Complex Variables and Transform techniques*
Problem Sheet 1

January 23, 2015

In the following $\mathbb{C}$ denotes the set of complex numbers, i.e. the complex plane.

1. For $z := (x, y), z_1 := (x_1, y_1), z_2 := (x_2, y_2)$ in $\mathbb{R}^2$, define

$$z_1 + z_2 := (x_1+, x_2, y_1 + y_2), \quad -z := (-x, -y),$$

$$z_1 z_2 := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1),$$

   Also define $i := (0, 1)$, and for $x \in \mathbb{R}$, let $\tilde{x} := (x, 0)$. Show that

   (a) $z = (x, y) \Rightarrow z = \tilde{x} + i\tilde{y}$;

   (b) $i^2 = -\tilde{1}$;

   (c) $(\tilde{x}_1 + i\tilde{y}_1) + (\tilde{x}_2 + i\tilde{y}_2) = (\tilde{x}_1 + \tilde{x}_2) + i(\tilde{y}_1 + \tilde{y}_2)$,

   (d) $(\tilde{x}_1 + i\tilde{y}_1)(\tilde{x}_2 + i\tilde{y}_2) = (\tilde{x}_1\tilde{x}_2 - \tilde{y}_1\tilde{y}_2) + i(\tilde{x}_1\tilde{y}_2 + \tilde{x}_2\tilde{y}_1)$.

2. For $x \in \mathbb{R}$, denote $\tilde{x}$ by x itself and $\mathbb{C}$ the set of all *complex numbers* $z := x + iy$ (with $x, y \in \mathbb{R}$). Define

$$Re(z) := x, \quad Im(z) := y, \quad \bar{z} := x - iy, \quad |z|^2 := \bar{z}z,$$

   and for $z \neq 0$, define $1/z := \bar{z}/|z|^2$. Show that

   (a) $|Re(z)| \leq |z|, \quad |Im(z)| \leq |z|, \quad |\bar{z}| = |z|$.

   (b) $|z_1+z_2| \leq |z_1|+|z_2|, \quad |z_1-z_2| \geq |z_1|-|z_2|, \quad |z-1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$,

   (c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

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3. Verify:
$$i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad 1/i = -i,$$
$$z \in \mathbb{R} \iff \bar{z} = z, \quad Re(iz) = -Im(z), \quad Im(iz) = Re(z)$$

4. Find
$$i^{4n}, \quad i^{4n+1}, \quad i^{4n+2}, \quad i^{4n+3} \quad \text{for } n \in \mathbb{N},$$
$$Re\left(\frac{3+4i}{7-i}\right), \quad Im\left(\frac{3+4i}{7-i}\right)$$

5. For a sequence (z_n) of complex numbers, define $z_n \to z$ iff $|z_n - z| \to 0$. Show that, for sequenes (z_n) and (w_n) of complex numbers,

   (a) $z_n \to z, \quad w_n \to w \quad \Rightarrow \quad z_n + w_n \to z + w,$

   (b) $z_n \to z, \quad w_n \to w \quad \Rightarrow \quad z_n w_n \to zw,$

   (c) $w_n \to w$ and $w \neq 0$ implies there exists $N \in \mathbb{N}$ such that $w_n \neq 0$ for all $n \geq N$;

   (d) $z_n \to z, \quad w_n \to w$ with $w \neq 0 \quad \Rightarrow \quad z_n/w_n \to z/w$

6. For nonzero $z \in \mathbb{C}$, let $z = r(\cos\theta + i\sin\theta)$ with $r = |z|$. Define $arg(z) := \theta$. Show that $\arg(z) = \theta + 2n\pi$ for any integer n.

7. Show, upto addition of multiples of 2π,
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \quad \arg(z_1/z_2) = \arg(z_1) - \arg(z_2).$$

8. Write $Arg(z) = \arg(z)$ whenever $-\pi \leq arg(z) < \pi$ and call $Arg9z)$ as the *principal value* of the argument of z. Find $arg(z)$ if z is:
$$-7, \quad 1 + i\sqrt{3}, \quad -2i.$$

9. Let $S \subseteq \mathbb{C}$. Show that the following are equivalent:

   (a) S contains all its limit points,

   (b) S contains all its boundary points,

   (c) complement of S is an open set.

10. Show that

   (a) Union of any two open sets is open.

   (b) Intersection of any two open sets is open.

   (c) Union of any two closed sets is closed.

   (d) Intersection of of any two closed sets is closed.