March 4, 2011

COMPLEX ANALYSIS: PROBLEMS SHEET - 4

M.THAMBAN NAIR

Integration

(1) A curve $\eta : [\alpha, \beta] \to \mathbb{C}$ is called a *reparameterization* of the curve $\gamma : [a, b] \to \mathbb{C}$ if there exists continuous increasing bijection $\varphi : [\alpha, \beta] \to [a, b]$ such that $\eta = \gamma \circ \varphi$, and in that case we may say that η is equivalent to γ and we write $\eta \sim \gamma$.

  (a) Given a curve $\gamma : [a, b] \to \mathbb{C}$ and a closed interval $[\alpha, \beta]$, find a curve $\eta : [\alpha, \beta] \to \mathbb{C}$ such $\gamma \sim \eta$.

  (b) If $\tilde{\gamma}$ is the reverse of $\gamma : [a, b] \to \mathbb{C}$ and $\eta : [-b, -a] \to \mathbb{C}$ is defined by $\eta(t) = \gamma(-t)$ for $-b \le t \le -a$, then show that $\eta \sim \tilde{\gamma}$.

  (c) If γ and η are piecewise smooth curves such that $\eta \sim \gamma$, then show that $\ell(\Gamma_\eta) = \ell(\Gamma_\gamma)$.

(2) Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ and a partition $\Pi_n := a = t_0, < t_1 < \ldots < t_n = b$ of $[a, b]$, let $S_n := \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|$. If $\max_{1\le j\le n} |t_j - t_{j-1}| \to 0$, then show that $S_n \to \int_a^b |\gamma'(t)|dt$.

(3) If η is a differentiable reparamterization of a piecewise smooth curve γ, and if f is continuous on Γ_γ, then show that $\int_\gamma f(z)dz = \int_\eta f(z)dz$.

(4) Prove: $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$

(5) Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, a partition $\Pi_n := a = t_0, < t_1 < \ldots < t_n = b$ of $[a, b]$, and a continuous function f on Γ, let $S_n(f) := \sum_{j=1}^n f(\gamma(t_j))[\gamma(t_j) - \gamma(t_{j-1})]$. If $\max_{1\le j\le n} |t_j - t_{j-1}| \to 0$, then show that $S_n(f) \to \int_\Gamma f(z)dz$.

(6) If $\varphi : [a, b] \to \mathbb{C}$ is continuously differentiable, then show that

$$\int_a^b \varphi'(t)\, dt = \varphi(b) - \varphi(a).$$

(7) Let Γ be a closed curve and f has a primitive on an open set Ω containing Γ, i.e., there exists a continuously differentiable function g such that $g' = f$ on Ω. Then $\displaystyle\int_\Gamma f(z)\, dz = 0$. Justify.

(8) Let Γ_n be the circle with center at z_0 and radius r traced n times, i.e., $\Gamma_n := \{z_0 + re^{it},\ 0 \le t \le 2n\pi\}$. Then, show that $\displaystyle\int_{\Gamma_n} \frac{dz}{z - z_0} = 2n\pi i,$.

(9) If $p(z)$ is a polynomial, then prove that $\displaystyle\int_{\Gamma_n} p(z)\,dz = 0$.

(10) If Γ is the circle with centre z_0 and radius r, then show that $\displaystyle\int_{\Gamma} \frac{dz}{(z-z_0)^n} = 0$ for every $n \in \mathbb{Z} \setminus \{-1\}$.

(11) Given distinct points α and β in $\mathbb{C}$, evaluate the integrals $\int_{[\alpha,\beta]} z^n dz$ and $\int_{[\alpha,\beta]} \bar{z}^n dz$, where $[\alpha,\beta]$ denotes the line segment joining α to β.

(12) If f is a real valued function defined on the interval $[a,b]$ and if $\gamma(t) = t$, $a \le t \le b$, then show that $\int_{[a,b]} f(z)dz = \int_a^b f(t)dt$.

(13) Let Γ be a closed piecewise smooth curve in the disc of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and let $f(z)$ represent this series in that disc. Then $\displaystyle\int_{\Gamma} f(z)\,dz = 0$. Justify.

(14) Evaluate the integrals $\int_\gamma f(z)dz$ in following, where

(a) γ is the curve joining $z_1 = -1-i$ to $z_2 = 1+i$ consisting of the line segment from $-1-i$ to 0 and the portion of the curve $y = x^2$ from 0 to $1+i$ and $f(z) = \begin{cases} 1, & y < 0, \\ 4y, & y > 0 \end{cases}$.

Answer: $-\frac{5}{8} + \frac{5}{2}i$

(b) γ is the curve consisting of the line segments joining the points 0 to 1 and 1 to $1+2i$ and $f(z) = 3x^2 - y + ix^3$.

Answer: $2+3i$

(c) $f(z) = \bar{z}$ and γ is the curve joining 1 to $1+i$ along the parabola $y = x^2$. Answer: $1 + \frac{1}{3}i$

(d) γ is the positive oriented circle $z-1| = 4$ traced once and $f(z) = \frac{1}{z-1} + \frac{6}{(z-1)^2}$ Answer: $-6\pi i$

(15) Show that $\left|\int_\gamma \frac{z}{z^2+1}dz\right| \le \frac{1}{2}$, where γ be the line segment joining 2 to $2+i$.

(16) For piecewise smooth curve $\gamma :: [a,b] \to \mathbb{C}$ and continuous function $f : \Gamma \to \mathbb{C}$, define $\displaystyle\int_\gamma f(z)|dz| := \int_a^b f(g(t))|\gamma'(t)|dt$. Show that

$$\Big|\int_\gamma f(z)dz\Big| \le \int_\gamma |f(z)|\,|dz|.$$

(17) If Γ_r is the circle $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$ and if f is continuous on and inside Γ_r, then prove that

$$\lim_{r\to 0} \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{z-z_0}dz = f(z_0).$$

(18) For every closed piecewise smooth curve Γ, $\displaystyle\int_\Gamma e^{-z^2}dz = 0$. Why?

(19) For positive real numbers, let I_1, I_2, I_3, I_4 be the integrals of e^{-z^2} over the line segments

$$[-a, a], \quad [a, a+ib], \quad [a+ib, -a+ib], \quad [-a+ib, -a],$$

respectively. Prove that

(i) $I_1 = \displaystyle\int_{-a}^{a} e^{-x^2}dx$,

(ii) $|I_2| \leq be^{-a^2+b^2}$.

(iii) $I_3 = -e^{b^2}\displaystyle\int_{-a}^{a} e^{-t^2}\cos(2bt)dt$,

(iv) $|I_4| \leq be^{-a^2+b^2}$.

(v) $e^{b^2}\displaystyle\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)dt = \int_{-\infty}^{\infty} e^{-x^2}dx$.

(20) Suppose $f_n \to f$ uniformly on Γ. Then show that

$$\int_\Gamma f_n(z)dz \to \int_\Gamma f(z)\,dz.$$

(21) Prove that $\displaystyle\int_0^\infty e^{-t^2}\cos t^2 dt = \frac{1}{4}\sqrt{\pi}\sqrt{1+\sqrt{2}}$, by integrating e^{-z^2} over the positive oriented triangle with vertices at $0, R, Re^{i\pi/8}$ for $R > 0$ and letting $R \to \infty$.

(22) Evaluate the integrals $\displaystyle\int_0^\infty \cos t^2 dt$, $\displaystyle\int_0^\infty \cos t^2 dt$ by integrating e^{-z^2} over the positive oriented triangle with vertices at $0, R, Re^{i\pi/4}$ for $R > 0$ and letting $R \to \infty$.

(23) Let (Ω_n) be a sequence of nonempty compact sets in $\mathbb{C}$ such that $\Omega_n \supseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$ and $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$. Let $z_0 \in \cap_{n=1}^\infty \Omega_n$. If f is a continuous function defined on Ω_1, show that

$$\max_{z\in\Omega_n} |f(z) - f(z_0)| \to 0 \quad \text{as} \quad n \to \infty.$$

(24) Suppose Ω be a simply connected domain and f be holomorphic on Ω. Suppose integral of f over every positively oriented triangle is zero. Prove that if Γ_1 and Γ_2 are two polygonal lines joining any two points z_0 and ζ_0 in Ω, then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$.

M. Thamban Nair, Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India.

E-mail address: mtnair@iitm.ac.in