in W_x$ for all $x \in U^c$ implies $y \in \overline{H_y}$, that is not true. By Theorem 75, there exists $u_1, \ldots, u_k$ in U^c such that $\cap_{i=1}^k W_{u_i} = \emptyset$, i.e., $U^c \cap \overline{G} \cap \left(\bigcap_{i=1}^k \overline{H_{u_i}}\right) = \emptyset$. Thus, taking $V = G \cap \left(\cap_{i=1}^k H_{u_i}\right)$ we have $\overline{V} \subseteq U$. Note that, since $\overline{V} \subseteq \overline{G} \cap \left(\cap_{i=1}^k \overline{H_{u_i}}\right)$ and $\overline{G} \cap \left(\cap_{i=1}^k \overline{H_{u_i}}\right)$ is a compact set, $\overline{V}$ is also compact. Thus, we obtained an open set V such that $\overline{V}$ compact and $K \subseteq V \subseteq \overline{V} \subseteq U$. □

4.3. **Riesz Representation Theorem.** The main result in this section is the Riesz representation theorem. For its statement, we require a few definitions.

Definition 78. Let X be a topological space. Then a function $f : X \to \mathbb{C}$ is said to be of **compact support** if $cl\{x \in X : f(x) \neq 0\}$ is compact. ◊

For topological space X, we denote by $C_c(X)$ the complex vector space of all continuous (complex valued) functions on X with compact support.

The proof of the following theorem is left as an exercise.

Theorem 79. *Let X be a Hausdorff topological space. Then $f \in C_c(X)$ if and only if there exists a compact set $K \subseteq X$ such that $f(x) = 0$ for all $x \in X \setminus K$.*

Definition 80. A linear functional $\varphi : C_c(X) \to \mathbb{C}$ is said to be a **positive linear functional** if $\varphi(f) \geq 0$ for every $f \in C_c(X)$ with $f \geq 0$. ◊

Theorem 81. **(Riesz representation theorem)** *Let X be a locally compact Hausdorff space and $\varphi : C_c(X) \to \mathbb{C}$ be a positive linear functional. Then there exists a σ-algebra $\mathcal{M}$ which contains all Borel sets of X and a unique measure μ on $\mathcal{M}$ such that the following are satisfied:*

(i) $\varphi(f) = \int_X f d\mu \quad \forall f \in C_c(X)$.
(ii) $\mu(K) < \infty$ *for every compact* $K \subseteq X$.
(iii) $\mu(E) = \inf\{\mu(V) : V \text{ open } E \subseteq V\}$ *for every* $E \in \mathcal{M}$.
(iv) *If either E is open or if $E \in \mathcal{M}$ with $\mu(E) < \infty$, then*

$$\mu(E) = \sup\{\mu(K) : K \text{ compact } K \subseteq E\}.$$

(v) *If $E \in \mathcal{M}$ and $\mu(E) = 0$, then $A \in \mathcal{M}$ for every $A \subseteq E$.*

Example 82. For $x_0 \in X$, the map $f \mapsto f(x_0)$ is a positive linear functional on $C_c(X)$. More generally, for $x_1, \ldots, x_k$ and $w_1, \ldots, w_k$ in $[0, \infty)$, the map $f \mapsto \sum_{i=1}^k f(x_i) w_i$ is a positive linear functional on $C_c(X)$. ◊

Example 83. The map $f \mapsto \int_{\mathbb{R}} f\, dm$ is a positive linear functional on $C_c(\mathbb{R})$. ◊

Example 84. Let $X = \mathbb{N}$ with metric induced by the usual metric on $\mathbb{R}$. Then the map $f \mapsto \sum_{n=1}^\infty f(n)$ is a positive linear functional on $C_c(\mathbb{N}) = c_{00}$. ◊

Example 85. If $\mathcal{M}$ is a σ-algebra on X which contains all open sets, and μ is a measure on $\mathcal{M}$ which is finite for every compact set, then the map $f \mapsto \int_X f d\mu$ is a positive linear functional on $C_c(X)$. $\Diamond$

In the due course, we make use of the following notations:

NOTATION: Suppose X is a locally compact Hausdorff space, K is a compact subset of X and V is an open subset of X.

For $f \in C_c(X)$ with $0 \le f \le 1$, we write

$$\begin{aligned} K \prec f \quad & \text{if} \quad f(x) = 1 \quad \forall x \in K, \\ f \prec V \quad & \text{if} \quad \text{supp} f \subseteq V, \\ K \prec f \prec V \quad & \text{if} \quad K \prec f,\ f \prec V. \end{aligned}$$

We note that for $f \in C_c(X)$ with $0 \le f \le 1$,

$$K \prec f \iff \chi_K \le f, \quad f \prec V \iff f \le \chi_V, \quad K \prec f \prec V \iff \chi_K \le f \le \chi_V.$$

4.4. **Urysohn's Lemma and Partition of Unity.** Now, we state and prove two more theorems, namely, Urysohn's Lemma and Partition of Unity which are not usually covered in a first course in topology, but we need them to prove our main theorem. These results are of independent interest as well.

Before stating Urysohn's lemma, we recall a few more definitions.

Definition 86. Let f be a real valued function defined on a topological space X.

(a) The function f is said to be **lower semi-continuous** (l.s.c) if for every $a \in \mathbb{R}$, the set $\{x \in X : f(x) > a\}$ is open in X, and

(a) The function f is said to be **upper semi-continuous** (u.s.c) if for every $b \in \mathbb{R}$, the set $\{x \in X : f(x) < b\}$ is open in X. $\Diamond$

Thus,

$$f \text{ is l.s.c.} \iff f^{-1}\{(a, \infty)\} \text{ open} \quad \forall\, a \in \mathbb{R}.$$

$$f \text{ is u.s.c.} \iff f^{-1}\{(-\infty, b)\} \text{ open} \quad \forall\, b \in \mathbb{R}.$$

Clearly, f is continuous $\iff$ f is l.s.c. & u.s.c. It can be seen that,

(1) G open implies χ_G is l.s.c,
(2) F closed implies χ_F is u.s.c

To see this, it is enough to observe that if G is open and F is closed in X and $a, b \in \mathbb{R}$, then

$$\{x : \chi_G(x) > a\} = \begin{cases} \emptyset, & a \ge 1, \\ G, & 0 < a < 1, \\ X, & a \le 0. \end{cases},$$

$$\{x : \chi_F(x) < b\} = \begin{cases} \emptyset, & b \le 0, \\ F^c, & 0 < b \le 1, \\ X, & b > 1. \end{cases}$$

We are going to make use of the following facts:

(1) If $\{f_\alpha : \alpha \in J\}$ is a family of l.s.c. functions then the function $f := \sup_{\alpha\in J} f_\alpha$ is also l.s.c.

(2) If $\{g_\alpha : \alpha \in J\}$ is a family of u.s.c. function then the function $g := \inf_{\alpha\in J} g_\alpha$ is also u.s.c.

The above two results follow since for $a, b \in \mathbb{R}$,

$$\{x : f(x) > a\} = \bigcup_{\alpha\in J}\{x : f_\alpha(x) > a\}, \quad \{x : g(x) < a\} = \bigcup_{\alpha\in J}\{x : g_\alpha(x) < b\}.$$

Theorem 87. (Urysohn's lemma) *Suppose X is a locally compact Hausdorff space, K is compact in X and V open in X such that $K \subseteq V$. Then there exists $f \in C_c(X)$ such that $K \prec f \prec V$.*

Proof. By Theorem 77, there exists an open set V_0 such that $\bar{V}_0$ is compact, and

$$K \subseteq V_0 \subseteq \overline{V}_0 \subseteq V.$$

Again applying the same theorem, there exists an open set V_1 such that $\overline{V}_1$ is compact, and

$$K \subseteq V_1 \subseteq \overline{V}_1 \subseteq V_0.$$

Thus,

$$K \subseteq V_1 \subseteq \overline{V}_1 \subseteq V_0 \subseteq \overline{V}_0. \subseteq V.$$

Now, let $r_1, r_2, \ldots$ be an enumeration of the set of all rational numbers in $[0, 1]$ with $r_1 = 0$, $r_2 = 1$, and let $\Delta := \{r_1, r_2, \ldots\}$. Thus, we obtained open sets V_{r_1} and V_{r_2} such that both $\overline{V}_{r_1}$ and $\overline{V}_{r_2}$ are compact, $K \subseteq V_{r_k}$, $\overline{V}_{r_k} \subseteq V$ for $k = 1, 2$ and

$$\overline{V}_{r_2} \subseteq V_{r_1}.$$

Note that $r_1 < r_3 < r_2$, and $\overline{V}_{r_2} \subseteq V_{r_1}$. Hence, there exists an open set, say V_{r_3} such that $\overline{V}_{r_3}$ is compact, and

$$\overline{V}_{r_2} \subseteq V_{r_3} \subseteq \overline{V}_{r_3} \subseteq V_{r_1}.$$

Now, either (i) $r_1 < r_4 < r_3$ or (ii) $r_3 < r_4 < r_2$ occur, and we have

$$\overline{V}_{r_3} \subseteq V_{r_1} \quad \text{and} \quad \overline{V}_{r_2} \subseteq V_{r_3}.$$

Thus, in case (i), there exists an open set, say V_{r_4} such that $\overline{V}_{r_4}$ is compact, and

$$\overline{V}_{r_3} \subseteq V_{r_4} \subseteq V_{r_1},$$

and in case (ii), there exists an open set, say V_{r_4} such that $\overline{V}_{r_4}$ is compact, and

$$\overline{V}_{r_2} \subseteq V_{r_4} \subseteq V_{r_3}.$$

In any case, we obtain an open set V_{r_4} such that $\overline{V}_{r_4}$ is compact, and

$$r, s \in \{r_1, r_2, r_3, r_4\} \text{ with } r < s \Rightarrow \overline{V}_s \subseteq V_r.$$

Now, having obtained open sets V_r for each $r \in \Delta_n := \{r_1, \ldots, r_n\}$ such that $\overline{V}_r$ compact for each $r \in \Delta_n$ and $\overline{V}_r \subseteq V_s$ whenever $s < r$, we find $V_{r_{n+1}}$ as follows: Let r, s in $\{k : 1 \leq k \leq n\}$ be such that

$$r = \max\{r_k : r_k < r_{n+1},\, 1 \leq k \leq n\},$$

$$s = \min\{r_k : r_k > r_{n+1},\, 1 \leq k \leq n\}.$$

Then $r < s$ so that $\overline{V}_s \subseteq V_r$ and hence by Theorem 77 there exists an open set U such that $\bar{U}$ is compact and

$$\overline{V}_s \subseteq U \subseteq \bar{U} \subseteq V_r.$$

We denote this set U by $V_{r_{n+1}}$. Thus, we have a family $\{V_r : r \in \Delta\}$ of open sets such that for each $r \in \Delta$, $\overline{V}_r$ is compact, $K \subseteq V_r$, $\overline{V}_r \subseteq V$ and

$$\overline{V}_s \subseteq V_r \quad \text{whenever} \quad r < s.$$

Now, for each $r \in \Delta$, we define a function f_r as

$$f_r(x) = \begin{cases} r & \text{if } x \in V_r, \\ 0 & \text{if } x \notin V_r. \end{cases}$$

It is easily seen that f_r is lower semi-continuous and hence

$$f = \sup_{r \in \Delta} f_r$$

is also lower semi-continuous. We show that $K \prec f \prec V$.

Observe that

$$x \notin \overline{V}_0 \Rightarrow f_r(x) = 0\, \forall\, r \in \Delta \Rightarrow f(x) = 0.$$

Hence, supp$f \subseteq \overline{V}_0$. In particular, f is with compact support. Also,

$$x \in K \Rightarrow x \in V_r\, \forall\, r \in \Delta \Rightarrow f_r(x) = r \quad \forall\, r \in \Delta.$$

Hence, $f(x) = 1$ for all $x \in K$. Thus, it remains to prove that that f is continuous; for which it is enough to show that f is upper semi-continuous as well. We show that f is equal to an upper semi-continuous function.

For $s \in \Delta$, let

$$g_s(x) = \begin{cases} 1 & \text{if } x \in \overline{V}_s, \\ s & \text{if } x \notin \overline{V}_s \end{cases} \qquad \text{and} \quad g = \inf_{s \in \Delta} g_s.$$

It can be seen that each g_s is upper semi-continuous so that g is also upper semi-continuous. We show that $f = g$.

First we observe that $f_r \leq g_s$ for every $r, s \in \Delta$: Note that if $r \leq s$, then we have $f_r(x) \leq r \leq s \leq g_s(x)$ for all $x \in X$. In case $s < r$, then $\overline{V}_r \subseteq V_s$. Thus, $x \in V_r$ implies

$x \in \overline{V}_s$ so that $f_r(x) = r \leq 1 = g_s(x)$, whereas if $x \notin V_r$ then $f_r(x) = 0 \leq g_s(x)$. Thus, $f_r \leq g_s$ for every $r, s \in \Delta$ which shows that $f \leq g$.

Next we assert that there is no $x \in X$ such that $f(x) < g(x)$. Suppose there exists $x \in X$ such that $f(x) < g(x)$. Let $r, s \in \Delta$ such that $f(x) < r < s < g(x)$. Then we have

$$f_r(x) \leq f(x) < r < s < g(x) \leq g_s(x).$$

Now, $f_r(x) < r$ implies that $x \notin V_r$, $s < g_s(x)$ implies that $x \in \overline{V}_s$, and $r < s$ implies that $\overline{V}_s \subseteq V_r$. Thus, we arrive at a contradiction. Hence, there is no $x \in X$ such that $f(x) < g(x)$. Consequently, $f(x) = g(x)$ for all $x \in X$. This completes the proof. □

Theorem 88. (Partition of unity) *Suppose X is a locally compact Hausdorff space, K is compact in X and $V_1, \ldots, V_n$ are open in X such that $K \subseteq \bigcup_{i=1}^n V_i$. Then there exists $h_1, \ldots, h_n$ in $C_c(X)$ such that*

(i) $h_i \prec V_i,\ i = 1, \ldots, n.$
(ii) $h_1(x) + \ldots + h_n(x) = 1 \quad \forall\, x \in K.$

Proof. Let $x \in K$. Then there exists i such that $x \in V_i$. By local compactness of X, there exists an open set W_x such that $x \in W_x$, $\overline{W}_x$ compact and $\overline{W}_x \subseteq V_i$. Since K is compact and $K \subseteq \cup_{x \in K} W_x \subseteq \cup_{i=1}^n V_i$, there exist $x_1, \ldots, x_m$ in K such that $K \subseteq \cup_{j=1}^m W_{x_j} \subseteq \cup_{i=1}^n V_i$. For each $i \in \{1, \ldots, n\}$, let

$$\Delta_i := \{j \in \{1, \ldots, m\} : \overline{W}_{x_j} \subseteq V_i\} \quad \text{and} \quad H_i := \bigcup_{j \in \Delta_i} \overline{W}_{x_j}.$$

Then each H_i is compact, $H_i \subseteq V_i$ and $K \subseteq \cup_{i=1}^n H_i$. By Urysohn's lemma, there exits $g_1, \ldots, g_n$ such that $H_i \prec g_i \prec V_i$ for $i = 1, \ldots, n$. Define

$$h_1 = g_1, \quad h_i = (1 - g_1)(1 - g_2) \ldots (1 - g_{i-1}) g_i,\ i = 2, \ldots, n.$$

Then it can be seen by induction that

$$h_1 + \ldots + h_k = 1 - (1 - g_1)(1 - g_2) \ldots (1 - g_k), \quad k = 1, \ldots, n.$$

In particular, $h_1 + \ldots + h_n = 1 - (1 - g_1)(1 - g_2) \ldots (1 - g_n)$.

Note that if $x \in K$, then $x \in H_i$ for some $i \in \{1, \ldots, n\}$ so that $g_i(x) = 1$. Hence, $(h_1 + \ldots + h_n)(x) = 1$. Thus, $K \prec h_1 + \ldots + h_n$. Also, $\operatorname{supp} h_i \subseteq \operatorname{supp} g_i \subseteq V_i$ for $i = 1, \ldots, n$. □

4.5. **Proof of Riesz Representation Theorem.** The proof involves many results.

Let X be a locally compact Hausdorff space and $\varphi : C_c(X) \to \mathbb{K}$ be a positive linear functional. First we define a set function $\mu : 2^X \to [0, \infty]$ as follows: For open sent $V \subseteq X$,

$$\mu(V) := \sup\{\varphi(f) : f \prec V\},$$

and for any $E \subseteq X$,

$$\mu(E) := \inf\{\mu(V) : V \text{ open}, \ E \subseteq V\}.$$

We may observe that $\{f \in C_c(X) : f \prec V, \ V \text{ open}\} = \{0\}$ so that $\mu(\phi) = 0$.

We show that μ is an outer measure. First the monotonicity:

Lemma 89. *For $E_1 \subseteq E_2 \subseteq X$, $\mu(E_1) \leq \mu(E_2)$.*

Proof. Note that $\{V : V \text{ open}, V \supseteq E_2\} \subseteq \{V : V \text{ open}, V \supseteq E_1\}$. Therefore,

$$\inf\{\mu(V) : V \text{ open}, V \supseteq E_1\} \leq \inf\{\mu(V) : V \text{ open}, V \supseteq E_1\}.$$

That is, $\mu(E_1) \leq \mu(E_2)$. □

Next, we show that μ is sub-additive. For showing this, we first prove sub-additivity on the family of all open sets:

Lemma 90. *For open sets V_n, $n \in \mathbb{N}$, $\mu(\bigcup_{i=1}^\infty V_i) \leq \sum_{i=1}^\infty \mu(V_i)$.*

Proof. Let $V = \bigcup_{i=1}^\infty V_i$. In order to prove the inequality in the lemma, it is enough to prove that

$$\varphi(f) \leq \sum_{i=1}^\infty \mu(V_i) \quad \forall\, f \prec V. \qquad (*)$$

So, let $f \prec V$ and $K_f := \text{supp} f$. Then $K_f \subseteq \bigcup_{i=1}^\infty V_i$. By compactness of K_f, there exists $n \in \mathbb{N}$ such that $K_f \subseteq \bigcup_{i=1}^n V_i$. Hence, by Partition of Unity (Theorem 88), there exist $h_1, \ldots, h_n \in C_c(X)$ such that $h_i \prec V_i$, $i = 1, \ldots, n$, and $h_1 + \ldots + h_n = 1$ on K_f. Thus, $h_1 f + \ldots + h_n f = f$ so that using the fact that $h_i f \prec V_i$, $i = 1, \ldots, n$,

$$\varphi(f) = \varphi(h_1 f) + \ldots + \varphi(h_n f) \leq \mu(V_1) + \ldots + \mu(V_n).$$

Thus, $(*)$ is proved. □

Proposition 91. *For $E_n \subseteq X$, $n \in \mathbb{N}$, $\mu(\bigcup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu(E_i)$.*

Proof. If atleast one of the $\mu(E_i)$ is infinity, then the inequality is obvious. So, assume that $\mu(E_i) < \infty$ for all $i \in \mathbb{N}$. By definition, for each $i \in \mathbb{N}$ and for every $\varepsilon > 0$, there exists an open set $V_i \supseteq E_i$ such that $\mu(V_i) < \mu(E_i) + \varepsilon/2^i$. Therefore, using Lemma 89 and Lemma 90,

$$\mu(\bigcup_{i=1}^\infty E_i) \leq \mu(\bigcup_{i=1}^\infty V_i) \leq \sum_{i=1}^\infty \mu(V_i) \leq \sum_{i=1}^\infty \mu(E_i) + \varepsilon.$$

This is true for every $\varepsilon > 0$. Hence the result follows. □

Lemma 89 and Proposition 91 show that μ is an outer measure. Hence, μ is a measure on the σ-algebra of all μ-measurable sets. However, it is not clear that this σ-algebra has all the required properties listed in RRT. So, our next endeavor is to obtain an appropriate σ-algebra $\mathcal{M}$. Let us observe some more properties of μ.

Let us observe some properties of μ which include some of the requirement of the μ in RRT.

Proposition 92. *For every open $V \subseteq X$,*

$$\mu(V) = \sup\{\mu(K) : K \subseteq V,\, K \,\text{compact}\}.$$

Proof. Let V be an open set. Clearly,

$$\sup\{\mu(K) : K \subseteq V,\, K \,\text{compact}\} \leq \mu(V).$$

To show the other way inequality, it is enough to show that for every $\varepsilon > 0$, there exists a compact set $K \subseteq V$ such that $\mu(V) - \varepsilon \leq \mu(K)$.

So, let $\varepsilon > 0$ be given. Then, by definition of $\mu(V)$, there exists $f \prec V$ such that $\mu(V) - \varepsilon \leq \varphi(f)$. Let $K_f := \text{supp} f$. Then $\varphi(f) \leq \mu(G)$ for all open sets $G \supseteq K_f$. Hence

$$\varphi(f) \leq \inf\{\mu(G) : K_f \subseteq G,\, G \text{ open}\} = \mu(K_f).$$

Thus, $\mu(V) - \varepsilon \leq \varphi(f) \leq \mu(K_f)$. □

Lemma 93. *If $K \prec f$, then $\mu(K) \leq \varphi(f)$. In particular, $\mu(K) < \infty$ for every compact set $K \subseteq X$.*

Proof. Let $K \prec f$. To see the first part, it is enough to show that

$$\alpha\mu(K) \leq \varphi(f) \quad \forall\, \alpha \text{ with } 0 < \alpha < 1. \qquad (*)$$

So let $0 < \alpha < 1$. Let $V_\alpha := \{x : f(x) > \alpha\}$. Then

$$K \subseteq V_\alpha \quad \text{so that} \quad \mu(K) \leq \mu(V_\alpha) := \sup\{\varphi(g) : g \prec V_\alpha\}.$$

Now,

$$g \prec V_\alpha \Rightarrow \alpha g \prec V_\alpha \Rightarrow \alpha g \leq f \Rightarrow \varphi(g) \leq \alpha^{-1}\varphi f.$$

Thus

$$\mu(K) \leq \mu(V_\alpha) := \sup\{\varphi(g) : g \prec V_\alpha\} \leq \alpha^{-1}\varphi(f).$$

Thus, $(*)$ is proved.

The particular case follows from the above by making use of Uryshohn's lemma, since for every compact K, there exists $f \in C_c(X)$ such that $K \prec f \prec X$, and hence $\mu(K) \leq \varphi(f) < \infty$. □

Lemma 94. *For every compact $K \subseteq X$,*

$$\mu(K) = \inf\{\varphi(f) : K \prec f\}.$$

Proof. Let K be a compact set. By Lemma 93,

$$\mu(K) \leq \inf\{\varphi(f) : K \prec f\}.$$

Thus, $\mu(K)$ is a lower bound for $\{\varphi(f) : K \prec f\}$. Therefore, it is enough to show that for every $\varepsilon > 0$, there exists $f \in C_c(X)$ such that $\varphi(f) \leq \mu(K) + \varepsilon$.

So, let $\varepsilon > 0$ be given. Recall that $\mu(K) = \inf\{\mu(V) : V \text{ open}, K \subseteq V\}$. Hence, there exists open $V \supseteq K$ such that $\mu(V) < \mu(K) + \varepsilon$. Also, by Urysohn's lemma, there exists f such that $K \prec f \prec V$. Hence

$$\varphi(f) \leq \mu(V) \leq \mu(K) + \varepsilon.$$

Thus, the proof is complete. □

Lemma 95. *For disjoint compact sets K_1, K_2,*

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2).$$

Proof. By Proposition 91, we have $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$. Since K_1 and K_2 are disjoint, by Urysohn's lemma, there exists f such that $K_1 \prec f \prec K_2^c$. Also, by Lemma 94, for every $\varepsilon > 0$, there exists g such that

$$K_1 \cup K_2 \prec g \quad \text{and} \quad \varphi(g) \leq \mu(K_1 \cup K_2) + \varepsilon.$$

Now, $K_1 \prec f$ and $K_2 \prec 1 - f$ (since $f = 0$ on K_2) implies $K_1 \prec fg$ and $K_2 \prec (1 - f)g$ so that using Lemma 93,

$$\mu(K_1) + \mu(K_2) \leq \varphi(fg) + \varphi((1 - f)g) = \varphi(g) \leq \mu(K_1 \cup K_2) + \varepsilon.$$

This is true for every $\varepsilon > 0$. Hence the result follows. □

Now, consider a candidate for the required σ-algebra: For that, let $\mathcal{M}_0$ be the family of all subsets such that

$$A \in \mathcal{M}_0 \iff \mu(A) < \infty \text{ and } \mu(A) = \sup\{\mu(K) : K \text{ compact}, K \subseteq A\}$$

and let

$$\mathcal{M} := \{E \subseteq X : E \cap K \in \mathcal{M}_0 \text{ for every compact } K\}.$$

We prove that $\mathcal{M}$ is a σ-algebra on $\mathcal{M}$ and μ is a unique measure on $\mathcal{M}$ satisfying all requirements in the statement of the theorem. Note that

- $\mathcal{M}_0$ contains all compact subsets of X.

Also, $E \subseteq X$, $\mu(E) = 0 \Rightarrow E \in \mathcal{M}_0 \cap \mathcal{M}$. Therefore,

Proposition 96. *For $E \in \mathcal{M}$ with $\mu(E) = 0$, $A \subseteq E \Rightarrow A \in \mathcal{M}$.*

As a corollary to Proposition 92, we also have

Proposition 97. *If V is open and $\mu(V) < \infty$, then $V \in \mathcal{M}_0$.*

In order to prove that $\mathcal{M}$ is a σ-algebra containing all Borel sets of X, we first prove the additivity of μ on $\mathcal{M}_0$.

Proposition 98. *Suppose $\{E_i : i \in \mathbb{N}\}$ is a disjoint family in $\mathcal{M}_0$. Then*

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Moreover, if $\mu(\bigcup_{i=1}^{\infty} E_i) < \infty$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}_0$.

Proof. In view of Proposition 91, it is enough to prove that $\sum_{i=1}^{\infty} \mu(E_i) \le \mu(\bigcup_{i=1}^{\infty} E_i)$. Let $E = \bigcup_{i=1}^{\infty} E_i$. If $\mu(E)$ is infinity, then clearly the results follows. Hence assume that $\mu(E) < \infty$ so that $\mu(E_i) < \infty$ for all $i \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $E_i \in \mathcal{M}_0$, there exists compact $K_i \subseteq E_i$ such that $\mu(E_i) \le \mu(K_i) + \varepsilon/2^i$. Thus, using Proposition 95,

$$\sum_{i=1}^{n} \mu(E_i) \le \sum_{i=1}^{n} \mu(K_i) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} = \mu\big(\bigcup_{i=1}^{n} K_i\big) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \le \mu(\bigcup_{i=1}^{\infty} E_i) + \varepsilon.$$

This is true for all $n \in \mathbb{N}$ and for every $\varepsilon > 0$. Hence we have $\sum_{i=1}^{\infty} \mu(E_i) \le \mu(\bigcup_{i=1}^{\infty} E_i)$.

Now, in the case of $\mu(E) < \infty$, the series $\sum_{i=1}^{\infty} \mu(E_i)$ converges in $\mathbb{R}$ so that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \le \sum_{i=1}^{N} \mu(E_i) + \varepsilon \le \mu\big(\bigcup_{i=1}^{N} K_i\big) + 2\varepsilon.$$

Thus, for every $\varepsilon > 0$, there exists a compact set $K := \bigcup_{i=1}^{N} K_i \subseteq E$ such that $\mu(E) \le \mu(K) + 2\varepsilon$. Hence it follows that $E \in \mathcal{M}_0$. $\square$

We require the following two lemmas too.

Lemma 99. *Suppose $E \in \mathcal{M}_0$. Then for every $\varepsilon > 0$, there exist an open $V \supseteq E$ and a compact $K \subseteq E$ such that*

$$\mu(V - K) < \varepsilon.$$

In particular,

$$\mu(V - E) < \varepsilon, \qquad \mu(E - K) < \varepsilon.$$

Proof. Let $E \in \mathcal{M}_0$ and $\varepsilon > 0$. Then by definitions of $\mathcal{M}_0$ and $\mu(E)$, there exist a compact set K and an open set V such that $K \subseteq E \subseteq V$ and $\mu(V) < \mu(E) + \varepsilon/2$ and $\mu(E) < \mu(K) + \varepsilon/2$. In particular, $\mu(V) < \mu(K) + \varepsilon < \infty$. By Proposition 97 the open sets V and $V - K$ belong to $\mathcal{M}_0$. Therefore, by Proposition 98,

$$\mu(K) + \mu(V - K) = \mu(V) < \mu(K) + \varepsilon.$$

Thus, $\mu(V - K) < \varepsilon$. $\square$

Lemma 100. *If A, B are in $\mathcal{M}_0$, then $A - B$, $A \cup B$, $A \cap B$ are also in $\mathcal{M}_0$.*

Proof. Let A, B be in $\mathcal{M}_0$ and $\varepsilon > 0$ be given. By Lemma 99, there exits compact sets K_1, K_2 and open sets V_1, V_2 such that $K_1 \subseteq A \subseteq V_1$, $K_2 \subseteq B \subseteq V_2$ and $\mu(V_i - K_i) < \varepsilon$ for $i = 1, 2$. We observe that[3]

$$A - B \ \subseteq\ V_1 - K_2 \ \subseteq\ (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2).$$

Hence,

$$\mu(A - B) \leq \varepsilon + \mu(K_1 - V_2) + \varepsilon.$$

Thus, we obtained a compact set $K := K_1 - V_2 \subseteq A - B$ such that $\mu(A - B) \leq \mu(K) + 2\varepsilon$ so that $A - B \in \mathcal{M}_0$.

Now, by Proposition 98, we have

$$A \cup B = (A - B) \cup B \in \mathcal{M}_0.$$

Also, we have $A \cap B = A - (A - B) \in \mathcal{M}_0$.

To see the last part, let $E \in \mathcal{M}_0$. Then, for every compact set K, $K \cap E \in \mathcal{M}_0$ so that $\in \mathcal{M}$. □

Proposition 101. *$\mathcal{M}$ is a σ-algebra containing all Borel sets of X.*

Proof. Suppose $A \in \mathcal{M}$. Then, by Lemma 100, for any compact set K,

$$A^c \cap K = K - (A \cap K) \in \mathcal{M}_0$$

so that $A^c \in \mathcal{M}$. Next, let $A_i \in \mathcal{M}$ for $i \in \mathbb{N}$. To show that $A := \cup_{i=1}^{\infty} A_i \in \mathcal{M}$. That is to show that for any compact set K, $A \cap K \in \mathcal{M}_0$. Note that

$$A \cap K = \cup_{i=1}^{\infty} A_i \cap K = \cup_{i=1}^{\infty} B_i,$$

where

$$B_1 = A_1 \cap K \quad \text{and} \quad B_n = (A_n \cap K) - \bigcup_{i=1}^{n-1} B_i \quad \text{for} \quad n = 2, 3, \ldots.$$

Then $\{B_n : n \in \mathbb{N}\}$ is a disjoint family in $\mathcal{M}_0$ with $\mu(\cup_{n=1}^{\infty} B_n) = \mu(A \cap K) < \infty$ so that by Proposition 98,

$$A \cap K = \cup_{n=1}^{\infty} B_n \in \mathcal{M}_0.$$

Thus $\mathcal{M}$ is a σ-algebra. It remains to show that $\mathcal{M}$ contains all open sets. For this, let F be a closed set. Then for any compact set K, $F \cap K$ is a compact set so that $F \in \mathcal{M}$. Thus, $\mathcal{M}$ contains all closed sets. Since $\mathcal{M}$ is a σ-algebra, it follows that $\mathcal{M}$ contains all Borel sets. □

We shall make use of the following fact also.

---

[3] Note that
$V_1 - K_2 = V_1 \cap K_2^c = [(V_1 \cap K_2^c) \cap K_1] \cup [(V_1 \cap K_2^c) \cap K_1^c] = (K_1 - K_2) \cup (V_1 - K_1)$
$K_1 - K_2 = (K_1 \cap K_2^c) \cap (V_2 \cup V_2^c) = (K_1 \cap K_2^c \cap V_2) \cup (K_1 \cap K_2^c \cap V_2^c) \subseteq (V_2 - K_2) \cup (K_1 - V_2)$.
Thus, $V_1 - K_2 \ \subseteq\ (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$.

Proposition 102. *$\mathcal{M}_0 \subseteq \mathcal{M}$, and $\mathcal{M}_0 = \{E \in \mathcal{M} : \mu(E) < \infty\}$.*

Proof. Let $E \in \mathcal{M}_0$. Then for any compact set K, we have $K \in \mathcal{M}_0$ so that by Lemma 100, $E \cap K \in \mathcal{M}_0$. Thus $E \in \mathcal{M}$.

Next, let $E \in \mathcal{M}$ such that $\mu(E) < \infty$, and let $\varepsilon > 0$ be given. Then to show that there exists a compact set $K \subseteq E$ such that $\mu(E) < \mu(K) + \varepsilon$. By definition of $\mu(E)$, there exists an open set $V \supseteq E$ such that $\mu(V) < \mu(E) + \varepsilon/2$. In particular, by Corollary 97, $V \in \mathcal{M}_0$. Hence, by Lemma 99, there exists a compact set $K_0 \subseteq V$ such that $\mu(V - K_0) < \varepsilon/2$. Thus, since $E \subseteq (E \cap K_0) \cup (V \cap K_0^c)$,

$$\mu(E) \leq \mu(E \cap K_0) + \mu(V - K_0) < \mu(E \cap K_0) + \varepsilon/2.$$

But, since $E \cap K_0 \in \mathcal{M}_0$, there exists a compact set K such that $\subseteq E \cap K_0$ and $\mu(E \cap K_0) < \mu(K) + \varepsilon/2$. Thus, we have

$$\mu(E) < \mu(E \cap K_0) + \varepsilon/2 < \mu(K) + \varepsilon.$$

This completes the proof. □

Proposition 103. *μ is a measure on $\mathcal{M}$.*

Proof. Suppose $\{E_i : i \in \mathbb{N}\}$ is a disjoint family in $\mathcal{M}$ and $E = \bigcup_{i=1}^{\infty} E_i$. If $\mu(E) = \infty$, then from $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$ it follows that $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$. If $\mu(E) < \infty$, then $\mu(E_i) < \infty$ for every $i \in \mathbb{N}$, so that by Propositions 102 and 98, $E_i \in \mathcal{M}_0$ for all $i \in \mathbb{N}$, $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$. □

Proposition 104. *For every $f \in C_c(X)$, $\varphi(f) = \int_X f d\mu$.*

Proof. We prove the required relation for real valued f's. The complex valued case will follow by using the linearity of φ and the linearity of the integral. So, let $f \in C_c(X)$ such that $f(x) \in \mathbb{R}$ for every $x \in X$.

First we note that it is enough to show that

$$\varphi(f) \leq \int_X f d\mu \tag{1}$$

for every real valued $f \in C_c(X)$. Because, once (1) is proved for every real valued $f \in C_c(X)$, then taking $g = -f$, we have $\Lambda g \leq \int_X g d\mu$, that is, $-\varphi(f) \leq -\int_X f d\mu$ so that $\varphi(f) \geq \int_X f d\mu$.

Now to prove (1), let $K = \mathrm{supp} f$ and $[a, b]$ such that $\mathrm{range}(f) \subseteq [a, b]$. Let $\varepsilon > 0$ and $t_0, t_1, \ldots, t_n$ in $\mathbb{R}$ be such that

$$t_0 < a < t_1 < t_2 < \ldots < t_n = b$$

and $t_i - t_{i-1} < \varepsilon$ for $i = 1, 2, \ldots, n$. Let $E_i = K \cap f^{-1}(t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. Clearly, $K = \cup_{i=1}^{n} E_i$ and $\{E_i : i = 1, 2, \ldots, n\}$ is a disjoint family. For each i let V_i be an open set such that $E_i \subseteq V_i$ and

$$\mu(V_i) < \mu(E_i) + \varepsilon/n, \quad i = 1, \dots, n$$

Without loss of generality, we can also assume that $f(x) < t_i + \varepsilon$ for all $x \in V_i$, possibly by replacing V_i by $V_i \cap f^{-1}(-\infty, t_i + \varepsilon)$. Now, since $K \subseteq \cup_{i=1}^n V_i$, by Partition of Unity, there exits $h_i \prec V_i$ such that $h_1 + \ldots + h_n = 1$ on K. Hence, $f = h_1 f + \ldots + h_n f$ on X. Thus, since $f \le t_i + \varepsilon$ on V_i, we have

$$\varphi(f) = \sum_{i=1}^n \varphi(h_i f) \le \sum_{i=1}^n \varphi((t_i + \varepsilon)h_i) = \sum_{i=1}^n (t_i + \varepsilon)\varphi(h_i) \le \sum_{i=1}^n (t_i + \varepsilon)\mu(V_i). \qquad (2)$$

Note that

$$\begin{aligned} (t_i + \varepsilon)\mu(V_i) &\le (t_i + \varepsilon)[\mu(E_i) + \varepsilon/n] \\ &\le (t_i + \varepsilon)\mu(E_i) + (b + \varepsilon)\varepsilon/n \\ &\le (t_{i-1} + 2\varepsilon)\mu(E_i) + (b + \varepsilon)\varepsilon/n \\ &\le \int_{E_i} f\, d\mu + 2\varepsilon\mu(E_i) + (b + \varepsilon)\varepsilon/n \end{aligned}$$

Hence, (2) implies

$$\begin{aligned} \varphi(f) &\le \sum_{i=1}^n \Big[\int_{E_i} f\, d\mu + 2\varepsilon\mu(E_i) \Big] + (b + \varepsilon)\varepsilon \\ &= \int_K f\, d\mu + 2\varepsilon\mu(K) + (b + \varepsilon)\varepsilon \\ &= \int_X f\, d\mu + 2\varepsilon\mu(K) + (b + \varepsilon)\varepsilon. \end{aligned}$$

This is true for all $\varepsilon > 0$. Thus, (1) is proved. □

Proposition 105. *The measure μ is a unique measure on $\mathcal{M}$ such that* (i)-(iv) *in Theorem 81 are satisfied.*

Proof. Suppose $\tilde{\mu}$ is another measure on $\mathcal{M}$ satisfying (i)-(iv) in Theorem 81. In order to show $\tilde{\mu} = \mu$ it is enough to show that $\tilde{\mu}(K) = \mu(K)$ for every compact set $K \subseteq X$ and $\varepsilon > 0$ be given. Then there exists an open set $V \supseteq K$ such that $\mu(V) < \mu(K) + \varepsilon$, and by Urysohn's lemma, there exists f such that $K \prec f \prec V$. Hence,

$$\tilde{\mu}(K) = \int_X \chi_K d\tilde{\mu} \le \int_X f d\tilde{\mu} = \varphi(f) \le \int_X \chi_V d\mu = \mu(V) \le \mu(K) + \varepsilon.$$

Thus, $\tilde{\mu}(K) \le \mu(K)$ for every compact set K. Similarly we can show that $\mu(K) \le \tilde{\mu}(K)$ for every compact set K. □

4.6. **Lebesgue measure on $\mathbb{R}$.** We observe that, if $f \in C_c(\mathbb{R})$, then $f \in C[a,b]$ for some $[a,b] \supseteq \mathrm{supp} f$ so that $\int_X f(x)dx$ is well-defined as Riemann integral. It can be easily seen that

$$\varphi(f) := \int_{\mathbb{R}} f(x)dx, \quad f \in C_c(\mathbb{R}),$$

defines a positive linear functional on $C_c(\mathbb{R})$.

Hence, by Riesz representation theorem, there exists a σ-algebra $\mathcal{M}$ containing all Borel sets of $\mathbb{R}$ and a unique measure μ on $\mathcal{M}$ satisfying (i)-(v) in Riesz representation theorem. This measure is, in fact, the Lebesgue measure on $\mathbb{R}$.

4.7. **Regularity of Measure.** Recall that the measure μ obtained in Riesz representation theorem has the properties

$$\mu(E) = \inf\{\mu(V) : V \text{ open } E \subseteq V\} \tag{$*$}$$

for all $E \in \mathcal{M}$ and

$$\mu(E) = \sup\{\mu(K) : K \text{ compact } K \subseteq E\} \tag{$**$}$$

for all open sets E and for all $E \in \mathcal{M}$ with $\mu(E) < \infty$.

Definition 106. Let X be a topological space $\mathcal{A}$ be a σ-algebra on X which contains all Borel sets, and μ be a measure on $\mathcal{A}$.

(a) $E \in \mathcal{A}$ is called **outer regular** w.r.t. μ if

$$\mu(E) = \inf\{\mu(V) : V \text{ open } E \subseteq V\}. \tag{$*$}$$

(b) $E \in \mathcal{A}$ is called **inner regular** w.r.t. μ if

$$\mu(E) = \sup\{\mu(K) : K \text{ compact } K \subseteq E\}. \tag{$**$}$$

(c) $E \in \mathcal{A}$ is called **regular** w.r.t. μ if ($*$) and ($**$) are satisfied.

(d) μ is said to be **outer regular** (resp. **inner regular**) if ($*$) (resp. ($**$)) is satisfied for every $E \in \mathcal{A}$.

(e) μ is said to be **regular** if it is outer regular and inner regular. ♢

We know that if X a locally compact Hausdorff space, then the measure in RRT is outer regular and every E with $\mu(E) < \infty$ is regular.

We shall impose additional condition on X so that the μ in RRT is regular as well.

Definition 107. A topological space is said to be **σ-compact** if it is a union of countable number of compact subsets. ♢

Theorem 108. *Let X be a locally compact Hausdorff space and $\mathcal{M}$ and μ have the properties as in Riesz representation theorem. If, in addition, X is σ-compact, then μ is regular.*

Proof. Since X is σ-compact, there exists compact sets K_i such that $X = \cup_{i=1}^{\infty} K_i$. Let $E \in \mathcal{M}$. We have to show that there exists a sequence (H_n) of compact sets such that $H_n \subseteq E$ for all $n \in \mathbb{N}$ and $\mu(H_n) \to \mu(E)$.

Note that for $E \in \mathcal{M}$, $E = \cup_{n=1}^{\infty} E_n$ where $E_n := E \cap (\cup_{i=1}^{n} K_i)$. We also have $\mu(E_n) \to \mu(E)$. Since $\mu(E_n) \leq \mu(\cup_{i=1}^{n} K_i) < \infty$, there exist compact sets $H_n \subseteq E_n$ and $\mu(E_n) < \mu(H_n) + 1/n$. Hence, it follows that $\mu(H_n) \to \mu(E)$ as $n \to \infty$. □

Theorem 109. *Let X be a locally compact Hausdorff space which is also σ-compact, and let $\mathcal{M}$ and μ have the properties as in Riesz representation theorem. Let $E \in \mathcal{M}$. Then we have the following:*

(a) For every $\varepsilon > 0$, there exists an open set $V \supseteq E$ and a closed set $F \subseteq E$ such that

$$\mu(V - E) < \varepsilon, \qquad \mu(E - F) < \varepsilon.$$

(b) There exists a G_δ-set $V \supseteq E$ and an F_σ-set $F \subseteq E$ such that

$$\mu(V - E) = 0 = \mu(E - F).$$

Proof. Since X is σ-compact, there exists compact sets K_i such that $X = \cup_{i=1}^{\infty} K_i$. Then we have $E = \cup_{i=1}^{\infty} E_i$ where $E_i := E \cap K_i$. Since $\mu(E_i) < \infty$, there exists open set $V_i \supseteq E_i$ such that $\mu(V_i - E_i) < \varepsilon/2^i$. Then we have

$$E \subseteq \bigcup_{i=1}^{\infty} V_i = V, \qquad V - E \subseteq \bigcup_{i=1}^{\infty} (V_i - E_i).$$

Thus, V is open and

$$\mu(V - E) \leq \sum_{i=1}^{\infty} \mu(V_i - E_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

Applying this result with E^c in place of E, there exists an open set $G \supseteq E^c$ such that $\mu(G - E^c) < \varepsilon$. But, $G - E^c = G \cap E = E - G^c$. Thus the closed set $F := G^c$ satisfies the relation $\mu(E - F) < \varepsilon$. Thus, (a) is proved.

By (a), for each n, there exists open sets $V_n \supseteq E$ and closed sets $F_n \subseteq E$ such that $\mu(V_n - E) < 1/n$ and $\mu(E - F_n) < 1/n$. Take $V = \cap_{n=1}^{\infty} V_n$ and $F = \cup_{n=1}^{\infty} F_n$. Then we have

$$\mu(V - E) \leq \mu(V_n - E) < 1/n, \qquad \mu(E - F) \leq \mu(E - F_n) < 1/n \quad \forall n \in \mathbb{N}.$$

Hence we have the result in (b). □

Here is a more general result than Theorem 108.

Theorem 110. *Let X be a locally compact Hausdorff space and λ be a positive Borel measure such that $\lambda(K) < \infty$ for every compact set K. If every open set in X is σ-compact, then λ is regular.*

Proof. Let $\varphi(f) := \int_X f d\lambda$ for $f \in C_c(X)$. Then it is seen that Λ is a positive linear functional on $C_c(X)$. Let $\mathcal{M}$ and μ be as in Riesz representation theorem. In particular we have

$$\int_X f d\lambda = \int_X f d\mu \quad \forall f \in C_c(X).$$

We have to show that $\lambda(E) = \mu(E)$ for all Borel set E. First we show that $\lambda(V) = \mu(V)$ for all open set V. By hypothesis, there exists compact sets $K_i, i \in \mathbb{N}$ such that $V = \cup_{i=1}^{\infty} K_i$. By Urysohn's lemma, there exists $f_i \in C_c(X)$ such that $K_i \prec f_i \prec V$ for every $i \in \mathbb{N}$. Let

$$g_n := \max\{f_1, \ldots, f_n\}, \quad n \in \mathbb{N}.$$

Then $g_n \in C_c(X)$ and $0 \leq g_n \leq g_{n+1}$ and $g_n \to \chi_V$. Hence, by Monotone convergence theorem,

$$\int_X \chi_V d\lambda = \lim_{n\to\infty} \int_X g_n d\lambda = \lim_{n\to\infty} \int_X g_n d\mu = \int_X \chi_V d\mu.$$

Thus $\lambda(V) = \mu(V)$.

Now, let E be a Borel set. Since X is σ-compact, by Theorem 108, we know that λ is a regular measure. Hence, for every $\varepsilon > 0$, there exists an open set V and a closed set F such that

$$F \subseteq E \subseteq V, \qquad \mu(V - F) < \varepsilon.$$

Since $V - F$ is open, we also have $\lambda(V - F) = \mu(V - F) < \varepsilon$. Thus,

$$\mu(V) < \mu(F) + \varepsilon \leq \mu(E) + \varepsilon, \qquad \lambda(V) < \lambda(F) + \varepsilon \leq \lambda(E) + \varepsilon$$

and hence

$$\lambda(E) \leq \lambda(V) = \mu(V) \leq \mu(E) + \varepsilon,$$
$$\mu(E) \leq \mu(V) = \lambda(V) \leq \lambda(E) + \varepsilon.$$

Thus, it follows that $\lambda(E) = \mu(E)$. ☐

4.8. **Lebesgue Measure on $\mathbb{R}^k$.** For $f : \mathbb{R}^k \to \mathbb{C}$ with compact support, let

$$\Lambda_n f := 2^{-nk} \sum_{x \in P_n} f(x),$$

where P_n is the set of all points in $\mathbb{R}^k$ with coordinates as integer multiples of 2^{-n}. Let Ω_n be the set of all *k-cells* of volume 2^{-nk} in $\mathbb{R}^k$ with corners in P_n. Here, by a k-cell we mean a set of the form $I_1 \times \ldots \times I_k$ where $J_j := [a_j, b_j)$ for some $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$, and the corner of $I_1 \times \ldots \times I_k$ is the point $(a_1, a_2, \ldots, a_k)$. If $W = I_1 \times \ldots \times I_k$ is a k-cell, then we denote $\text{vol}(W) := (b_1 - a_1) \times \ldots \times (b_k - a_k)$.

Now, if $f \in C_c(\mathbb{R}^k)$, then by uniform continuity of f it can be shown that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and functions g and h with compact support and which are constants on cells in Ω_N and satisfying

$$g \leq f \leq h, \qquad h - g < \varepsilon.$$

Clearly, $\Lambda_N g = \Lambda_n g$, $\Lambda_N h = \Lambda_n h$ for all $n \geq N$ so that

$$\Lambda_N g = \Lambda_n g \leq \Lambda_n f \leq \Lambda_n h = \Lambda_N h \quad \forall\, n \geq N.$$

Therefore $\varphi(f) := \lim_{n\to\infty} \Lambda_n f$ exists. It can be shown that Λ is a positive linear functional on $C_c(\mathbb{R}^k)$. Hence, by Riesz representation theorem, there exists a σ-algebra $\mathcal{M}_k$ which contains all Borel sets in $\mathbb{R}^k$ and a measure m_k on $\mathcal{M}$ which satisfy all the conditions in Riesz representation theorem. Since $\mathbb{R}^k$ is also σ-compact, this measure is regular as well.

Members of the a σ-algebra $\mathcal{M}_k$ are called *Lebesgue measurable sets* and m_k as the Lebesgue measure on $\mathbb{R}^k$.

The measure m_k has many more properties than already described.

Theorem 111. *The σ-algebra $\mathcal{M}_k$ and the measure m_k on $\mathbb{R}^k$ have the following properties:*

(a) *For every k-cell W, $m_k(W) = \mathrm{vol}(W)$.*

(b) *m_k is regular and complete, and $E \in \mathcal{M}_k$ if and only if there exists an F_σ-set F and a G_δ-set G such that $F \subseteq E \subseteq G$ and $m_k(G - F) = 0$.*

(c) *m_k is translation invariant, that is, for every $E \in \mathcal{M}_k$ and $x \in \mathbb{R}^k$, $m_k(E+x) = m_k(E)$.*

(d) *If μ is any translation invariant Borel measure on $\mathbb{R}^k$ such that $\mu(K) < \infty$ for every compact set K, then there exists $c_K > 0$ such that $\mu(E) = c_K m_k(E)$ for all Borel set E.*

(e) *If $T : \mathbb{R}^k \to \mathbb{R}^k$ is a linear transformation then*

$$m_k(T(E)) = |\det(T)| m_k(E) \quad \forall\, E \in \mathcal{M}_k.$$

4.9. **Some Approximation Theorems.** In this section we assume that X is a locally compact Hausdorff space, $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $\mathcal{M}$ having the properties stated in Riesz Representation Theorem 81.

Theorem 112. (Lusin's theorem) *Suppose f is a complex measurable function on X such that there exists $A \in \mathcal{M}$ with*

$$\mu(A) < \infty \quad \text{and} \quad f(x) = 0 \quad \text{for} \quad x \notin A.$$

Then for every $\varepsilon > 0$, there exists $g \in C_c(X)$ such that

$$\mu(\{x \in X : f(x) \neq g(x)\}) < \varepsilon.$$

In fact, we can have g such that $dis\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|$.

Proof. **Step (i)**: *Assume that A is compact and $0 \le f < 1$.*

We know that the sequence (φ_n) of simple functions defined by

$$\varphi_n := \sum_{i=1}^{2^n} \frac{i-1}{2^n} \chi_{E_{i,n}}, \quad E_{i,n} = \left\{ x : \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n} \right\}$$

satisfies $\varphi_n \le \varphi_{n+1}$ for all $n \in \mathbb{N}$, and $\varphi_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. We may also observe that $\varphi_n = \varphi_1 + (\varphi_2 - \varphi_1) + \cdots + (\varphi_n - \varphi_{n-1})$ so that

$$f = \sum_{n=1}^{\infty} \psi_n, \quad \text{where} \quad \psi_n = \varphi_n - \varphi_{n-1}, \quad \varphi_0 = 0.$$

Moreover, $2^n \psi_n = \chi_{T_n}$ for some $T_n \subseteq A$: Indeed, if $x \in E_i$, then

$$\varphi_n(x) = (i-1)/2^n = (2i-2)/2^{n+1} \quad \text{and} \quad \varphi_{n+1}(x) \in \{(2i-2)/2^{n+1}, (2i-1)/2^{n+1}\}$$

so that $\varphi_{n+1}(x) - \varphi_n(x) \in \{0, 1/2^{n+1}\}$, i.e., $2^{n+1}\psi_{n+1}(x) \in \{0, 1\}$.

Now, since $\mu(T_n) \le \mu(A) < \infty$, there exists compact $K_n \subseteq T_n$ and open $V_n \supseteq T_n$ such that $\mu(V_n - K_n) < \varepsilon/2^n$. Also, since A is compact there exists an open set $V \supseteq A$ with $\bar{V}$ compact. Hence, without loss of generality, we may assume that $V_n \subseteq V$ (replacing V_n by $V_n \cap V$, if necessary). By Urysohn's lemma, there exists $h_n \in C_c(X)$ such that $K_n \prec h_n \prec V_n$. Note that $2^n \psi_n = h_n$ except on $V_n - K_n$, that is,

$$2^n \psi_n = h_n \text{ on } K_n \cup V_n^c.$$

Indeed, $x \in K_n \subset T_n$ implies $h_n = 1$, $2^n \psi_n = 1$, and $x \in V_n^c \subseteq T_n^c$ implies $h_n = 0$, $\psi_n = 0$. Hence, defining

$$g = \sum_{n=1}^{\infty} 2^{-n} h_n$$

we see that g, being the uniform limit of a series of continuous functions, is continuous and $\text{supp}(g) \subseteq \cup_{n=1}^{\infty} V_n \subseteq \overline{V}$. Hence, $g \in C_c(X)$ and

$$g = f \quad \text{except on} \quad E := \bigcup_{n=1}^{\infty} (V_n - K_n).$$

(Since $g = 0$ on $\bigcap_{n=1}^{\infty}(K_n \cup V_n^c$.) Note that $\mu(E) \le \sum_{n=1}^{\infty} \mu(V_n - K_n) < \varepsilon$. Thus, this g satisfies the requirements.

Step (ii): *Assume that A is compact and $0 \le f < b$ for some $b > 0$.*

In this case we may apply step (i) with f/b in place of f.

Step (iii): *Assume that A is compact and f is bounded real valued.*

We apply the earlier part to f^+ and f^-, and then derive the result for f. [Indeed, there exists g_1 and g_2 in $C_c(X)$ such that

$$\mu(\{x : f^+(x) \ne g_1(x)\}) < \varepsilon/2, \qquad \mu(\{x : f^-(x) \ne g_2(x)\}) < \varepsilon/2.$$

Then it follows that $\mu(\{x : f(x) \neq (g_1 - g_2)(x)\}) < \varepsilon$.]

Step (iv): *Suppose f is a bounded real valued function and A is not necessarily compact.*

Then, since $\mu(A) < \infty$, there exits a compact set $A_0 \subseteq A$ such that $\mu(A - A_0) < \varepsilon$. Let $f_0(x) = \begin{cases} f(x) & \text{if } x \in A_0, \\ 0 & \text{if } x \notin A_0. \end{cases}$. Clearly $f_0(x) = f(x)$ except possibly on $A - A_0$ where $\mu(A - A_0) < \varepsilon$. By step (iii) there exists $g \in C_c(X)$ and $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $f_0 = g$ except possibly on E. Hence, $f = g$ except possibly on $E \cup (A - A_0)$ which is of measure less than 2ε.

Step (v): *Suppose that f is an unbounded real valued function.*

Consider the sets $B_n = \{x : |f(x)| > n\}$ for $n \in \mathbb{N}$. Since f is real valued, we have $\bigcap_{n=1}^{\infty} B_n = \emptyset$ so that $\mu(B_n) \to 0$ as $n \to \infty$. Let N be such that $\mu(B_N) < \varepsilon$. Note that if $x \notin B_N$, then $|f(x)| \leq N$. Let $f_N(x) = \begin{cases} f(x) & \text{if } x \in B_N^c, \\ 0 & \text{if } x \in B_N. \end{cases}$. By step (iv) there exists $g \in C_c(X)$ and $E \in \mathcal{M}$ with $\mu(E) < \varepsilon$ such that $f_N = g$ except possibly on E. Thus, $f = g$ except possibly on $E \cup B_N$ which is of measure less than 2ε.

Step (vi): *Suppose f is a complex measurable function.*

We write $f = f_1 + i f_2$ where f_1 and f_2 are real and imaginary parts of f, and we can apply the result in step (iv) and then (v) for f_1 and f_2.

Finally, suppose $R := \sup_{x \in X} |f(x)| < \infty$. Let g be as above. Then we may define $\tilde{g}$ as

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } |g(x)| \leq R \\ R\frac{g(x)}{|g(x)|} & \text{if } |g(x)| > R. \end{cases}$$

Clearly $\tilde{g} \in C_c(X)$. Note that if $|g(x)| \leq R$ then $\tilde{g}(x) = g(x)$, and if $|g(x)| > R$, then $f(x) \neq g(x)$. Hence, $\{x : f(x) \neq \tilde{g}(x)\} = B_1 \cup B_2$, where

$$B_1 = \{x : f(x) \neq \tilde{g}(x), \, |g(x)| \leq R\} \subseteq \{x : f(x) \neq g(x)\},$$

$$B_2 = \{x : f(x) \neq \tilde{g}(x), \, |g(x)| > R\} \subseteq \{x : f(x) \neq g(x)\}$$

Thus, $|\tilde{g}(x)| \leq R$ for all $x \in X$ and $\mu(\{x : f(x) \neq \tilde{g}(x)\}) < \varepsilon$. □

For the next theorem we require the following general result.

Lemma 113. *Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space and $\{E_n\}$ is a denumerable family in $\mathcal{A}$ such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Let S be the set of all x which belongs to infinitely many E_n, i.e., $S^c := \{x : \exists\, N \text{ with } x \notin E_n \,\forall\, n \geq N\}$. Then $\mu(S) = 0$.*

Proof. Let $g := \sum_{n=1}^{\infty} \chi_{E_n}$. Note that $x \in S$ if and only if $g(x) = \infty$, and $\int_X g d\mu = \sum_{n=1}^{\infty} \mu(E_n)$. Hence, if $dis\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then we have $g(x) < \infty$ a.e., so that $\mu(S) = 0$. □

Theorem 114. *Suppose f and A are as in Lusin's theorem, and suppose f is bounded almost everywhere. Then there exists a sequence (g_n) in $C_c(X)$ such that $g_n \to f$ a.e. Moreover, if $|f| \leq b$ for some $b > 0$, then $|g_n| \leq b$ for all $n \in \mathbb{N}$.*

Proof. Without loss of generality, assume that $|f(x)| \leq 1$ for all $x \in X$. By Lusin's theorem, there exists $g_n \in C_c(X)$ such that $|g_n| \leq 1$ and $\mu(\{x : f(x) \neq g_n(x)\}) < 1/2^n$. Let $E_n = \{x : f(x) \neq g_n(x)\}$. Then we have $\sum_{n=1}^{\infty} \mu(E_n) \leq 1$, so that by the above lemma, $\mu(E^c) = 0$, where $E = \{x : \exists\, N \text{ with } x \notin E_n \,\forall\, n \geq N\}$. Thus, $x \in E$ implies there exists $N \in \mathbb{N}$ such that $g_n(x) = f(x)$ for all $n \geq N$. Hence $g_n(x) \to f(x)$ as $n \to \infty$ for every $x \in E$. □

4.10. **Completion of $C_c(X)$.** Suppose X is a locally compact Hausdorff space, $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $\mathcal{M}$ having the properties stated in Riesz Representation Theorem 81.

Proposition 115. *Let $1 \leq p < \infty$. Then for every $f \in L^p(\mu)$ there exists a sequence (φ_n) of simple measurable functions in $L^p(\mu)$ such that $\|f - \varphi_n\|_p \to 0$ as $n \to \infty$.*

Proof. Assume that $f \geq 0$. Let (φ_n) be a sequence of non-negative simple measurable functions such that $0 \leq \varphi_n \leq \varphi_{n+1} \leq f$ for all $n \in \mathbb{N}$ and $\varphi_n \to f$. Then it follows that

$$\varphi_n \in L^p(\mu), \quad (f - \varphi_n)^p \leq f^p \quad \forall\, n \in \mathbb{N}$$

and $(f - \varphi_n)^p \to 0$ as $n \to \infty$. Hence by dominated convergence theorem, $\|f - \varphi_n\|_p \to 0$ as $n \to \infty$. The case of real valued f follows by considering f^+ and f^- and the case of complex valued f will follow by taking $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. □

Theorem 116. *Let $1 \leq p < \infty$. Then $C_c(X)$ is dense in $L^p(\mu)$.*

Proof. Let $f \in L^p(\mu)$ and $\varepsilon > 0$. Let φ be a simple measurable function in $L^p(\mu)$ such that $\|f - \varphi\|_p < \varepsilon$ (by Proposition 4.10). Now, $\varphi \in L^p(\mu)$ implies $\mu(\{x : \varphi(x) \neq 0\}) < \infty$. Hence, by Lusin's theorem, there exists $g \in C_(X)$ such that $\varphi = g$ except possibly on a set E with $\mu(E) < \varepsilon$, and $|g| \leq \|\varphi\|_\infty$. Therefore,

$$\|\varphi - g\|_p^p = \int_X |\varphi - g|^p d\mu = \int_E |\varphi - g|^p d\mu \leq (2\|\varphi\|_\infty)^p \mu(E) < (2\|\varphi\|_\infty)^p \varepsilon.$$

Thus,

$$\|f - g\|_p \leq \|f - \varphi\|_p + \|\varphi - g\|_p < \varepsilon + 2\|\varphi\|_\infty \varepsilon^{1/p}.$$

This completes the proof. □

By the above theorem $L^p(\mu)$ is a completion of $C_c(X)$ with respect to for $1 \leq p < \infty$.

In view of the above, one may look for the closure of $C_c(X)$ in $L^\infty(\mu)$. For this purpose, we consider the space $C_0(X)$ which consists of all continuous functions $f : X \to \mathbb{C}$ such that for every $\varepsilon > 0$ there exits a compact set $K \subseteq X$ with the property that $|f(x)| < \varepsilon$ for all $x \notin K$. We shall show that $C_c(X)$ is dense in $C_0(X)$ with respect to the supremum norm, and $C_0(X)$ is a closed subspace of $L^\infty(\mu)$. Thus, $C_c(X)$ is dense in $L^\infty(\mu)$ if and only if $C_0(X) = L^\infty(\mu)$.

Theorem 117. *The space $C_c(X)$ is dense in $C_0(X)$.*

Proof. Let $f \in C_0(X)$ and $\varepsilon > 0$. Let $K \subseteq X$ be compact such that $|f| < \varepsilon$ on K^c. Then by Urysohn's lemma, there exists $g \in C_c(X)$ such that $K \prec g \prec X$. Since $f = fg$ on K we have

$$\|f - fg\|_\infty = \sup_{x \in K^c} |f(x)(1 - g(x))| < \varepsilon.$$

Clearly, $fg \in C_c(X)$. □

In fact, $C_0(X)$ is a completion of $C_c(X)$ as $C_0(X)$ is a Banach space according to the following theorem.

We recall that for any set Ω, $\ell^\infty(\Omega)$ denotes the vector space of all bounded (complex valued) functions on Ω, and it is a Banach space with respect to the norm

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)|, \quad f \in \ell^\infty(\Omega).$$

Theorem 118. *Let Ω be a topological space. Then $C_0(\Omega)$ is a closed subspace of $\ell^\infty(\Omega)$. In particular, $C_0(\Omega)$ is Banach space with respect to the supremum norm.*

Proof. Let (f_n) be a sequence in $C_0(\Omega)$ which converges to an $f \in \ell^\infty(\Omega)$. We have to show that $f \in C_0(\Omega)$. For this, let $x_0 \in \Omega$ and let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\|f - f_n\|_\infty < \varepsilon$ for all $n \geq N$. Then, for any $x \in \Omega$, we have

$$|f(x) - f(x_0)| \leq \|f - f_N\|_\infty + |f_N(x) - f_N(x_0)| < \varepsilon + |f_N(x) - f_N(x_0)|.$$

Since f_N is continuous, there exists an open set V containing x_0 such that for all $x \in V$, $|f_N(x) - f_N(x_0)| < \varepsilon$. Therefore, for all $x \in V$, we have $|f(x) - f(x_0)| < 2\varepsilon$. Thus, f is continuous at x_0. Also, since $f_N \in C_0(\Omega)$, there exits compact $K \subseteq \Omega$ such that $|f_N(x)| < \varepsilon$ for all $x \in K^c$. Therefore, for all $x \in K^c$, $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 2\varepsilon$. Thus we have proved that $f \in C_0(\Omega)$. □

Corollary 119. *Let X be a locally compact Hausdorff space and μ be a measure as in Riesz representation theorem. Then $C_0(X)$ is a closed subspace of $L^\infty(\mu)$.*

Proof. By Theorem 118, $C_0(X)$ is a Banach space. Clearly, $C_0(X)$ is a subspace of the Banach space $L^\infty(\mu)$. Hence, $C_0(X)$ is closed in $L^\infty(\mu)$. □

Department of Mathematics, I.I.T. Madras, Chennai-600 036, INDIA
E-mail address: mtnair@iitm.ac.in