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# COMPACT EMBEDDING

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Recall the following result from [1]:

**THEOREM 1.** *Let $X$ and $Y$ be Hilbert spaces, $(u_n)$ and $(v_n)$ be orthonormal sequences in $X$ and $Y$, respectively, and $(\lambda_n)$ be a sequence of scalars such that $\lambda_n \to 0$. Then the following hold:*

(1) *For every $x \in X$, the series $\sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle v_n$ converges in $Y$.*
(2) *The map $T : X \to Y$ defined by*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle v_n, \quad x \in X,$$

*is a compact operator.*

Let $X$ be a Hilbert space, and $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of $X$, and $(\sigma_n)$ be a sequence of scalars such that $\sigma_n \to 0$. Let

$$D := \Big\{ x \in X : \sum_{n=1}^{\infty} \frac{|\langle x, u_n \rangle|^2}{\sigma_n^2} < \infty \Big\}.$$

Then, $D$ is a subspace of $X$. Define

$$\langle x, y \rangle_0 := \sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} \langle x, u_n \rangle \langle u_n, y \rangle, \quad x, y \in X_0.$$

Then $\langle \cdot, \cdot \rangle_0$ is an inner product on $D$, and $X_0$ with $\langle \cdot, \cdot \rangle_0$ is a Hilbert space. Let us denote this Hilbert space by $X_0$. Note that, for $x \in X_0$,

$$\langle x, u_k \rangle_0 = \sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} \langle x, u_n \rangle \langle u_n, u_k \rangle = \frac{1}{\sigma_k^2} \langle x, u_k \rangle \quad \forall\, k \in \mathbb{N},$$

and

$$\langle u_\ell, u_k \rangle_0 = \frac{1}{\sigma_k^2} \langle u_\ell, u_k \rangle = \frac{\delta_{k\ell}}{\sigma_k^2} \quad \forall\, k, \ell \in \mathbb{N}.$$

Let

$$\varphi_n := \sigma_n u_n, \quad n \in \mathbb{N}.$$

Then we have $(\varphi_n)$ is an orthonrmal sequence in $X_0$ with respect to $\langle\cdot,\cdot\rangle_0$. Note that for every $x \in X_0$,

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n = \sum_{n=1}^{\infty} \langle x, \varphi_n\rangle_0 \varphi_n$$

and

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n = \sum_{n=1}^{\infty} \sigma_n \langle x, \varphi_n\rangle_0 u_n.$$

Thus, we have proved the following theorem.

**THEOREM 2.** *The identity operator $I_0 : X_0 \to X$ is a compact operator with singular value decomposition*

$$I_0 x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n = \sum_{n=1}^{\infty} \sigma_n \langle x, \varphi_n\rangle_0 u_n, \quad x \in X_0.$$

In view of the last theorem, $X_0$ is said to be **compactly imbedded** in $X$.

## References

[1] M.T. Nair, *Functional Analysis: A First Course*, Prentice-Hall of India, New Delhi, 2002.

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