February 25, 2011

**COMPLEX ANALYSIS: PROBLEMS SHEET - 3**

M.THAMBAN NAIR

## Power Series

Note: Problems from 1-3 and 9-12 are discussed in class, either by proving them, or by way of indicating their proofs.

(1) Suppose a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z$ with $|z-z_0|<r$ for some $r>0$. Then, prove that for any $\rho$ with $0<\rho<r$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on the set $\{z : |z-z_0| \leq \rho\}$.

(2) Let $R$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Prove the following:
   (i) If the series converges at $z_1$, then $R \geq |z_1 - z_0|$.
   (ii) If the series diverges at $z_2$, then $R \leq |z_2 - z_0|$.
   (iii) If $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ diverges at $z_2$, then $R \leq |z_2 - z_0|$.

(3) If $(a_n)$ and $(b_n)$ are sequences of complex numbers such that $|a_n| \leq M|b_n|$ for all $n \in \mathbb{N}$, and if $R_1$ nd $R_2$ are the radius of convergence of $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} b_n(z-z_0)^n$ respectively, then prove that $R_1 \leq R_2$.

(4) Using Problem 3, show that radius of convergence of $\sum_{n=1}^{\infty} n^{-n} z^n$ is $\infty$.

(5) Prove that radius of convergence of $\sum_{n=1}^{\infty} n^n z^n$ is 0.

(6) Find a power series in a neighborhood of $z_0 = 1$ which represents the function $f(z) := 1/z$.

(7) Find the radius of convergence for each of the following series:

$$\text{(i)} \sum_{n=0}^{\infty} n^2 z^n, \quad \text{(ii)} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n, \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n, \quad \text{(iv)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

$$\text{(v)} \sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^{3n}, \quad \text{(vi)} \sum_{n=1}^{\infty} \frac{z^{n!}}{n}, \quad \text{(vii)} \sum_{n=0}^{\infty} n^n z^{n^2}, \quad \text{(viii)} \sum_{n=0}^{\infty} \frac{n+1}{n!} z^{n^3}.$$

(8) Give one example each of a power series which
   (a) converges only on the interior of the disc of convergence,
   (b) converges diverges on a proper subset of the boundary of the disc of convergence,
   (c) converges on the closure of the disc of convergence.

(9) Show that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1}$ have the same radius of convergence.

(10) If $(a_n)$ and $(b_n)$ are sequences of complex numbers such that $\limsup_n |b_n|^{1/n}$, then show that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} na_nb_n(z-z_0)^{n-1}$ have the same radius of convergence.

(11) For a sequence $(a_n)$ of nonzero complex numbers, let $\gamma := \limsup_n \left|\frac{a_{n+1}}{a_n}\right|$. Then show that $R := 1/\gamma$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

(12) If $f$ represents a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on its disc of convergence, then $a_k = \frac{f^{(k)}(z_0)}{k!}$ for every $k \in \mathbb{N}$. Justify.

(13) Let $f$ be a holomorphic function in an open set $\Omega$ such that $f' = f$ and $f(0) = 1$. Then show that $f(z) = e^z$. Deduce that, for all $z \in \mathbb{C}$,

(i) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

(ii) $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

(iii) $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

(14) Prove that, for $|z| < 1$,

(i) $$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

(ii) $$\mathrm{Log}\frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

(15) Find the function represented by the series $\sum_{n=1}^{\infty} n^2 z^n$.

M. Thamban Nair, Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India.

*E-mail address*: mtnair@iitm.ac.in