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COMPLEX ANALYSIS: PROBLEMS SHEET - 5

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(1) Let f be continuous in a neighbourhood of z_0 and $\Gamma_r := \{z \in \mathbb{C} : |z - z_0| = r\}$. Show that
$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{z - z_0} dz \to f(z_0) \quad \text{as} \quad r \to 0.$$

(2) Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}$ (using complex integrals).

(3) Let f be an entire function such that for some $n \in \mathbb{N}$ and $R > 0$, $\left|\frac{f(z)}{z^n}\right|$ is bounded for $|z| > R$. Prove tht f is a polynomial of degree atmost n.

(4) Let f be holomorphic and map $\Omega := \{z \in \mathbb{C} : |z| < 1\}$ into itself. Prove that $|f'z)| \leq 1/(1 - |z|)$ for all $z \in \Omega$.

(5) Prove that there is no analytic function f on $\Omega := \{z \in \mathbb{C} : |z| < 1\}$ such that
   (i) $f(1/n) = (1/2^n$ for $n \in \mathbb{N} \setminus \{1\}$.
   (ii) $f(1/n) = (-1)^n/n^2$ for $n \in \mathbb{N} \setminus \{1\}$.

(6) Let f be a nonconstant holomorphic function in a connected opens set Ω. If $z_0 \in \Omega$ is such that $|f(z_0)| \leq |f(z)|$ for all $z \in \Omega$, then prove that $f(z_0) = 0$.

(7) Let u be a (real valued) harmonic function in a connected opens set Ω. Let $g := u_x - iu_y$ on Ω. Justify the following:
   (i) g is holomorphic on Ω.
   (ii) There exists a holomorphic function f on Ω such that $\Re f = u$.
   (iii) u is infinitely differentiable.

(8) For $0 < a < 1$, find the annulus of convergence of the series $\sum_{n=-\infty}^{\infty} a^{n^2} z^n$.

(9) Locate and classify the isolated singularities of the following functions:
$$\text{(i)}\ \frac{z^5}{1 = z + z^2 = z^3 + z^4}, \quad \text{(ii)}\ \frac{1}{\sin^2 z}, \quad \text{(iii)}\ \sin(1/z).$$
   Also, check whether $z_0 = \infty$ is an isolated singularity (i.e., $w_0 = 0$ is an isolated singularity of $f(1/z)$) in each case.

(10) If f and g are holomorphic functions having z_0 a pole of the same order for both, then prove that
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}.$$

(11) Find the residues of the following functions:

(i) $\dfrac{z^3}{z-1}$ (ii) $\dfrac{z^3}{(z-1^2)}$

(12) If f and g are holomorphic in a neighbourhood of z_0, and z_0 is a simple a pole of g, then prove that $Res(f/g, z_0) = f(z_0)/g'(z_0)$.

(13) Determine the residues of each of the following functions at each of their singularities:

$$\text{(i) } \frac{z^3}{1-z^4}, \quad \text{(ii) } \frac{z^5}{(z^2-1)^2}, \quad \text{(iii) } \frac{\cos z}{1+z+z^2}.$$

(14) If f is holomorphic in a neighbourhood of z_0, and z_0 is a zero of f order m, then prove that $Res(f'/f, z_0) = m$.

(15) Evaluate the following using complex integrals:

$$\text{(i) } \int_0^\infty \frac{e^{ix}}{x} dx, \quad \text{(ii) } \int_0^\infty \frac{dx}{1+x^2},$$

$$\text{(iii) } \int_0^\infty \frac{\sin^2 x}{x} dx, \quad \text{(iii) } \int_0^\infty \frac{\cos ax}{x^2+b^2} dx, \quad a \geq 0,\ b > 0.$$

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