MA2030: Linear Algebra and Numerical Analysis
Assignment Sheet 9

In the following $\|x\|$ denotes a norm for $x \in \mathbb{R}^n$, and $\|A\|$ denotes the matrix (operator) norm induced by the corresponding vector norm. .

(1) Let $C \in \mathbb{R}^{n\times n}$ and (x_k) be a sequence of vectors in $\mathbb{R}^n$. Suppose $\|x_k - x\| \to 0$ as $k \to \infty$. Prove the following:
  (a) $\|Cx_k - Cx\| \to 0$ as $k \to \infty$.
  (b) For every $d \in \mathbb{R}^n$, $\|(Cx_k + d) - (Cx + d)\| \to 0$ as $k \to \infty$.

(2) Let $C \in \mathbb{R}^{n\times n}$ and $d \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Define the sequence (x_k) of vectors in $\mathbb{R}^n$ iteratively as

$$x_k = Cx_{k-1} + d, \qquad k = 1, 2, \ldots.$$

If $\|x_k - x\| \to 0$ as $k \to \infty$ for some $x \in \mathbb{R}^n$, then prove that $x = Cx + d$.

[*Hint*: Use Problem 1 and the fact that $\|x_k - x\| \to 0$ and $x_k - y\| \to 0$ as $k \to \infty$ imply $x = y$.]

(3) Let $C \in \mathbb{R}^{n\times n}$, $d \in \mathbb{R}^n$ and (x_k) in $\mathbb{R}^n$ be as in Problem 3. If $\|C\| < 1$, then prove that (x_k) is a Cauchy sequence, i.e., for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_k - x_m\| < \varepsilon \quad \text{whenever} \quad k, m \geq N.$$

(4) Let $A \in \mathbb{R}^{n\times n}$, and let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be such that $A = A_1 + A_2$. Prove:
  (a) If A_1 is invertible, then for $x \in \mathbb{R}^n$,

$$Ax = b \iff x = Cx + d, \quad \text{where} \quad C = -A_1^{-1}A_2, \quad d = A_1^{-1}b.$$

  (b) If $\|C\| < 1$ and for $x_0 \in \mathbb{R}^n$ if (x_k) is defined iteratively as

$$x_k = Cx_{k-1} + d, \qquad k = 1, 2, \ldots,$$

  then there exists $x \in \mathbb{R}^n$ such that $\|x_k - x\| \to 0$ as $k \to \infty$, and in that case $Ax = b$.
  (c) Under the assumptions in (4b),

$$\|x_k - x\| \leq \frac{\|C\|^k}{1 - \|C\|}\|x_1 - x_0\| \qquad \forall\, k \in \mathbb{N}.$$

(5) Let $A \in \mathbb{R}^{n\times n}$, and let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be such that $A = A_1 + A_2$. If A_1 is invertible and $\|A_1^{-1}A_2\| < 1$, then prove that A is invertible.

[*Hint*: Use the following facts:
(i) $A = A_1 + A_2 \implies A_1^{-1}A = I + A_1^{-1}A_2$;
(ii) $\|B\| < 1 \implies I - B$ is one-one and hence invertible,
(iii) BC invertible $\implies C$ one-one and B onto.]

(6) Describe Jacobi method and Gauss-Siedel method for finding iterative approximations for the solution of $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ is invertible.