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COMPLEX ANALYSIS: PROBLEMS SHEET - 2

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Elementary Functions

(1) For $y \in \mathbb{R}$, show that the series $e^{iy} := \sum_{n=1}^{\infty} \frac{(iy)^n}{n!}$ converges.

(2) For $y \in \mathbb{R}$, $e^{iy} = \cos y + i \sin y$. Why?

(3) Show that the function $z \mapsto e^z$ is holomorphic on the entire complex plane and satisfies the following:

(i) $e^{z_1+z_2} = e^{z_1} e^{z_2}$ for every $z_1, z_2 \in \mathbb{C}$,

(ii) $e^{z+2\pi i} = e^z$ for every $z \in \mathbb{C}$,

(iii) if $x = \Re(z)$ and $y = \Im(z)$, then $|e^z| = e^x$ and $\arg(z) = y$, and

(iv) the strip $\{z \in \mathbb{C} : -\pi \leq \Im(z) < \pi\}$ is mapped onto the entire complex plane.

(4) Given θ_1 and $\theta_2 \in [0, 2\pi)$ with $\theta_1 < \theta_2$, , describe the image of the strip $\{z \in \mathbb{C} : \theta_1 \leq \Im(z) < \theta_2\}$ under the map $z \mapsto e^z$.

(5) Given $r, R0$ with $0 < r < R$, describe the image of the strip $\{z \in \mathbb{C} : r \leq \Re(z) < R\}$ under the map $z \mapsto e^z$.

(6) For the function $f(z) = e^z$, describe the curves

$$|f(z)| = \text{constant}, \qquad \arg(f(z)) = \text{constant}.$$

(7) If f is is holomorphic on $\mathbb{C}$ satisfying $f'(z) = f(z)$, and g is defined by $g(z) = e^{-z} f(z)$, then show that $g'(z) = 0$, so that $f(z) = ce^z$ for some constant $c \in \mathbb{C}$.

(8) Find the most general form of a holomorphic f on $\mathbb{C}$ satisfying $f'(z) = cf(z)$ for some constant $c \in \mathbb{C}$.

(9) Derive the identities:

(i) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

(ii) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.

(iii) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

(iv) $\sin z = \sin x \cosh y + i \cos x \sinh y$.

(v) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

(vi) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

(10) Show that

$$\cos(z+2\pi) = \cos z, \qquad \sin(z+2\pi) = \sin(z)$$

for all $z \in \mathbb{C}$.

(11) Find zeros of $\sin z$ and $\cos z$.

(12) Find all roots of the equation $\cos z = 1$.

(13) Show that if b is a logarithm of a, then $b + 2n\pi i$ is also a logarithm of a for every $n \in \mathbb{Z}$.

(14) Show that if $b = \log a$, $\log a = \ln|a| + i\arg(a)$.

(15) Show that, for $n \in \mathbb{Z}$,

(i) $2n\pi i = \log 1$.

(ii) $(2n+1)\pi i = \log(-1)$.

(iii) $\left(2n + \frac{1}{2}\right)\pi i = \log i$.

(16) Find all values of

(i) $\cosh(\log 2)$,

(ii) $\log(\log i)$.

(17) Does the relation $\log a_1 a_2 = \log a_1 + \log a_2$ hold for all nonzero a_1, a_2 in $\mathbb{C}$?

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