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COMPLEX ANALYSIS: PROBLEMS SHEET -1

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(1) Show that $\mathbb{C}$ is a field under the addition and multiplication defined for complex numbers.

(2) Show that the map $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = (x, 0)$ is a field isomorphism.

(3) For a nonzero complex number x, show that $z^{-1} = \bar{z}/|z|$.

(4) Show that for z_1, z_2 in $\mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$.

(5) Show that $d(z_1, z_2) := |z_1 - z_2|$ defines a metric on $\mathbb{C}$, and it is a complete metric.

(6) Show that $|z_1 - z_2| \geq |z_1| - |z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

(7) Suppose α, β, γ are nonzero complex numbers such that $|\alpha| = |\beta| = |\gamma|$.

Show that $\alpha + \beta + \gamma = 0 \iff \dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 0.$

(8) Suppose z_1, z_2, z_3 are vertices of an equilateral triangle. Show that $z_1^2 + z_2^2 + z_3^2 + = z_1z_2 + z_2z_3 + z_3z_1$.

(9) Show that the equation of a straight line using complex variable z is given by $\bar{\alpha}z + \alpha\bar{z} + \gamma = 0$ for some $\alpha \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

(10) If $|z| = 1$ and $z \neq 1$, then show that $\dfrac{1+z}{1-z} = ib$ for some $b \in \mathbb{R}$.

(11) For $n \in \mathbb{N}$, derive a formula for the n^{th} root of a complex number z using its polar representation.

(12) Let S^2 be the unit sphere in $\mathbb{R}^3$ with centre at the origin, i.e., $S^2 := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}$. Show that the *steriographic projection*

$$z := x + iy \mapsto \left(\frac{2x}{1+|z|^2}, \frac{2x}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right)$$

is a bijective continuous function from $\mathbb{C}$ onto $S^2 \setminus \{(1, 0, 0)\}$ with its inverse

$$(\alpha, \beta, \gamma) \mapsto \frac{\alpha + i\beta}{1-\gamma},$$

which is also continuous.

(13) Show that the functions $z \mapsto \Re(z)$, $z \mapsto \Im(z)$, $z \mapsto |z|$ are continuous functions on $\mathbb{C}$.

(14) Show that $\lim_{z\to 0} \dfrac{z}{|z|}$ does not exist.

(15) Suppose u is a real valued function defined on an open set $\Omega \subseteq \mathbb{R}^2$. Let $(x_0, y_0) \in \Omega$.
  (i) When do you say that u has partial derivatives u_x and u_y at (x_0, y_0)?
  (ii) When do you say that u is differentiable at (x_0, y_0)?
  (iii) What is gradient of u at (x_0, y_0)?
  (iv) What is the relation between gradient and derivative of u?

(16) Show that a function f is differentiable at $z_0 \in \Omega$ if and only if its real part u and imaginary part v are differentiable at (x_0, y_0) and u_x, u_y, v_x, v_y satisfy the equations

$$u_x(z_0) = v_y(z_0), \qquad u_y(z_0) = -v_x(z_0),$$

and in that case

$$f'(z_0) = u_x(z_0) + iv_x(z_0) = v_y(z_0) - iu_y(z_0).$$

(17) Find points at which the following functions are differentiable:

  (i) $f(z) = x$, (ii) $f(z) = \bar{z} = x - iy$,

  (iii) $f(z) = |z|^2$, (iv) $f(z) = \bar{z}^2$.

(18) Find points at which the functions in the last problem satisfy CR-equations.

(19) Prove that the CR-equations in polar coordinates are $qru_r = v_\theta$, $u_\theta = -rv_r$.

(20) Suppose f is holomorphic on an open set Ω. Prove that if f satisfies any of the following conditions, then f is a constant function.
  (i) f' is constant on Ω,
  (ii) f is real valued on Ω,
  (iii) $|f|$ is constant on Ω,
  (iv) $\arg(f)$ is constant on Ω.

(21) Suppose f is holomorphic on an open set Ω. Prove that the function $z \mapsto g(z) = \overline{f(\bar{z})}$ is holomorphic in $\Omega^* := \{\bar{z} : z \in G\}$.

(22) (i) Show that the equation of a straight line is given by

$$\bar{\alpha}z + \alpha\bar{z} + \gamma = 0$$

for some α, β in $\mathbb{C}$ and $\gamma \in \mathbb{C}$.
  (ii) Show that the above line passes through 0 if and only if $\gamma = 0$.

(23) Show that the equation of a circle with centre at z_0 and radius $r > 0$ is given by

$$|z|^2 - (\bar{z}_0 z + z_0\bar{z}) + |z_0|^2 - r^2 = 0$$

(24) Prove the following:

(i) For nonzero $a \in \mathbb{C}$, the function $z \mapsto az$ maps a straight line into a straight line and a circle into a circle.

(ii) For nonzero $b \in \mathbb{C}$, the function $z \mapsto z + b$ maps a straight line into a straight line and a circle into a circle.

(25) For nonzero $a, b \in \mathbb{C}$, the function $z \mapsto az+b$ maps a straight line into a straight line and a circle into a circle - Why?

(26) Given a curve $\gamma : [0,1] \to \mathbb{C}$, let $\tilde{\gamma} : [a,b] \to \mathbb{C}$ be defined by

$$\tilde{\gamma}(t) = \gamma(a + (b-a)t).$$

If Γ and $\widetilde{\Gamma}$ are the images of γ and $\tilde{\gamma}$ respectively, then show that Γ and $\widetilde{\Gamma}$ are homeomorphic.

(27) Find the points at which the curve $\gamma : [0,4] \to \mathbb{C}$ defined in the following are not regular. Justify your answer:

$$\gamma(t) := \begin{cases} t, & 0 \leq t \leq 1, \\ 1 + (t-1)i, & 1 < t \leq 2, \\ 3 - t + i, & 2 < t \leq 3, \\ (4-t)i, & 3 < t \leq 4. \end{cases}$$

(28) Let f be defined on an open set Ω and differentiable at a point $z_0 \in \Omega$ and $f'(z_0) \neq 0$. Then prove that f preserves angle between curves which are regular at z_0 and intersecting at z_0.

(29) Define $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, and show that the real and imaginary parts of f satisfy the CR-equations if and only if

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

(30) Define the operators $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \bar{z}}$ and show that the real and imaginary parts of f satisfy the CR-equations if and only if $\dfrac{\partial f}{\partial \bar{z}} = 0$.

(31) Show that

$$\frac{\partial^2 f}{\partial \bar{z} \partial z} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

and deduce that f harmonic on $\Omega \iff \dfrac{\partial^2 f}{\partial \bar{z} \partial z} = 0$.

(32) Prove that v is a harmonic conjugate of u if and only if $-u$ is a harmonic conjugate of v.

(33) Prove that v_1 and v_2 are harmonic conjugates of u if and only $v_1 - v_2$ is a constant.

(34) Prove that if u is a real valued harmonic function on an open set Ω, then any two harmonic conjugates of u differ by a constant.

(35) Prove that if u is a real valued on an open set Ω such that both u and u^2 are harmonic on Ω, then u is a constant function.

(36) Prove that if u and v are harmonic functions on an open set Ω such that v is a harmonic conjugate of u, then uv and $u^2 - v^2$ are harmonic.

(37) Prove that the Laplace equation $\triangle u = 0$ can be written in polar coordinates as $r^2 u_{rr} + r u_r + u_{\theta r} = 0$.

(38) Prove that if u is a real valued harmonic function on an open set Ω, then $\dfrac{\partial u}{\partial z}$ is holomorphic on Ω.

(39) Suppose φ_1 and φ_2 are fractional linear transformations. Prove that $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_1$ are fractional linear transformations.

(40) Consider a fractional linear transformation φ given by

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad z \in \widetilde{\mathbb{C}}.$$

(i) Show that φ is one-one and onto and its inverse is given by

$$\varphi^{-1}(z) = \frac{-dz+b}{cz-a}, \qquad z \in \widetilde{\mathbb{C}}.$$

(ii) Show that φ is differentiable at every $z \in \mathbb{C}$ and its derivative is given by

$$\varphi'(z) = \frac{ad-bc}{(cz+d)^2}, \qquad z \in \mathbb{C}.$$

(41) Show that the set of all fractional linear transformations is in one-one correspondence with the set of all 2×2 nonsingular matrices with complex entries.

(42) Let $\mathcal{F}$ be the set of all fractional linear transformations. Define a binary operation on $\mathcal{F}$ so that $\mathcal{F}$ becomes a group.

(43) Show a fractional linear transformation maps every circle and straight line in $\mathbb{C}$ onto either a circle or a straight line.

(44) Prove that the identity function is the only fractional linear transformation having more than two distinct fixed points.

(45) Given distinct points z_1, z_2, z_3 and distinct points w_1, w_2, w_3 in the plane $\widetilde{\mathbb{C}}$, show that the fractional linear transformation $w = \varphi(z)$ defined by

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

map z_1, z_2, z_3 onto w_1, w_2, w_3, respectively.

(46) Find the fractional linear transformation φ that maps -1, 0, i onto the points 0, 1, $-i$, respectively.

What is the image of the circle passing through -1, 0, i? A circle or a straight line? Why?

(47) Find the fractional linear transformation φ that maps 0, 1, ∞ onto the points 1, i, -1, respectively.

What is the image of the real axis under this transformation? Why?

(48) Find the fractional linear transformation φ that maps 1, i, -1 onto the points i, 0, $-i$, respectively.

What is the image of the unit circle (with centre at 0) under this transformation? Why?

(49) Suppose

$$\varphi(z) = \frac{az + b}{cz + d}, \qquad z \in \widetilde{\mathbb{C}}.$$

is the fractional linear transformation that maps the real axis onto the unit circle (with centre 0). Show that

$$|a| = |c| \neq 0, \qquad |b| = |d| \neq 0.$$

(50) If the fractional linear transformation in the last problem maps the upper half plane onto the open unit disk, then show that it is of the form

$$\varphi(z) = \alpha \, \frac{z - z_0}{z - \bar{z}_0}$$

for some α, z_0 in $\mathbb{C}$ such that $|\alpha| = 1$ and $\Im(z_0) > 0$.

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