Topics in Fourier Analysis

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Preface

These notes are prepared for the use of students of IIT Madras for an elective course (specifically meant for MSc Mathematics) on *Fourier Analysis*.

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Fourier Series: An Introduction

1.1 Motivation through heat equation

The consideration of *Fourier Series* can be traced back to the situation which Fourier[1] encountered in the beginning of last century while solving heat equation:

Consider a thin metallic wire of length ℓ. Suppose an initial temperature is supplied to it, and suppose the temperature at both the end points kept at 0. Then one would like to know the temperature at each point of the string at a particular time.

Let us represent the string as an interval $[0, \ell]$. Let $u(x,t)$ be the temperature at the point $x \in [0, \ell]$ at time t. It is known that $u(\cdot, \cdot)$ satisfies the partial equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \quad 0 < t < \infty, \tag{1.1}$$

where $c > 0$ is the heat conductivity of the material. Since the temperature at both the end points kept at 0, we have

$$u(0,t) = 0 = u(\ell, t), \quad t > 0. \tag{1.2}$$

Let the initial temperature at the point x be $f()x)$, i.e.,

$$u(x, 0) = f(x), \quad 0 \le x \le \ell. \tag{1.3}$$

[1] Jean-Baptiste Joseph Fourier (21 March 1768 16 May 1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. In 1822 Fourier published his work on heat flow in Thorie analytique de la chaleur (The Analytical Theory of Heat),in which he based his reasoning on Newton's law of cooling, namely, that the flow of heat between two adjacent molecules is proportional to the extremely small difference of their temperatures.

Exercise 1.1.1 *Equation (1.1) satisfying (1.2) and (1.3) cannot have more than one solution.*

In order to find $u(x,t)$, we use a procedure called *method of separation of variables.* In this method, we assume first that $u(x,t)$ is of the form:

$$u(x,t) = \phi(x)\psi(t).$$

Then we have

$$\frac{\partial u}{\partial t} = \phi(x)\psi'(t), \quad \frac{\partial^2 u}{\partial t^2} = \phi''(x)\psi(t).$$

Hence, from (1.1),

$$\phi(x)\psi'(t) = c^2\phi''(x)\psi(t), \tag{1.4}$$

and from (1.2),

$$\phi(0)\psi(t) = 0, \qquad \phi(\ell)\psi(t) = 0. \tag{1.5}$$

If $\psi(t) \equiv 0$, then $u(x,t) \equiv 0$ - not desirable. Hence, $\psi(t) \neq 0$ for some t, and hence by (1.5),

$$\phi(0) = 0, \qquad \phi(\ell) = 0. \tag{1.6}$$

By (1.4),

$$\frac{\psi'(t)}{c^2\psi(t)} = \frac{\phi''(x)}{\phi(x)} = K, \text{const.}$$

Hence,

$$\psi'(t) = Kc^2\psi(t), \quad \phi''(x) = K\phi(x). \tag{1.7}$$

Let us consider different cases:

Suppose $K = 0$: In this case, $\phi''(x) = 0$ so that ϕ is of the form

$$\phi(x) = ax + b.$$

By (1.6), $\phi(0) = 0$ so that $b = 0$ and $\phi(\ell) := a\ell + b = a\ell = 0$ so that $a = 0$. Hence, $\phi(x) \equiv 0$ so that $u(x,t) \equiv 0$ - not desirable.

Therefore, $K = 0$ not possible.

Suppose $K > 0$: In this case, $K = \alpha^2$ for some $\alpha \neq 0$. Then we have

$$\phi''(x) - \alpha^2\phi(x) = 0$$

so that ϕ is of the form

$$\phi(x) = ae^{\alpha x} + be^{-\alpha x}.$$

By (1.6), $\phi(0) = 0$ and $\phi(\ell) = 0$. Hence $a+b = 0$ and $ae^{\alpha\ell}+be^{-\alpha\ell} = 0$ leading to $a = 0$, $b = 0$. Therefore, $\phi(x) \equiv 0$ so that $u(x,t) \equiv 0$ - not desirable.

Therefore, $K > 0$ not possible.

Suppose $K < 0$: In this case, $K = -\alpha^2$ for some $\alpha \neq 0$. Then we have

$$\phi''(x) + \alpha^2\phi(x) = 0$$

so that ϕ is of the form

$$\phi(x) = a\cos\alpha x + b\sin\alpha x.$$

By (1.6), $\phi(0) = 0 = \phi(\ell$ so that $a = 0$ and $b\sin\alpha\ell = 0$. Thus, $\alpha = \{n\pi/\ell$ for some $n \in \mathbb{Z}\}$ so that

$$\phi(x) \in \{b_n \sin\frac{n\pi x}{\ell} : n \in \mathbb{N}\}$$

for some some (b) with $b_n \neq 0$ for all $n \in \mathbb{N}$.

Now, from (1.7),

$$\psi'(t) = -\alpha^2c^2\psi(t)$$

with $\alpha \in \{n\pi/\ell : n \in \mathbb{Z}\}$. Hence,

$$\psi(t) = ae^{-\alpha^2c^2t},$$

and hence, u is of the form

$$u(x,t) = ae^{-\alpha^2c^2t}\sin\alpha x, \quad \alpha \in \{n\pi/\ell : n \in \mathbb{N}\}.$$

Note that, for each $n \in \mathbb{N}$,

$$u_n(x,t) = a_ne^{-\lambda_n^2c^2t}\sin\lambda_n x, \quad \lambda_n := \frac{n\pi}{\ell},$$

satisfies (1.1) and (1.2), where (a_n) is any sequence in $\mathbb{R}$. But, this u_n need not satisfy (1.3), unless $f(x) = a_n\sin(n\pi x/\ell)$ for some $n \in \mathbb{Z}$. In particular, we have proved the following:

> If f is of the form $f(x) = a_n\sin(n\pi x/\ell)$ for some $n \in \mathbb{N}$ and $a_n \in \mathbb{R}$, then $u_n(x,t) := a_ne^{-\lambda_n^2c^2t}\sin(n\pi x/\ell)$ is a solution of (1.1) satisfying (1.2) and (1.3).

At this point, we may observe that if $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are solutions of (1.1) satisfying (1.2), then $\alpha u(\cdot,\cdot)+\beta v(\cdot,\cdot)$ is also a solution of (1.1) satisfying (1.2) for any $\alpha,\beta\in\mathbb{R}$. However, if $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ also satisfy (1.3), then $\alpha\, u(\cdot,\cdot)+\beta\, v(\cdot,\cdot)$ need not satisfy (1.3). In fact, if $u(x,0)=f(x)$ and $v(x,0)=f(x)$, then

$$\alpha\, u(x,0)+\beta\, v(x,0)=(\alpha+\beta)f(x).$$

However, if $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ do not satisfy (1.3), then $\alpha\, u(\cdot,\cdot)+\beta\, v(\cdot,\cdot)$ can satisfy (1.3). That is the case, if $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are solutions of (1.1) satisfying (1.2) and if f is of the form $f(x)=\alpha\, u(x,0)+\beta\, v(x,0)$ for some $\alpha,\beta\in\mathbb{R}$.

The above discussion, motivates the following consideration:

Suppose f is of the form

$$f(x)=\sum_{n=1}^{k}a_n\sin(n\pi x/\ell) \tag{1.8}$$

for some $k\in\mathbb{N}$. Then, it can be easily seen that

$$u(x,t):=\sum_{n=1}^{k}a_n e^{-\lambda_n^2c^2t}\sin(n\pi x/\ell)$$

satisfies (1.1), (1.2) and (1.3). In this case $a_1,\ldots,a_k$ can be determined in terms of f as follows: Note that

$$\int_0^\ell f(x)\sin(\frac{m\pi x}{\ell})dx=\sum_{n=1}^{\infty}a_n\int_0^\ell\sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})dx.$$

Since

$$\int_0^\ell\sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})dx=0\quad\text{for}\quad m\neq n,$$

and

$$\int_0^\ell\sin^2(\frac{n\pi x}{\ell})dx=\int_0^\ell\frac{1-\cos 2(\frac{n\pi x}{\ell})}{2}dx=\frac{\ell}{2},$$

we obtain

$$\int_0^\ell f(x)\sin(\frac{m\pi x}{\ell})dx=\sum_{n=1}^{\infty}a_n\int_0^\ell\sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})dx=a_n\frac{\ell}{2}.$$

Thus,

$$a_n = \frac{2}{\ell} \int_0^\ell f(x) \sin(\frac{n\pi x}{\ell}) dx. \tag{1.9}$$

Thus, we have proved the following theorem.

Theorem 1.1.2 *If f is of the form (1.8) for some $k \in \mathbb{N}$, then a_n is given by (1.9), and*

$$u(x,t) := \sum_{n=1}^{k} a_n e^{-\lambda_n^2 c^2 t} \sin(n\pi x/\ell)$$

satisfies (1.1), (1.2) and (1.3).

What can we say if f is, in some sense, arbitrary? The consideration of the functions of the form in (1.8) suggests the following query:

> If f is of the form $f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/\ell)$, can we say that
> $u(x,t) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 c^2 t} \sin(n\pi x/\ell)$
> is a solution of (1.1) satisfying (1.2) and (1.3) with appropriate notion of convergence?

As a first step, let us assume that f is of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{\ell}). \tag{1.10}$$

Assume further that, the series in (1.10) can be integrated term by term along $[0, \ell]$. Then, we have

$$\int_0^\ell f(x) \sin(\frac{m\pi x}{\ell}) dx = \sum_{n=1}^{\infty} a_n \int_0^\ell \sin(\frac{n\pi x}{\ell}) \sin(\frac{m\pi x}{\ell}) dx = a_n \frac{\ell}{2}.$$

Hence,

$$a_n = \frac{2}{\ell} \int_0^\ell f(x) \sin(\frac{n\pi x}{\ell}) dx. \tag{1.11}$$

If f is as in (1.10), we may define

$$u(x,t) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 c^2 t} \sin(\lambda_n x), \tag{1.12}$$

where $\lambda_n := n\pi/\ell$ and a_n is as in (1.11), and check whether the above series converges and it satisfies (1.1), (1.2) and (1.3):

Formally,

$$\sum_{n=1}^{\infty} |a_n e^{-\lambda_n^2 c^2 t} \sin(\lambda_n x)| \le \sum_{n=1}^{\infty} |a_n| e^{-\lambda_n^2 c^2 t} \le \sum_{n=1}^{\infty} |a_n| e^{-\lambda_n^2 c^2 t_0}$$

for each $t_0 > 0$ and for every $t \ge t_0$. It can be seen that the scalar series $\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n^2 c^2 t_0}$ is convergent. Note that

$$|a_n| \le \frac{2}{\ell} \int_0^{\ell} |f(x)| dx.$$

Assume that $f \in L^1[0,\ell]$ so that (a_n) is bounded. Hence, the series in (1.12) converges uniformly, and hence it represents a continuous function on the set $[0,\ell] \times [t_0,\infty)$. Also, for any fixed $t_0 > 0$, and for every $t \ge t_0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \left|\frac{\partial}{\partial t}\Big(a_n e^{-\lambda_n^2 c^2 t} \sin(\lambda_n x)\Big)\right| &\le \sum_{n=1}^{\infty} |a_n| \lambda_n^2 c^2 e^{-\lambda_n^2 c^2 t} \\ &\le \sum_{n=1}^{\infty} |a_n| \lambda_n^2 c^2 e^{-\lambda_n^2 c^2 t_0}. \end{aligned}$$

It can also be seen that the scalar series $\sum_{n=1}^{\infty} |a_n| \lambda_n^2 c^2 e^{-\lambda_n^2 c^2 t_0}$ is convergent. Hence, for each $x \in [0,\ell]$, the series

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial t}\Big(a_n e^{-\lambda_n^2 c^2 t} \sin(\lambda_n x)\Big) := \sum_{n=1}^{\infty} \Big(a_n(-\lambda_n^2 c^2) e^{-\lambda_n^2 c^2 t} \sin(n\pi x/\ell)\Big)$$

is uniformly convergent for $t \in [t_0,\infty)$. Therefore, we can assert that, for each $x \in [0,\ell]$ and for each $t > t_0$, $\frac{\partial}{\partial t} u(x,t)$ exists and

$$\frac{\partial}{\partial t} u(x,t) = \sum_{n=1}^{\infty} \Big(a_n(-\lambda_n^2 c^2) e^{-\lambda_n^2 c^2 t} \sin(n\pi x/\ell)\Big). \tag{1.13}$$

Also,

$$\sum_{n=1}^{\infty} \Big|\frac{\partial^2}{\partial x^2}\Big(a_n e^{-\lambda_n^2 c^2 t}\sin(\lambda_n x)\Big)\Big| \le \sum_{n=1}^{\infty} |a_n|\lambda_n^2 e^{-\lambda_n^2 c^2 t}.$$

Again, it can be seen that the scalar series $\sum_{n=1}^{\infty} |a_n|\lambda_n^2 e^{-\lambda_n^2 c^2 t}$ is convergent for each $t > 0$. Hence, for each $t > 0$, the series

$$\sum_{n=1}^{\infty} \frac{\partial^2}{\partial x^2}\Big(-a_n e^{-\lambda_n^2 c^2 t}\sin(\lambda_n x)\Big) := \sum_{n=1}^{\infty}\Big(a_n\lambda_n^2 e^{-\lambda_n^2 c^2 t}\sin(\lambda_n x)\Big)$$

is uniformly convergent for $x \in [0, \ell]$. Therefore, we can assert that, for every $t > 0$, $\frac{\partial^2}{\partial x^2}u(x,t)$ exists and

$$\frac{\partial^2}{\partial x^2}u(x,t) = \sum_{n=1}^{\infty}\Big(-a_n\lambda_n^2 e^{-\lambda_n^2 c^2 t}\sin(\lambda_n x)\Big). \tag{1.14}$$

The equalities in (1.13) and (1.14) imply that

$$\frac{\partial}{\partial t}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t)$$

for every $(x, t) \in [0, \ell] \times [t_0, \infty)$. This is true for every $t_0 > 0$. Hence, (1.1) holds for every $(x, t) \in [0, \ell] \times (0, \infty)$. Clearly, $u(x, t)$ defined in (1.12) satisfies (1.2) and (1.3) as well.

Thus, we have proved the following:

Theorem 1.1.3 *If $f \in L^1[0, \ell]$ and if f can be represented as in (1.10) with a_n is as in (1.11), then $u(x, t)$ as in (1.12) is well defined and it satisfies the heat equation (1.1), boundary conditions in (1.2) and the initial condition (1.3).*

Note that the series in (1.10) is valid for all $x \in \mathbb{R}$. Suppose it is represented by $f(x)$. Then we see that

$$f(-x) = -f(x) \quad \forall\, x \in \mathbb{R}$$

and

$$f(x \pm 2\ell) = f(x) \quad \forall\, x \in \mathbb{R}.$$

In particular, if the series is known in the interval $[0, \ell]$, then we know it in the interval $[-\ell, \ell]$, and also throughout $\mathbb{R}$.

1.2 Fourier Series of 2π-Periodic functions

In the last section, we assumed that the function f can be represented as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \lambda_n x, \quad \lambda_n := \frac{n\pi}{\ell}.$$

If $\ell = \pi$, then the above series takes the form

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx, \quad A_n := \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

The above series is a special case of the *Fourier series* that we are going to introduce. Let us consider a few definitions.

Definition 1.2.1 A function of the form

$$c_0 + \sum_{n=1}^{k} (a_n \cos nx + b_n \sin nx).$$

where c_0, $a_n, b_n \in \mathbb{R}$, is called a **trigonometric polynomial**, and a series of the form

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with c_0, $a_n, b_n \in \mathbb{R}$ is called a **trigonometric series**. ◊

Note that a trigonometric polynomial is a special case of a trigonometric series.

We observe that trigonometric polynomials are 2π-*periodic* on $\mathbb{R}$, i.e., if $f(x)$ is a trigonometric polynomial, then

$$f(x + 2\pi) = f(x) \quad \forall\, x \in \mathbb{R}.$$

From this, we can infer that, if the trigonometric series

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges at a point $x \in \mathbb{R}$, then it has to converge at $x + 2\pi$ as well; and hence at $x + 2n\pi$ for all integers n. This shows that we can restrict the discussion of convergence of a trigonometric series to an interval of length 2π. Hence, we cannot expect to have a trigonometric series expansion for a function $f : \mathbb{R} \to \mathbb{R}$ if it is not a 2π-periodic function.

We know that a convergent trigonometric series is 2π-periodic. What about the converse?

> Suppose that f is a 2π-periodic function. Is it possible to represent f as a trigonometric series?

Suppose, for a moment, that we can write

$$f(x) = c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

for all $x \in \mathbb{R}$. Then what should be the coefficients c_0, a_n, b_n? To answer this question, let us further assume that f *is integrable on* $[-\pi, \pi]$ *and the series can be integrated term by term.*

For instance if the above series is uniformly convergent to f in $[-\pi, \pi]$, then term by term integration is possible. By Weierstrass test,we have the following result:

> If $\sum_{n=0}^{\infty}(|a_n| + |b_n|)$ converges, then $c_0+\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is a dominated series on $\mathbb{R}$ and hence it is uniformly convergent to a continuous function.

For $n, m \in \mathbb{N} \cap \{0\}$, we observe the following *orthogonality relations*:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos nx \cos mx dx &= \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m \neq 0, \\ 2\pi, & \text{if } n = m = 0, \end{cases} \\
\int_{-\pi}^{\pi} \sin nx \sin mx dx &= \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m, \end{cases} \\
\int_{-\pi}^{\pi} \cos nx \sin mx dx &= 0.
\end{aligned}$$

Thus, under the assumption that f is integrable on $[-\pi, \pi]$ and the series can be integrated term by term, we obtain

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

Motivated by this, we define the *Fourier series*:

Definition 1.2.2 The **Fourier series** of a 2π-periodic function f is the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right),$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$ and this fact is written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right).$$

The numbers a_n and b_n are called the **Fourier coefficients** of f. ♢

> If f is a trigonometric polynomial, then its Fourier series is itself.

Writing

$$\cos nx = \frac{1}{2}[e^{inx} + e^{-inx}], \quad \sin nx = \frac{1}{2i}[e^{inx} - e^{-inx}],$$

we have

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \frac{a_n}{2}[e^{inx} + e^{-inx}] + \frac{b_n}{2i}[e^{inx} - e^{-inx}] \\ &= \left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-inx}. \end{aligned}$$

Thus, writing

$$c_n := \frac{a_n}{2} + \frac{b_n}{2i}, \quad c_{-n} := \frac{a_n}{2} - \frac{b_n}{2i},$$

we have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) = \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Now, suppose $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$ with $c_n \in \mathbb{C}$, and this series can be integrated term by term. Then, we have

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n\in\mathbb{Z}} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}dx.$$

But,

$$\int_{-\pi}^{\pi} e^{i(n-m)}dx = \begin{cases} 0 & if\, m = m, \\ 2\pi & if\, m \neq m. \end{cases}$$

Hence, $\int_{-\pi}^{\pi} e^{i(n-m)}dx = 2\pi c_m$, i.e.,

$$c_n = \frac{1}{2\pi}\int_{-2\pi}^{\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}.$$

The following theorem shows that there is a large class of functions which can be represented by their Fourier series (see Bhatia [**?**]). We shall come back to this theorem at a later stage.

Theorem 1.2.3 (Dirichlet's theorem) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function which is piecewise differentiable on $(-\pi,\pi)$, and $f(x+)$, $f(x-)$, $f'(x+)$, $f'(x-)$ exist. Then the Fourier series of f converges, and the limit function $\tilde{f}(x)$ is given by*

$$\tilde{f}(x) = \frac{1}{2}[f(x-) + f(x+)], \quad x \in \mathbb{R}.$$

Remark 1.2.4 It is known that there are continuous functions f defined on $[-\pi,\pi]$ whose Fourier series does not converge pointwise to f. Its proof relies on UBP (see [1]). We shall consider this at a later occasion. ◊

Although each term and the partial sums of a Fourier series are infinitely differentiable, the sum function need not be even continuous at certain points. This fact is illustrated by the following example.

Example 1.2.5 Let $f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ 1, & 0 < x \le \pi. \end{cases}$ By Dirichlet's theorem (Theorem 1.2.3), the Fourier series of f converges to $f(x)$ for every $x \neq 0$, and at the point 0, the series converges to $1/2$. Note that

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos nx dx = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

and for $n \in \mathbb{N}$,

$$b_n = \frac{1}{\pi}\int_0^\pi \sin nx dx = \frac{1}{\pi}\left[\frac{1-\cos n\pi}{n}\right] = \frac{1}{\pi}\left[\frac{1-(-1)^n}{n}\right] = \begin{cases} \dfrac{2}{\pi n}, & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

Thus, Fourier series of f is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}.$$

In particular, for $x = \pi/2$,

$$1 = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin[(2n+1)\pi/2]}{(2n+1)} = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

which leads to the *Madhava–Nilakaṅtha series*

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}. \qquad \Diamond$$

1.3 Fourier Series for Even and Odd Functions

The following can be verified easily:

- Suppose f is an even function, i.e.,

  $$f(-x) = f(x) \quad \forall\, x \in X.$$

  Then $f(x)\cos nx$ is an even function and $f(x)\sin nx$ is an odd function, so that

  $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$$

  $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

- Suppose f is an odd function, i.e.,

$$f(-x) = -f(x) \quad \forall\, x \in X.$$

Then $f(x)\cos nx$ is an odd function and $f(x)\sin nx$ is an even function, so that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx.$$

Thus, we have the following:

(1) Suppose f is an even function. Then the Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{with} \quad a_n := \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx.$$

In particular,

$$f(0) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n, \quad f(\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (-1)^n a_n,$$

respectively.

(2) Suppose f is an odd function. Then the Fourier series of f is

$$\sum_{n=1}^{\infty} b_n \sin nx \quad \text{with} \quad b_n := \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx,$$

In particular,

$$f(\pi/2) = \sum_{n=0}^{\infty} (-1)^n b_{2n+1}.$$

Example 1.3.1 Consider the function f defined by

$$f(x) = |x|, \quad x \in [-\pi, \pi].$$

In this case, f is an even function. Hence, the Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [-\pi, \pi]$$

with

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$$

and for $n = 1, 2, \ldots$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx dx = \frac{2}{\pi}\left\{\left[x\,\frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi \frac{\sin nx}{n}dx\right\} \\ &= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^\pi = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] \end{aligned}$$

Thus,

$$a_{2n} = 0, \quad a_{2n+1} = \frac{-4}{\pi(2n+1)^2}, \quad n = 1, 2, \ldots$$

so that

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}, \quad x \in [-\pi, \pi].$$

Taking $x = 0$ (using Dirichlet's theorem), we obtain

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

◊

Example 1.3.2 Let $f(x) = x, \quad x \in [-\pi, \pi]$. In this case, f is an odd function. Hence, the Fourier series is

$$\sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [-\pi, \pi]$$

with

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{\pi}\left\{\left[-x\frac{\cos nx}{n}\right]_0^\pi + \int_0^\pi \frac{\cos nx}{n}dx\right\} \\ &= \frac{2}{\pi}\left\{-\pi\frac{\cos n\pi}{n}\right\} = \frac{(-1)^{n+1}2}{n}. \end{aligned}$$

Thus the Fourier series is

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

In particular (using Dirichlet's theorem), with $x = \pi/2$ we obtain the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \qquad \Diamond$$

Example 1.3.3 Let $f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 1, & 0 \le x \le \pi. \end{cases}$ In this case, f is an odd function. Hence, the Fourier series is

$$\sum_{n=1}^{\infty} b_n \sin nx,$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{\pi}(1 - \cos n\pi) = \frac{2}{\pi}[1 - (-1)^n].$$

Thus

$$f(x) \sim \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}.$$

Taking $x = \pi/2$, again we obtain the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \qquad \Diamond$$

Example 1.3.4 Let $f(x) = x^2, \quad x \in [-\pi, \pi]$. Since f is an even function, its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [-\pi, \pi], \quad a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx.$$

It can be see that $a_0 = 2\pi^2/3$, and $a_n = (-1)^n 4/n^2$. Thus

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad x \in [-\pi, \pi].$$

Taking $x = 0$ and $x = \pi$ (using Dirichlet's theorem), we have

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \qquad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

respectively. $\Diamond$

1.4 Sine and Cosine Series Expansions

Suppose a function f is defined on $[0,\pi]$. By extending it to $[-\pi,\pi]$ so that the extended function is an odd function, we obtain *Forier sine series* of f, and by extending it to $[-\pi,\pi]$ so that the extended function is an even function, we obtain *Fourier cosine series* of f.

The *odd extension* and *even extension* of f, denoted by f_{odd} and f_{even} are defined by

$$f_{odd}(x) = \begin{cases} f(x) & \text{if } 0 \le x < \pi, \\ -f(-x) & \text{if } -\pi \le x < 0, \end{cases},$$

$$f_{even}(x) = \begin{cases} f(x) & \text{if } 0 \le x < \pi, \\ f(-x) & \text{if } -\pi \le x < 0, \end{cases}$$

respectively. Therefore,

$$f(x) = f_{odd}(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [0,\pi]$$

and

$$f(x) = f_{even}(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [0,\pi]$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Example 1.4.1 Let $f(x) = x^2$, $x \in [0,\pi]$. The even extension of f is itself. Its odd extension is:

$$f_{odd}(x) = \begin{cases} x^2, & \text{if } 0 \le x < \pi, \\ -x^2, & \text{if } -\pi \le x < 0. \end{cases},$$

Hence,

$$f(x) = f_{odd}(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [0,\pi],$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx = \frac{2}{\pi}\left\{\left[-x^2\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} 2x\frac{\cos nx}{n}\,dx\right\}.$$

Note that

$$\left[-x^2\frac{\cos nx}{n}\right]_0^\pi = -\pi^2\frac{\cos n\pi}{n} = \pi^2\frac{(-1)^{n+1}}{n},$$

$$\begin{aligned}\int_0^\pi 2x\frac{\cos nx}{n}\,dx &= \left[2x\frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi 2\frac{\sin nx}{n}dx \\ &= 2\left[\frac{\cos nx}{n^2}\right]_0^\pi = 2\left[\frac{(-1)^n-1}{n^2}\right].\end{aligned}$$

Thus,

$$\begin{aligned}b_n &= \frac{2}{\pi}\left\{\pi^2\frac{(-1)^{n+1}}{n} + 2\left[\frac{(-1)^n-1}{n^2}\right]\right\} \\ &= 2\pi\frac{(-1)^{n+1}}{n} + \frac{4}{\pi}\left[\frac{(-1)^n-1}{n^2}\right].\end{aligned}$$

$\diamond$

Example 1.4.2 Let $f(x) = x$, $x \in [0,\pi]$. Its odd extension is itself, and

$$f_{even}(x) = |x|, \quad x \in [-\pi,\pi].$$

From Examples 2.5.13 and 1.3.1, we obtain

$$x \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx, \quad x \in [0,\pi]$$

and

$$x \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}, \quad x \in [0,\pi].$$

$\diamond$

Example 1.4.3 Let us find the sine series expansion of the function

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2, \\ 1, & \text{if } \pi/2 \le x < \pi. \end{cases}$$

The sine series of f is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [0,\pi],$$

where

$$b_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin nx\,dx = -\frac{2}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi} = \frac{2}{\pi}\left[\frac{\cos n\pi/2 - \cos n\pi}{n}\right].$$

Note that $b_{2n-1} = \dfrac{2}{(2n-1)\pi}$ and

$$b_{2n} = \frac{2}{2n\pi}[(-1)^n - 1] = \begin{cases} -\frac{2}{n\pi} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Thus, for $x \in [0, \pi]$, we have

$$\begin{aligned} \frac{\pi}{2} f(x) \quad \sim \quad & \frac{\sin x}{1} - \frac{\sin 2x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(4n-3)x}{4n-3} \\ & - \frac{\sin(4n-2)x}{4n-2} + \frac{\sin(4n-1)x}{4n-1} + \frac{\sin(4n+1)x}{4n+1} + \cdots. \end{aligned}$$

◊

1.5 Fourier Series of 2ℓ-Periodic Functions

Suppose f is a T-periodic function. We may write $T = 2\ell$. Then we may consider the change of variable $t = \pi x/\ell$ so that the function

$$f(x) := f(\ell t/\pi),$$

as a function of t is 2π-periodic. Hence, its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) \cos nt dt = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) \sin nt dt = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx. \end{aligned}$$

In particular,

$$f \quad \text{even} \quad \Rightarrow \quad b_n = 0 \quad \text{and} \quad a_n = \frac{2}{\ell}\int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx,$$

$$f \quad \text{odd} \quad \Rightarrow \quad a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}dx.$$

Example 1.5.1 Let $f(x) = 1 - |x|, \quad -1 \le x \le 1$. Taking $\ell = 1$, we have

$$a_n = \int_{-1}^{1}(1-|x|)\cos n\pi x\,dx = 2\int_0^1 (1-|x|)\cos n\pi x\,dx$$

and

$$b_n = \int_{-1}^{1}(1-|x|)\sin n\pi x\,dx = 0.$$

Now,

$$\int_0^1 \cos n\pi x\,dx = \left[\frac{\sin n\pi x}{n}\right]_0^1 = 0,$$

$$\int_0^1 x\cos n\pi x\,dx = \left[x\frac{\sin n\pi x}{n}\right]_0^1 - \int_0^1 \frac{\sin n\pi x}{n}\,dx = \left[\frac{\cos n\pi x}{n}\right]_0^1 = \frac{(-1)^n - 1}{n}.$$

Hence,

$$a_n = 2\int_0^1 (1-|x|)\cos n\pi x\,dx = \frac{2}{n}[1-(-1)^n] = \begin{cases} 0, & n\text{even}, \\ 4/n, & n \text{ odd.}\end{cases}$$

Thus,

$$f(x) \sim \sum_{n=0}^{\infty}\frac{4}{2n+1}\cos n\pi x. \qquad \Diamond$$

1.6 Fourier Series on Arbitrary Intervals

Suppose a function f is defined in an interval $[a,b]$. We can obtain Fourier expansion of it on $[a,b]$ as follows:

Method 1: Let us consider a change of variable as $y = x - \frac{a+b}{2}$. Let

$$\varphi(y) := f(x) = f(y + \frac{a+b}{2}), \qquad \text{where} \quad -\ell \le y \le \ell$$

with $\ell = (b-a)/2$. We can extend φ as a 2ℓ-periodic function and obtain its Fourier series as

$$\varphi(y) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{\ell}y + b_n\sin\frac{n\pi}{\ell}y\right)$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(y)\cos\frac{n\pi y}{\ell}dy, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(y)\sin\frac{n\pi y}{\ell}dy.$$

Thus,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{\ell}y + b_n\sin\frac{n\pi}{\ell}y\right)$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi y}{\ell}dx, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi y}{\ell}dx$$

with $\ell = (b-a)/2$ and $y = x - \frac{a+b}{2}$.

Method 2: Considering the change of variable as $y = x - a$ and $\ell := b - a$, we define $\varphi(y) := f(x) = f(y+a)$ where $0 \le y < \ell$. We can extend φ as a 2ℓ-periodic function in any manner and obtain its Fourier series. Here are two specific cases:

(a) For $y \in [-\ell, 0]$, define $\tilde{f}_e(y) = \varphi f(-y)$. Thus $\tilde{f}_e$ on $[-\ell,\ell]$ is an even function. In this case,

$$\varphi(y) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{\ell}y$$

where $\ell = (b-a)/2$ and

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(y)\cos\frac{n\pi y}{\ell}dy.$$

(b) For $y \in [-\ell, 0]$, define $\tilde{f}_o(y) = -\varphi(-y)$. Thus $\tilde{f}_o$ on $[-\ell,\ell]$ is an odd function. In this case,

$$\varphi(y) \sim \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{\ell}y$$

where

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(y)\sin\frac{n\pi y}{\ell}dy.$$

From the series of φ we can recover the corresponding series of f on $[a,b]$ by writing $y = x - a$.

1.7 Problems

The following are taken from the book by MTN[2]

1. Find the Fourier series of the 2π- period function f such that:

   (a) $f(x) = \begin{cases} 1, & \frac{-\pi}{2} \le x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \frac{3\pi}{2}. \end{cases}$

   (b) $f(x) = \begin{cases} x, & \frac{-\pi}{2} \le x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \frac{3\pi}{2}. \end{cases}$

   (c) $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi. \end{cases}$

   (d) $f(x) = \frac{x^2}{4}$, $-\pi \le x \le \pi$.

2. Using the Fourier series in Exercise 1, find the sum of the following series:

   (a) $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots,$ (b) $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \ldots.$

   (c) $1 - \dfrac{1}{4} + \dfrac{1}{9} - \dfrac{1}{16} + \ldots,$ (d) $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots.$

3. If $f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$, then show that

$$f(x) \sim \frac{8}{\pi} \cos \frac{\pi}{4} \left[\frac{\sin x}{1.3} + \frac{\sin 3x}{5.7} + \frac{\sin 10x}{9.11} + \ldots \right].$$

4. Show that for $0 < x < 1$,

$$x - x^2 = \frac{8}{\pi^2} \left[\frac{\sin x\pi}{1^3} + \frac{\sin 3\pi x}{3^3} + \frac{\sin 5\pi x}{5^3} + \ldots \right].$$

5. Show that for $0 < x < \pi$,

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots = \frac{\pi}{4}.$$

[2] *Calculus of One Variable*, Anne Publishers, New Delhi, 2015

6. Show that for $-\pi < x < \pi$,

$$x \sin x = 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \dots,$$

and find the sum of the series

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \dots.$$

7. Show that for $0 \le x \le \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots\right],$$

$$x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\right].$$

8. Assuming that the Fourier series of f converges uniformly on $[-\pi, \pi)$, show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

9. Using Exercises 7 and 8 show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$ (*d*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$

10. Write down the Fourier series of $f(x) = x$ for $x \in [1, 2)$ so that it converges to $1/2$ at $x = 1$.

2

More on Fourier Series

2.1 Trigonometric series and Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $f \in L^1[-\pi, \pi]$. Recall that the Fourier series of f is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx. \tag{2.2}$$

Now, writing

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \cos nx = \frac{e^{inx} - e^{-inx}}{2i}$$

the series in (2.1) takes the form (verify)

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{with} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt \quad \text{for} \quad n \in \mathbb{Z}.$$

Notation 2.1.1 We shall denote by $\mathbb{T}$ the set of all complex numbers z such that $|z| = 1$. Note that, every $z \in \mathbb{T}$ can be represented as $z = e^{it}$ for some $t \in \mathbb{R}$. Thus,

$$\mathbb{T} = \{e^{it} : t \in \mathbb{R}\}$$

and the map $\varphi : \mathbb{R} \to \mathbb{C}$ defined by

$$\varphi(t) := e^{it}, \quad t \in \mathbb{R},$$

is continuous, 2π-periodic and onto.

If $f : \mathbb{T} \to \mathbb{C}$, then we obtain a new function $F : \mathbb{R} \to \mathbb{C}$ defined by

$$F(t) = f(e^{it}), \quad t \in \mathbb{R}.$$

Note that $F : \mathbb{R} \to \mathbb{C}$ is 2π-periodic. In view of this, any 2π-periodic function on $\mathbb{R}$ is considered as a function defined on $\mathbb{T}$. $\Diamond$

With the above convention, the space all 2π-periodic measurable functions $f : \mathbb{R} \to \mathbb{C}$ with $f|_{[-\pi,\pi]} \in L^1[-\pi, \pi]$ will be denoted by $L^1(\mathbb{T})$. We may write this fact as "f is a 2π-periodic function such that $f \in L^1[-\pi, \pi]$".

Now, for $f \in L^1(\mathbb{T})$ we define the Fourier series.

Definition 2.1.2 Let $f \in L^1(\mathbb{T})$. Then the **Fourier series** (FS) of f is the series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \tag{2.3}$$

with

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}, \tag{2.4}$$

and we write this fact as

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

The coefficients $\hat{f}(n)$ are called the **Fourier coefficients** of f. $\Diamond$

Notation 2.1.3 In the above and in the following, the integrals are w.r.t. the Lebesgue measure, and we use the notation $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\Diamond$

Note that, for $f \in L^1(\mathbb{T})$,

$$|\hat{f}(n)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| dt \quad \forall\, n \in \mathbb{Z}.$$

Thus,

- $(\hat{f}(n))$ is a bounded sequence, in fact, bi-squence.

Definition 2.1.4 Let $f \in L^1(\mathbb{T})$. For $N \in \mathbb{N}$, the sum

$$S_N(f,x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

is called the N**-th partial sum** of the Fourier series (2.3). $\Diamond$

Definition 2.1.5 For $f \in L^1(\mathbb{T})$, we say that Fourier series of f converges to a function $g : [-\pi,\pi] \to \mathbb{C}$ at $x \in [-\pi,\pi]$ if

$$S_N(f,x)) \to g(x) \quad \text{as} \quad N \to \infty. \qquad \Diamond$$

Note that $S_N(f,\cdot)$ is a trigonometric polynomial; in particular, it is continuously differentiable infinite many times. However, there here is no guarantee that the limit function even continuous. Of course, if the convergence is uniform, then g has to be continuous.

Since e^{inx} are 2π-periodic functions, we can talk about Fourier series of 2π-periodic functions. If (2.3) (resp. (2.1)) converges at a point $x \in [-\pi,\pi]$, then it converges at $x + 2k\pi$ for every $k \in \mathbb{Z}$.

In the due course we shall also use the following notations:

Notation 2.1.6 For $1 \le p \le \infty$, we shall denote by $L^p(\mathbb{T})$ the space of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ which are 2π-periodic, that is, $f(x+2\pi) = f(x)$ a.a. $x \in \mathbb{R}$, and $f \in L^p(-\pi,\pi)$, and by $C(\mathbb{T})$ the space of all continuous 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. $\Diamond$

- For $1 \le p < \infty$,

  $$f \mapsto \|f\|_p := \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p dx\Big)^{1/p}$$

  defines a complete norm on $L^p(\mathbb{T})$, that is, $L^p(\mathbb{T})$ is a Banach space[1] w.r.t. this norm.

- The space $L^2(\mathbb{T})$ is also a Hilbert space[2] with inner product

  $$(f,g) \mapsto \langle f,g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

[1] A Banach space is a normed linear space which is complete w.r.t. the metric induced by the norm.

[2] A Hilbert space is an inner product space which is complete w.r.t. the metric induced by the inner product.

- The space $C(\mathbb{T})$ is a Banach space w.r.t. the norm

$$\|f\|_\infty := \sup_{-\pi\le t\le\pi} |f(t)|, \quad f \in C(\mathbb{T}).$$

Using the above notation, we have

$$\|\hat{f}\|_\infty := \sup_{n\in\mathbb{Z}} |\hat{f}(n)| \le \|f\|_1 \quad \forall\, f \in L^1(\mathbb{T}).$$

Theorem 2.1.7 *Let $f \in L^1(\mathbb{T})$. If $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges, then the FS of f converges uniformly to a function $g \in C(\mathbb{T})$ and*

$$\hat{g}(n) = \hat{f}(n) \quad \forall\, n \in \mathbb{Z}.$$

Proof. Suppose $f \in L^1(\mathbb{T})$ such that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges. Then, by Weierstrass test, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ converges uniformly to a continuous function, say g on $[-\pi, \pi]$. It can be also seen that $g(-\pi) = g(\pi)$ so that g can be extended to the whole of $\mathbb{R}$ to obtain $g \in C(\mathbb{T})$. Also, for every $m \in \mathbb{Z}$,

$$g(x)e^{-imx} = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}e^{-imx},$$

and the convergence is uniform. Therefore, term by term integration is possible, so that we obtain $\hat{g}(m) = \hat{f}(m)$ for all $m \in \mathbb{Z}$. ▮

In the due course, we shall also show that, if g is as in the above theorem, then $g = f$ a.e. In particular, we would get:

- If $f \in C(\mathbb{T})$, and if $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges, then the FS of f converges to f uniformly.

Notation 2.1.8 For $k \in \mathbb{N}$, we denote by $C^k(\mathbb{T})$ the space of all 2π-periodic functions on $\mathbb{R}$ which are k-times continuously differentiable on $\mathbb{R}$. ♢

The following theorem gives a sufficient condition under which the assumptions in Theorem 2.1.7 are satisfied.

Theorem 2.1.9 *For $k \in \mathbb{N}$, if $f \in C^k(\mathbb{T})$, then*

$$\hat{f}(n) = \frac{1}{(in)^k}\widehat{f^{(k)}}(n) \forall\, n \in \mathbb{Z} \setminus \{0\}.$$

In particular, if $f \in C^2(\mathbb{T})$, then $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|$ converges.

Proof. Since $f(-\pi) = f(\pi)$,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\Big\{\Big[f(x)\,\frac{e^{-inx}}{-in}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\Big[\frac{e^{-inx}}{-in}\Big]dx\Big\} \\ &= \frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \frac{1}{in}\hat{f}'(n).\end{aligned}$$

Following the same arguments, we have $\hat{f}'(n) = \frac{1}{in}\hat{f}''(n)$. Repeated application of the above gives

$$\hat{f}(n) = \frac{1}{(in)^k}\widehat{f^{(k)}}(n).$$

In particular, if $f \in C^2(\mathbb{T})$, then $|\hat{f}(n)| \le \frac{1}{n^2}\|\hat{f^{(2)}}\|_\infty$ so that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|$ converges. ▮

Remark 2.1.10 *By the above theorem,*

$$f \in C^k(\mathbb{T}) \quad \Rightarrow \quad |\hat{f}(n)| = \frac{|\widehat{f^{(k)}}(n)|}{|n|^k}.$$

We shall soon see that $\hat{f}(n) \to 0$ as $n \to \pm\infty$, so that we would get

$$|\hat{f}(n)| = O\Big(\frac{1}{|n|^k}\Big)$$

whenever $f \in C^k(\mathbb{T})$. ◊

Let us recall that for $f \in L^1(\mathbb{T})$, the sequence $(\hat{f}(n))$ bounded. In fact, $|\hat{f}(n))| \le \|f\|_1$ for all $n \in \mathbb{Z}$. Hence, $(\hat{f}(n)) \in \ell^\infty(\mathbb{Z})$ for every $f \in L^1(\mathbb{T})$. Further, the map $f \mapsto (\hat{f}(n))$ is a linear operator from $L^1(\mathbb{T})$ to $\ell^\infty(\mathbb{Z})$, that is,

$$\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n), \quad \widehat{\alpha f}(n) = \alpha\hat{f}(n)$$

for every $f, g \in L^1(\mathbb{T})$ and $\alpha \in \mathbb{C}$. In fact,

- $f \in L^1(\mathbb{T})$ implies $\hat{f}(n) \to 0$ as $|n| \to \infty$, that is, $(\hat{f}(n)) \in c_0(\mathbb{Z})$, where $c_0(\mathbb{Z}) :=$ the space of all sequences $(a_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$ such that $a_n \to 0$ as $n \to \pm\infty$.

This result is known as *Riemann Lebesgue Lemma.*

2.2 Riemann Lebesgue Lemma

Since $C[-\pi,\pi]$ is dense in $L^1[-\pi,\pi]$ for $1 \le p < \infty$, we have the following:

Theorem 2.2.1 *Let $f \in L^p(\mathbb{T})$ for $1 \le p < \infty$. Then for every $\varepsilon > 0$, there exists $g \in C(\mathbb{T})$ such that*

$$\int_{-\pi}^{\pi} |f(x) - g(x)|^p dx < \varepsilon.$$

In the above theorem and in what follows the integral is w.r.t. the Lebesgue measure.

For $f : \mathbb{R} \to \mathbb{R}$ and $\tau \in \mathbb{R}$, let

$$f_\tau(x) := f(x-\tau), \quad x \in \mathbb{R}.$$

Theorem 2.2.2 *Let f be a 2π-periodic extension of a function in $L^p[-\pi,\pi]$, where $1 \le p < \infty$. Then*

$$\int_{-\pi}^{\pi} |f(x)|^p dx = \int_{-\pi}^{\pi} |f_\tau(x)|^p dx \quad \forall\, \tau \in \mathbb{R}.$$

Proof. Follows by change of variables. ∎

Theorem 2.2.3 *If $g \in C(\mathbb{T})$, then for $1 \le p < \infty$,*

$$\int_{-\pi}^{\pi} |g(x) - g_\tau(x)|^p dx \to 0 \quad as \quad |\tau| \to 0.$$

Proof. Follows from the uniform continuity of g. ∎

From Theorem 2.2.1 and Theorem 2.2.3, we obtain the following theorem.

Theorem 2.2.4 *Let $f \in L^p(\mathbb{T})$ for $1 \le p < \infty$. Then*

$$\int_{-\pi}^{\pi} |f(x) - f_\tau(x)|^p dx \to 0 \quad as \quad |\tau| \to 0.$$

Theorem 2.2.5 (Riemann Lebesgue lemma) *For $f \in L^1(\mathbb{T})$,*

$$\hat{f}(n) \to 0 \quad as \quad n \to \pm\infty.$$

Proof. Since $e^{i\pi} = -1$, we have

$$\int_{-\pi}^{\pi} f(x)e^{-inx}dx = -\int_{-\pi}^{\pi} f(x)e^{-in(x+\pi/n}dx = -\int_{-\pi}^{\pi} f(y-\pi/n)e^{-iny}dy.$$

Hence,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)e^{-inx}dx &= \frac{1}{2}\int_{-\pi}^{\pi}[f(x) - f(x-\pi/n)]e^{-inx}dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi}[f(x) - f_{\pi/n}(x)]e^{-inx}dx\end{aligned}$$

Thus,

$$|\hat{f}(n)| = \frac{1}{2\pi}\Big|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\Big| \le \frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x) - f_{\frac{\pi}{n}}(x)|dx.$$

By Theorem 2.2.3, $\frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x) - f_{\frac{\pi}{n}}(x)|dx \to 0$ as $n \to \infty$. Consequently, $|\hat{f}(n)| \to 0$ as $n \to \pm\infty$. ∎

Corollary 2.2.6 (Riemann Lebesgue lemma) *Let $f \in L^1[a,b]$. Then*

$$\int_{-\pi}^{\pi} f(t)\cos(nt)dt \to 0 \quad \textit{and} \quad \int_{-\pi}^{\pi} f(t)\sin(nt)dt \to 0 \quad \textit{as} \quad n \to \infty.$$

2.3 Dirichlet kernel

For $f \in L^1(\mathbb{T})$ and $N \in \mathbb{N}$, we have

$$S_N(f,x) := \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx}\int_{-\pi}^{\pi} f(t)e^{-int}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt,$$

where

$$D_N(t) := \sum_{n=-N}^{N} e^{int}.$$

Using change of variables[3], we obtain

$$S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt.$$

---

[3] $\int_{-\pi}^{\pi} g(t)dt = \int_{-\pi+x}^{\pi+x} g(x-y)dy = \int_{-\pi}^{\pi} g(x-y)dy + \int_{\pi}^{\pi+x} g(x-y)dy - \int_{-\pi}^{-\pi+x} g(x-y)dy$; $\int_{-\pi}^{-\pi+x} g(t)dt = \int_{-\pi}^{-\pi+x} g(t+2\pi)dt = \int_{\pi}^{\pi+x} g(t)dt$

Hence,

$$f(x) - S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x) - f(x-t)]D_N(t)dt.$$

Our idea is to specify conditions on f and points $x \in [-\pi, \pi]$ under which

$$|S_N(f,x) - f(x)| \to 0 \quad \text{as} \quad N \to \infty.$$

Definition 2.3.1 The function $D_N(\cdot)$ is called the **Dirichlet kernel**. $\diamondsuit$

We observe that,

- $D_N(-t) = D_N(t)$ for all $t \in [-\pi, \pi]$ and
- $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} D_N(t)dt = 1.$
- $\widehat{D}_N(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} D_N(t)e^{int}dt = 1$ for all $n \in \mathbb{Z}$.
- $D_N(t) = \displaystyle\sum_{n=-N}^{N} e^{int} = 1 + \sum_{n=1}^{N}[e^{int} + e^{-int}] = 1 + 2\sum_{n=1}^{N}\cos nt.$

Theorem 2.3.2 *For each $N \in \mathbb{N}$, $D_N(0) = 2N + 1$, and for $t \neq 0$,*

$$D_N(t) = \frac{\sin(N + \frac{1}{2})t}{\sin(\frac{t}{2})}$$

Proof. Clearly, $D_N(0) = 2N + 1$. So, let $t \neq 0$. Note that

$$(e^{it} - 1)D_N(t) = \sum_{n=-N}^{N}[e^{i(n+1)t} - e^{int}] = e^{i(N+1)t} - e^{-iNt}.$$

But,

$$(e^{it} - 1) = e^{it/2}(e^{it/2} - e^{-it/2}) = 2ie^{it/2}\sin(t/2).$$

Thus, for $t \neq 2k\pi$,

$$D_N(t) = \frac{e^{i(N+1)t} - e^{-iNt}}{(e^{it} - 1)} = \frac{e^{-it/2}}{\sin(t/2)}\frac{e^{i(N+1)t} - e^{-iNt}}{2i} = \frac{\sin(N + \frac{1}{2})t}{\sin(\frac{t}{2})}.$$

Thus, we have obtained the required expression for $DN(t)$. ∎

2.4 Dirichlet-Dini criterion for convergence

We investigate the convergence: $S_N(f,x) \to f(x)$ as $N \to \infty$.

Theorem 2.4.1 (Dirichlet-Dini criterion) *Suppose $f \in L^1(\mathbb{T})$ is such that*

$$\int_{-\pi}^{\pi} \left| \frac{f(x) - f(x-t)}{t} \right| dt < \infty \qquad (*)$$

at a point $x \in [-\pi, \pi]$. then

$$S_N(f,x) \to f(x).$$

If $(*)$ hods uniformly for $x \in [-\pi, \pi]$, then

$$S_N(f,x) \to f(x) \quad \text{uniformly for} \quad x \in [-\pi, \pi].$$

Proof. Since $\displaystyle\int_{-\pi}^{\pi} D_N(t)dt = 1$ and $S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt$, we have

$$f(x) - S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) - f(x-t)]D_N(t)dt.$$

Hence, by Theorem 2.3.2,

$$\begin{aligned} f(x) - S_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) - f(x-t)]\frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \left\{\frac{f(x) - f(x-t)}{t}\right\}\left\{\frac{t/2}{\sin(\frac{t}{2})}\right\}\sin(N+\frac{1}{2})tdt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\varphi(t)}{g(t)}\sin(N+\frac{1}{2})tdt, \end{aligned}$$

where

$$\varphi(t) = \begin{cases} \frac{f(x)-f(x-t)}{t}, & t \neq 0 \\ 0, & t = 0. \end{cases}, \quad g(t)\begin{cases} \frac{\sin(t/2)}{t/2}, & 0 < |t| \leq \pi, \\ 1, & |t| = 0. \end{cases}$$

Note that $\varphi \in L^1(\mathbb{T})$ and g is continuous, bounded on $[-\pi, \pi]$, and $g(t) \to 1$ as $|t| \to 0$. Hence, there exists $\delta > 0$ such that $|g(t)| \geq 1/2$ for all t with $0 < |t| < \delta$. Also, for $\delta \leq |t| \leq \pi$, $1/g(t)$ is bounded. Hence, the function $t \mapsto \varphi(t)/g(t)$ belongs to $L^1(\mathbb{T})$, and hence, by Riemann Lebesgue lemma,

$$f(x) - S_N(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\varphi(t)}{g(t)}\sin(N+\frac{1}{2})tdt \to 0 \quad \text{as} \quad N \to \infty.$$

Also, if $(*)$ hods uniformly for $x \in [-\pi, \pi]$, then the above convergence is also uniform. ∎

The following corollaries are immediate from Theorem 2.4.1.

Corollary 2.4.2 *Suppose $f \in L^1(\mathbb{T})$.*

(i) *If f is Lipschitz at a point $x \in [-\pi, \pi]$, then*

$$S_N(f, x) \to f(x) \quad as \quad N \to \infty.$$

(ii) *If f is Lipschitz on $[-\pi, \pi]$, then*

$$S_N(f, x) \to f(x) \quad as \quad N \to \infty$$

uniformly on $[-\pi, \pi]$.

In particular, if $f \in C^1(\mathbb{T})$, then

$$S_N(f, x) \to f(x) \quad as \quad N \to \infty$$

uniformly on $\mathbb{R}$.

Theorem 2.4.3 (Dirichlet's theorem) *Suppose $f \in L^1(\mathbb{T})$ is such that the following limits exist at a point $x \in \mathbb{R}$, $f(x+)$, $f(x-)$, $f'(x+)$, $f'(x-)$ exist, where*

$$f(x+) := \lim_{t \to 0+} f(x+t), \qquad f(x-) := \lim_{t \to 0+} f(x-t),$$
$$f'(x+) := \lim_{t \to 0+} \frac{f(x+t) - f(x+)}{t}, \quad f'(x-) := \lim_{t \to 0+} \frac{f(x-) - f(x-t)}{t}.$$

Then

$$S_N(f, x) \to \frac{f(x+) + f(x-)}{2} \quad as \quad N \to \infty.$$

In particular, if f is continuous at x and $f'(x+)$, $f'(x-)$ exist, then

$$S_N(f, x) \to f(x) \quad as \quad N \to \infty.$$

Proof. Since $D_N(t) = D_N(-t)$, we have

$$\begin{aligned} S_N(f, x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{0} f(x-t) D_N(t) dt + \frac{1}{2\pi} \int_{0}^{\pi} f(x-t) D_N(t) dt \\ &= \frac{1}{2\pi} \int_{0}^{\pi} f(x+t) D_N(t) dt + \frac{1}{2\pi} \int_{0}^{\pi} f(x-t) D_N(t) dt \\ &= \frac{1}{2\pi} \int_{0}^{\pi} [f(x+t) + f(x-t)] D_N(t) dt \end{aligned}$$

and

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)dt = \frac{2}{2\pi}\int_0^{\pi} D_N(t)dt.$$

Hence, for any $\beta \in \mathbb{R}$,

$$S_N(f,x) - \beta = \frac{1}{2\pi}\int_0^{\pi}[f(x+t) + f(x-t) - 2\beta]D_N(t)dt.$$

Taking $\beta = \frac{f(x+)+f(x-)}{2}$, we have

$$f(x+t) + f(x-t) - 2\beta = [f(x+t) - f(x+)] - [f(x-) - f(x-t)].$$

Thus,

$$S_N(f,x) - \beta = A_N + B_N,$$

where

$$A_N = \frac{1}{2\pi}\int_0^{\pi}[f(x+t)-f(x+)]D_N(t)dt, \quad B_N = \frac{1}{2\pi}\int_0^{\pi}[f(x-)-f(x-t)]D_N(t)dt.$$

Note that

$$\begin{aligned} A &= \frac{1}{2\pi}\int_0^{\pi}[f(x+t) - f(x+)]D_N(t)dt \\ &= \frac{1}{2\pi}\int_0^{\pi}[f(x+t) - f(x+)]\frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}dt \\ &= \frac{1}{\pi}\int_0^{\pi}\left\{\frac{f(x+t) - f(x+)}{t}\right\}\left\{\frac{t/2}{\sin(\frac{t}{2})}\right\}\sin(N+\frac{1}{2})tdt \end{aligned}$$

Since $\frac{f(x+t)-f(x+)}{t} \to f'(x+)$ as $t \to 0+$, there exists $\delta > 0$ such that

$$\begin{aligned} 0 < t < \delta \quad &\Rightarrow \quad \left|\frac{f(x+t) - f(x+)}{t} - f'(x+)\right| \le 1 \\ &\Rightarrow \quad \left|\frac{f(x+t) - f(x+)}{t}\right| \le 1 + |f'(x+)|. \end{aligned}$$

Hence, the function

$$t \mapsto \left\{\frac{f(x+t) - f(x+)}{t}\right\}\left\{\frac{t/2}{\sin(\frac{t}{2})}\right\}, \quad t \ne 0,$$

is bounded on $(0,\delta)$, and hence, belongs to $L^1(\mathbb{T})$. Therefore, by Riemann Lebesgue lemma, $A_N \to 0$ as $N \to \infty$. Similarly, we see that, $B_N \to 0$ as $N \to \infty$. ▮

The following result, known as *localization lemma*, has already been observed in the course of the proof of Theorem 2.4.1. However, we repeat it proof here.

Lemma 2.4.4 (Localization lemma) *For* $0 < r < \pi$ *and* $x \in [-\pi, \pi]$,

$$\int_{r\le|t|\le\pi} f(x-t)D_N(t)dt \to 0 \quad \text{as} \quad N \to \infty.$$

Proof. Observe that

$$\int_{r\le|t|\le\pi} f(x-t)D_N(t)dt = \int_{r\le|t|\le\pi} g(x,t)\sin(N+1/2)tdt,$$

where

$$g(x,t) = \begin{cases} f(x-t)/\sin(t/2), & r \le |t| \le \pi, \\ 0, & |t| \le r. \end{cases}$$

Since $g(x,\cdot)$ is integrable, by Riemann Lebesgue lemma,

$$\int_{r\le|t|\le\pi} g(x,t)\sin(N+1/2)tdt \to 0 \quad \text{as} \quad N \to \infty.$$

Hence, the result. ∎

Proof of Corollary 2.4.2 using localization lemma. Suppose f is Lipschitz at a point $x \in [-\pi, \pi]$ with Lipschitz constant K_x, i.e., there exists $\delta > 0$ such that

$$|f(x) - f(x-t)| \le K_x|t| \quad \text{whenever} \quad |t| < \delta.$$

Now,

$$\begin{aligned} f(x) - S_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x) - f(x-t)]D_N(t)dt \\ &= \frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x) - f(x-t)]D_N(t)dt \\ &\quad + \frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x) - f(x-t)]D_N(t)dt \end{aligned}$$

By Lemma 2.4.4,

$$\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt \to 0 \quad \text{as} \quad N\to\infty.$$

Hence, for a given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$,

$$\left|\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt\right| < \varepsilon/2.$$

Also,

$$\left|\frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt\right| \le \frac{1}{2\pi}\int_{0\le|t|<\delta}|f(x)-f(x-t)|\,|D_N(t)|dt,$$

$$\frac{1}{2\pi}\int_{0\le|t|<\delta}|f(x)-f(x-t)|\,|D_N(t)|dt \le K_x\frac{1}{2\pi}\int_{0\le|t|<\delta}|t|\,|D_N(t)|dt,$$

$$|t|\,|D_N(t)| = |t|\left|\frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}\right| = 2\left|\frac{t/2}{\sin(\frac{t}{2})}\right| |\sin(N+\frac{1}{2})t| \le 2M,$$

where M is a bound for $\left|\frac{t/2}{\sin(\frac{t}{2})}\right|$| on $0<|t|\le\delta$. Hence,

$$\left|\frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt\right| \le \frac{4MK_x\delta}{2\pi} = \frac{2MK_x\delta}{\pi}.$$

We may take δ such that $\frac{2MK_x\delta}{\pi} < \varepsilon/2$. Hence,

$$\begin{aligned}|f(x)-S_N(f,x)| &\le \left|\frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt\right| \\ &\quad + \left|\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt\right| \\ &< \varepsilon.\end{aligned}$$

for all $N \ge N_0$. ∎

Exercise 2.4.5 *Suppose f is 2π-periodic and Hölder continuous at x, i.e., there exist $M>0$ and $\alpha>0$ such that $|f(x)-f(y)|\le M|x-y|^\alpha$ for all $y\in[-\pi,\pi]$. Then show that $S_N(f,x)\to f(x)$ as $N\to\infty$.*

Exercise 2.4.6 *Suppose f is 2π-periodic and Hölder continuous on $[-\pi,\pi]$, i.e., there exist $M>0$ and $\alpha>0$ such that $|f(x)-f(y)|\le M|x-y|^\alpha$ for all $x,y\in[-\pi,\pi]$. Then show that $S_N(f,x)\to f(x)$ uniformly.*

2.5 Cesàro summablity of Fourier series

Definition 2.5.1 A series $\sum_{n=1}^{\infty} a_n$ of complex numbers a_n is said to be **Cesàro summable** to a number s, if the sequence of arithmetic means of its partial sums converges. That is, if $s_n := \frac{1}{n}\sum_{n=1}^{\infty} a_n$ for $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ is **Cesàro summable** to s if

$$\sigma_n := \frac{1}{n}\sum_{n=1}^{\infty} s_n \to s \quad \text{as} \quad n \to \infty. \qquad \Diamond$$

In the case of Fourier series, we have the following definition.

Definition 2.5.2 Let $f \in L^1(\mathbb{T})$. Then the Fourier series of f is **Cesàro summable** to $f(x)$ at a point $x \in (-\pi, \pi)$ if

$$\sigma_N(f,x) := \frac{1}{N+1}\sum_{k=0}^{N} S_k(f,x) \to f(x) \quad \text{as} \quad N \to \infty. \qquad \Diamond$$

Since $S_k(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_k(t)dt$, we have

$$\sigma_N(f,x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(f,x) = \int_{-\pi}^{\pi} f(x-t)\left\{\frac{1}{N+1}\sum_{k=0}^{N} D_k(t)\right\}.$$

Thus,

$$\sigma_N(f,x) = \int_{-\pi}^{\pi} f(x-t)K_N(t)dt \quad \text{with} \quad K_N(t) := \frac{1}{N+1}\sum_{k=0}^{N} D_k(t).$$

Definition 2.5.3 The function

$$K_N(t) := \frac{1}{N+1}\sum_{k=0}^{N} D_k(t), \quad t \in [-\pi, \pi],$$

is called the **Fejér** kernel. $\Diamond$

Since $D_N(\cdot)$ is an even function and $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1$, $K_N(\cdot)$ is also even and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1.$$

Hence,

$$f(x) - \sigma_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x) - f(x-t)]K_N(t)dt.$$

We prove the following theorem.

Theorem 2.5.4 (Fejér's theorem) *If $f \in C(\mathbb{T})$, then the Fourier series of f is uniformly Cesàro summable on $[-\pi, \pi]$, that is,*

$$\sigma_N(f,x) := \frac{1}{N+1}\sum_{k=0}^{N} S_k(f,x) \to f(x) \quad \text{as} \quad N \to \infty$$

uniformly on $[-\pi, \pi]$.

For the proof of Theorem 2.5.4, we shall make use of the following lemma.

Lemma 2.5.5 *For $t \neq 0$,*

$$K_N(t) = \frac{1}{N+1}\frac{1-\cos(N+1)t}{1-\cos t} = \frac{1}{N+1}\frac{\sin^2[(N+1)t/2]}{\sin^2(t/2)}.$$

In particular, $K_N(t) \geq 0$ for all $t \in \mathbb{R}$. Further, for $0 < r \leq \pi$ and $r \leq |t| \leq \pi$,

$$K_N(t) \leq \frac{1}{N+1}\left(\frac{1}{\sin^2(r/2)}\right).$$

In particular, $K_N(t) \to 0$ uniformly for $r \leq |t| \leq \pi$.

Proof. Recall that

$$K_N(t) := \frac{1}{N+1}\sum_{k=0}^{N} D_k(t) \quad \text{where} \quad D_k(t) = \frac{\sin(k+1/2)t}{\sin t/2}$$

Hence,

$$(N+1)K_N(t) = \sum_{k=0}^{N}\frac{\sin(k+1/2)t}{\sin t/2} = \sum_{k=0}^{N}\frac{e^{i(k+1/2)t} - e^{-i(k+1/2)t}}{e^{it/2} - e^{-it/2}}$$

But,

$$\frac{e^{i(k+1/2)t} - e^{-i(k+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{e^{i(k+1)t} - e^{-ikt}}{e^{it} - 1},$$

$$\frac{e^{i(k+1/2)t} - e^{-i(k+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{e^{ikt} - e^{-i(k+1)t}}{1 - e^{-it}},$$

Therefore,

$$[e^{it} - 1](N+1)K_N(t) = \sum_{k=0}^{N}[e^{i(k+1)t} - e^{-ikt}], \tag{1}$$

$$[1 - e^{-it}](N+1)K_N(t) = \sum_{k=0}^{N}[e^{ikt} - e^{-i(k+1)t}] \tag{2}$$

Subtracting the (2) from (1),

$$[2\cos t - 2](N+1)K_N(t) = 2\sum_{k=0}^{N}[\cos(k+1)t - \cos kt] = 2[\cos(N+1)t - 1]$$

Thus,

$$K_N(t) = \frac{1}{N+1}\frac{\cos(N+1)t - 1}{\cos t - 1} = \frac{1}{N+1}\frac{\sin^2[(N+1)t/2]}{\sin^2(t/2)}.$$

Thus, we have proved (1). It is clear that $K_N(t)$ is even and non-negative. Now, for $0 < \delta \leq \pi$, $\sin^2(t/2) \geq \sin\delta/2$, so that for $\delta \leq |t| \leq \pi$,

$$K_N(t) = \frac{1}{N+1}\frac{\sin^2[(N+1)t/2]}{\sin^2(t/2)} \leq \frac{1}{N+1}\frac{1}{\sin^2(\delta/2)}.$$

Thus,

$$\int_\delta^\pi K_N(t)dt \leq \frac{\pi - \delta}{(N+1)\sin^2(\delta/2)} \to 0 \quad \text{as} \quad N \to \infty.$$

In particular, $K_N(t) \to 0$ uniformly for $r \leq |t| \leq \pi$. ∎

Proof of Theorem 2.5.4. Sine $K_N(t)$ is a non-negative function (see Lemma 2.5.5), we have

$$|f(x) - \sigma_N(f,x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x) - f(x-t)|K_N(t)dt.$$

Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta \in (0,\pi]$ such that

$$|f(x) - f(x-t)| < \varepsilon \quad \text{whenever} \quad |t| < \delta.$$

Hence,

$$\frac{1}{2\pi}\int_{|t|<\delta}|f(x)-f(x-t)|K_N(t)dt < \frac{\varepsilon}{2\pi}\int_{|t|<\delta}K_N(t)dt = \varepsilon.$$

Also, there exists $M>0$ such that $|f(y)|\le M$ for all $y\in[-\pi,\pi]$. Hence,

$$\frac{1}{2\pi}\int_{|t|\ge\delta}|f(x)-f(x-t)|K_N(t)dt \le \frac{2M}{2\pi}\int_{|t|\ge\delta}K_N(t)dt.$$

We have observed in Lemma 2.5.5 that $K_N(t)$ is an even function and $K_N(t)\to 0$ as $N\to\infty$ uniformly on $[\delta,\pi]$. Hence, there exists N_0 such that

$$\frac{1}{2\pi}\int_{|t|\ge\delta}|f(x)-f(x-t)|K_N(t)dt \le \frac{4M}{2\pi}\int_{\delta}^{\pi}K_N(t)dt < \varepsilon \quad \text{for all} \quad N\ge N_0.$$

Hence,

$$|f(x)-\sigma_N(f,x)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-f(x-t)|K_N(t)dt < 2\varepsilon$$

for all $N\ge N_0$. Note that N_0 is independent of the point x. Thus, we have proved that $S_N(f,x)\to f(x)$ as $N\to\infty$ uniformly for $x\in[-\pi,\pi]$. ∎

Corollary 2.5.6 *Let $f\in C(\mathbb{T})$ be such that $\hat{f}(n)=0$ for all $n\in\mathbb{Z}$. Then $f(x)=0$ for every $x\in\mathbb{R}$.*

In particular, if $f,g\in C(\mathbb{T})$ is such that $\hat{f}(n)=\hat{g}(n)$ for every $n\in\mathbb{Z}$, then $f=g$, and the map $f\mapsto(\hat{f}(n))$ is an injective continuous linear operator from $C(\mathbb{T})$ into $c_0(\mathbb{Z})$.

Proof. By assumption $\sigma_N(f,x)=0$ for every $N\in\mathbb{N}$. Hence, by Theorem 2.5.4, $f(x)=0$ for every $x\in\mathbb{R}$. ∎

The above corollary shows:

The Fourier coefficients of $f\in C(\mathbb{T})$ determines f uniquely.

Remark 2.5.7 *The proof of Theorem 2.5.4 reveals more:*

If f is bounded, peace-wise continuous, 2π-periodic, and continuous at x, then $\sigma_N(f,x)\to f(x)$ as $N\to\infty$. ◊

As a corollary to Theorem 2.5.4, we also have the following.

Theorem 2.5.8 *The space of all trigonometric polynomials is dense in $C(\mathbb{T})$ with respect to the uniform norm, and hence dense in $L^p(T)$ w.r.t. $\|\cdot\|_p$ for $1 \le p < \infty$.*

Proof. By Theorem 2.5.4, space of all trigonometric polynomials is dense in $C(\mathbb{T})$ with respect to the uniform norm $\|\cdot\|_\infty$. Hence, for any $f \in C(\mathbb{T})$, there exists a sequence (f_n) of trigonometric polynomials such that

$$\|f - f_n\|_p^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f_n(x)|^p dx \le \|f - f_n\|_\infty^p \to 0$$

as $n \to \infty$. ∎

Corollary 2.5.9 *If $f \in L^2(T)$ for some $1 \le p < \infty$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.*

In particular, if $f, g \in C(\mathbb{T})$ is such that $\hat{f}(n) = \hat{g}(n)$ for every $n \in \mathbb{Z}$, then $f = g$ a.e., and the map $f \mapsto (\hat{f}(n))$ is an injective continuous linear operator from $L^2(T)$ into $c_0(\mathbb{Z})$.

Proof. Suppose $f \in L^2(T)$ for some $1 \le p < \infty$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, i.e., $\langle f, u_n\rangle = 0$ for all $n \in \mathbb{Z}$, where $u_n(x) := e^{inx}$. Hence, $\langle f, g\rangle = 0$ for all trigonometric polynomials. By Theorem 2.5.8, there exists a sequence (f_n) of trigonometric polynomials such that $\|f - f_n\|_2 \to 0$ as $n \to \infty$. Hence,

$$\|f\|_2^2 = \langle f, f\rangle = \langle f, \lim_{n\to\infty} f_n\rangle = \lim_{n\to\infty}\langle f, f_n\rangle = 0.$$

Hence, $f = 0$ a.e. ∎

Theorem 2.5.10 *Suppose $f \in C^2(\mathbb{T})$. Then the Fourier series of f converges to f uniformly.*

Proof. By Theorem 2.1.9 and Theorem 2.1.7, the Fourier series of f converges uniformly, to say $g \in C(\mathbb{T})$. Hence, we also have $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$, and therefore, by Corollary 2.5.6, $g = f$. ∎

Exercise 2.5.11 *Suppose f is piecewise continuous and 2π-periodic. If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(x) = 0$ for all x at which f is continuous.*

Example 2.5.12 Let $f(x) = x^2$, $|x| \le \pi$. Note that

$$2\pi \hat{f}(0) = \int_{-\pi}^{\pi} x^2 dx = 2\frac{\pi^3}{3}$$

so that $\hat{f}(0) = \pi^2/3$, and for $n \neq 0$,

$$\begin{aligned}
2\pi \hat{f}(n) &= \int_{-\pi}^{\pi} x^2 e^{-inx} dx \\
&= \left[x^2 \frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{e^{-inx}}{-in} dx \\
&= \left[x^2 \frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \left[2x \frac{e^{-inx}}{(-in)^2}\right]_{-\pi}^{\pi} \\
&= -\left[2x \frac{e^{-inx}}{(-in)^2}\right]_{-\pi}^{\pi} = \left[2x \frac{e^{-inx}}{n^2}\right]_{-\pi}^{\pi} = 4\pi \frac{e^{inx}}{n^2} \\
&= 4\pi \frac{(-1)^n}{n^2}
\end{aligned}$$

Hence, for $n \neq 0$,

$$\hat{f}(n) = 2\frac{(-1)^n}{n^2}.$$

Thus,

$$x^2 \approx \frac{\pi^2}{3} + 2\sum_{n \neq 0}^{\infty} \frac{(-1)^n}{n^2} e^{inx} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Since the series of coefficients converges absolutely, we have

$$f(x) = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Taking $x = 0$,

$$0 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Taking $x = \pi$,

$$\pi^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

♢

Example 2.5.13 Let $f(x) = x$, $x \in [-\pi, \pi]$. Note that $\hat{f}(0) = 0$ and for $n \neq 0$,

$$2\pi\hat{f}(n) = \int_{-\pi}^{\pi} xe^{-inx}dx = \left[x\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in}dx = \left[x\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi}.$$

Thus,

$$2\pi\hat{f}(n) = \left[x\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} = \frac{1}{-in}[\pi e^{-in\pi} + \pi e^{in\pi}] = 2\pi\frac{e^{in\pi}}{-in}$$

so that

$$\hat{f}(n) = \frac{(-1)^n}{-in\pi} = \frac{(-1)^{n+1}}{in\pi}.$$

Hence,

$$x = \sum_{n\neq 0}^{\infty} \frac{(-1)^{n+1}}{in} e^{inx} = \sum_{n=1} \frac{(-1)^{n+1}}{in}[e^{inx} - e^{-inx}] = 2\sum_{n=1} \frac{(-1)^{n+1}}{n} \sin nx$$

Taking $x = \pi/2$ we obtain the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$ ♢

We also have another important theorem on convergence.

Theorem 2.5.14 (Jordan) *If $f \in L^1(\mathbb{T})$ is of bounded variation*[4]*, then for every $x \in \mathbb{R}$,*

$$S_N(f, x) \to \frac{1}{2}(f(x+) + f(x-) \quad as \quad N \to \infty.$$

[4]A function $f : [a, b] \to \mathbb{C}$ is of *bounded variation* if there exits $\kappa > 0$ such that for every partition $x_0 < x_1 < \cdots < x_n = b$, $\sum_{k=1}^{n} |f(x_{k+1}) - f(x_k| \leq \kappa$.

In particular, if $f \in AC(\mathbb{T})$, then

$$S_N(f,x) \to f(x) \quad as \quad N \to \infty$$

for every $x \in \mathbb{R}$.

If f is a function of bounded variation, then the following theorem gives rate of decay of Fourier coefficients.

Theorem 2.5.15 (Taibleson[5]) *Let $f \in L^1(\mathbb{T})$ be a function of bounded variation. Then*

$$|\hat{f}(n)| \leq \frac{V(f)}{n},$$

$V(f)$, the total variation of f in $[0, 2\pi]$.

Proof. Let $a_k := \frac{2k\pi}{n}$, $k = 0, 1, \ldots, n$, and define

$$g(x) := f(a_k) \quad \text{for} \quad x \in (a_{k-1}, a_k).$$

Then $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Hence,

$$2\pi \hat{f}(n) = \int_0^{2\pi} [f(x) - g(x)]e^{-inx}dx = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} [f(x) - g(x)]e^{-inx}dx$$

so that

$$2\pi|\hat{f}(n)| \leq \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |f(x) - g(x)|dx \leq \sum_{k=1}^{n} V_k(f)(a_k - a_{k-1}),$$

where $V_k(f)$ denotes the total variation of f in $[a_{k-1}, a_k]$. Thus,

$$2\pi|\hat{f}(n)| \leq \frac{2\pi}{n} \sum_{k=1}^{n} V_k(f) \leq \frac{V(f)}{n},$$

where $V(f)$ is the the total variation of f in $[0, 2\pi]$. ▮

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[5]Mitchel Taibleson, *Fourier coefficients of function of bounded variation*, Proc. AMS, Vol. 18 (1967)766-766

2.6 Uniqueness theorem

Theorem 2.6.1 (Uniqueness of Fourier series) *Let $f \in L^1(\mathbb{T})$. If $\hat{f}(n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ a.e. In particular, the map $\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ be defined by*

$$\mathcal{F}(f) = (\hat{f}(n)), \quad f \in L^1(\mathbb{T})$$

is injective.

Proof. Let

$$g(t) = \int_{-\pi}^{t} f(x)dx, \quad t \in [-\pi, \pi].$$

Then, by Fundamental Theorem of Lebesgue Integration (FTLI), g is absolutely continuous, g' exists a.e. and $g' = f$ a.e. Note that

$$g(t + 2\pi) - g(t) = \int_{t}^{t+2\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx = 2\pi\hat{f}(0) = 0.$$

Hence g is 2π-periodic. Let

$$h(t) = \int_{-\pi}^{t} g(x)dx, \quad t \in [-\pi, \pi].$$

Then for $t \in [-\pi, \pi]$,

$$h(t + 2\pi) - h(t) = \int_{t}^{t+2\pi} g(x)dx = \int_{-\pi}^{\pi} g(x)dx = 2\pi\hat{g}(0),$$

so that

$$G(t) = \int_{-\pi}^{t} [g(x) - g(0)]dx, \quad t \in [-\pi, \pi],$$

satisfies

$$G(t + 2\pi) - G(t) = \int_{t}^{t+2\pi} [g(x) - \hat{g}(0)dx = 2\pi[\hat{g}(0) - \hat{g}(0)] = 0,$$

and hence G is 2π-periodic. Thus, $G' = g - \hat{g}(0)$ a.e. and $G'' = g' = f$ a.e. Hence, by Theorem 2.1.9,

$$\hat{f}(n) = \widehat{G''}(n) = (in)^2\hat{G}(n) \quad \text{for all} \quad n \neq 0.$$

Therefore, $\hat{G}(n) = 0$ for all $n \neq 0$. Hence, by Theorem 2.5.10, $G(x) = \hat{G}(0)$. Hence $G'' = 0$, so that $f = 0$ a.e. ∎

2.7 Non-surjectivity

We show that the map $\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ be defined by

$$\mathcal{F}(f) = (\hat{f}(n)), \quad f \in L^1(\mathbb{T})$$

is not onto. For this we shall make use of the following lemma and a theorem, called *bounded inverse theorem* from Functional Analysis.

Recall that, for $N \in \mathbb{N}_0$,

$$D_N(t) := \sum_{k=-N}^{N} e^{ikt} = \frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}.$$

Lemma 2.7.1 *For every $N \in \mathbb{N}_0$,*

$$\int_{-\pi}^{\pi} |D_N(t)|dt \geq \frac{8}{\pi} \sum_{k=0}^{2N} \frac{1}{k+1}.$$

In particular, $(\|D_N\|_1)$ is unbounded.

Proof. Since $|\sin t| \leq |t|$ for $|t| \leq \pi/2$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_N(t)|dt &= 2\int_0^{\pi} \Big|\frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}\Big|dt \geq 4\int_0^{\pi} \Big|\frac{\sin(N+\frac{1}{2})t}{t}\Big|dt \\
&= 4\int_0^{(2N+1)\pi/2} \Big|\frac{\sin\tau}{\tau}\Big|d\tau \\
&= 4\sum_{k=1}^{2N+1} \int_{(k-1)\pi/2}^{k\pi/2} \Big|\frac{\sin\tau}{\tau}\Big|d\tau \\
&\geq \frac{8}{\pi}\sum_{k=1}^{2N+1} \int_{(k-1)\pi/2}^{k\pi/2} \frac{|\sin\tau|}{k}d\tau \\
&= \frac{8}{\pi}\sum_{k=1}^{2N+1} \frac{1}{k} = \frac{8}{\pi}\sum_{k=0}^{2N} \frac{1}{k+1}.
\end{aligned}$$

This,

$$\|D_N\|_1 := \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt \geq \frac{4}{\pi^2}\sum_{k=0}^{2N} \frac{1}{k+1},$$

and $(\|D_N\|_1)$ is unbounded. ∎

We may recall the following definition from Functional Analysis:

Definition 2.7.2 A linear operator $F : X \to Y$ between two normed linear spaces X and Y is called a **bounded linear operator** if image of every bounded set in X is bounded in Y, that is, for every $S \subseteq X$. $\Diamond$

You may verify the equivalence of the following statements:

- F is a bounded linear operator.
- F is continuous at 0.
- F is continuous at every $u \in X$.
- $\exists M > 0$ such that $\|F(u)u\| \le M\|u\|$ for all $u \in X$.

We shall make use of the following important theorem, which is usually proved in a first course in Functional Analysis[6].

Theorem 2.7.3 (Bounded inverse theorem) *Let X and Y be Banach spaces and $F : X \to Y$ be a bounded linear linear operator. If F is bijective, then its inverse $F^{-1} : Y \to X$ is also a bounded linear operator.*

Theorem 2.7.4 *The map $\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$, defined by*

$$\mathcal{F}(f) = (\hat{f}(n)), \quad f \in L^1(\mathbb{T}),$$

is not onto.

Proof. We know that $\mathcal{F}$ is a bounded linear operator. By Theorem 2.6.1, $\mathcal{F}$ is one-one. Assume for a moment that $\mathcal{F}$ is onto. Then by Theorem 2.7.3, its inverse is also a bounded linear operator.

Recall that $\widehat{D}_N(n) = 1$ for $-N \le n \le N$ so that

$$\|(\widehat{D}_N(n))\|_\infty := \sup \|\widehat{D}_N(n))| = 1 \quad \forall\, N \in \mathbb{N}_0.$$

Thus, $(\widehat{D}_N(n))$ is a bounded sequence in $c_0(\mathbb{Z})$. Hence, $(D_N) = (\mathcal{F}^{-1}(\widehat{D}_N(n)))$ is also a bounded sequence which is not true, by Lemma 2.7.1. ∎

[6]see: M.T. Nair, *Functional Analysis: A First Course*, PHI-Learning, New Delhi, 2002 (Fourth Print: 2014).

By the above theorem there exists $(c_n) \in c_0(\mathbb{Z})$ such that there is no $f \in L^1(\mathbb{T})$ satisfying $c_n = \hat{f}(n)$ for all $n \in \mathbb{Z}$. It is a natural urge to have an example of such a sequence $c_n)$. We shall show that (c_n) with

$$c_n = \begin{cases} 1/\log(n), & n \geq 2, \\ 0, & n \leq 1, \end{cases}$$

is such a sequence. For showing this, we shall use the first part of the following theorem.

Theorem 2.7.5 *Let $f \in L^1(\mathbb{T})$. The following are true.*

(i) *The the series $\sum_{n\neq 0} \frac{\hat{f}(n)}{n} e^{inx}$ converges at every $x \in \mathbb{R}$.*

(ii) *, $\int_a^b f(x)dx = \sum_{n\in\mathbb{Z}} \int_a^b \hat{f}(n)e^{inx}\,dx$.*

Proof. (i) Let

$$g(t) = \int_{-\pi}^{t} [f(x) - \hat{f}(0)]dx.$$

Then g is absolutely continuous and g is 2π-periodic. Hence, g is differentiable a.e., $g' \in L^1(\mathbb{T})$ and $g' = f - \hat{f}(0)$ a.e. Since $\hat{g'}(n) = in\hat{g}(n)$ for all $n \neq 0$, it follows that

$$\hat{g}(n) = \frac{\hat{f}(n)}{in}, \quad n \neq 0.$$

Since g is absolutely continuous, it is of bounded variation. hence, by Jordan's theorem (Theorem 2.5.14), Fourier series of g converges. Thus,

$$g(x) = \hat{g}(0) + \sum_{n\neq 0} \hat{g}(n)e^{inx} = \hat{g}(0) + \sum_{n\neq 0} \frac{\hat{f}(n)}{in}\, e^{inx}.$$

In particular, $\displaystyle\sum_{n\neq 0} \frac{\hat{f}(n)}{n} e^{inx}$ converges.

(ii) Let g be as above. Then

$$g(x) - g(y) = \sum_{n\neq 0} \frac{\hat{f}(n)}{in}\, [e^{inx} - e^{inx}] = \sum_{n\neq 0} \hat{f}(n) \int_y^x e^{int} dt.$$

But,

$$g(x)-g(y)=\int_y^x g'(t)dt=\int_y^x [f(t)-\hat{f}(0)]dt=\int_y^x f(t)dt-\hat{f}(0)(x-y).$$

Hence,

$$\int_y^x f(t)dt=\hat{f}(0)(x-y)+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_a^b \hat{f}(n)e^{int}\,dt=\sum_{n\in\mathbb{Z}}\int_a^b \hat{f}(n)e^{int}\,dt.$$

This competes the theorem. ∎

Corollary 2.7.6 *Let $(c_n)\in c_0(\mathbb{Z})$ be such that $\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{c_n}{n}$ diverges. Then there is no $f\in L^1(\mathbb{T})$ satisfying $c_n=\hat{f}(n)$ for all $n\in\mathbb{N}$.*

Proof. We shall prove it by the method of contradiction: Suppose there exists $f\in L^1(\mathbb{T})$ such that $c_n=\hat{f}(n)$ for all $n\in\mathbb{N}$. Then, by Theorem 2.7.5 (i), the series $\sum_{n\in\mathbb{Z}\setminus\{0\}}^{\infty}\frac{\hat{f}(n)}{n}e^{inx}$ converges. In particular, taking $x=0$, $\sum_{n\in\mathbb{Z}\setminus\{0\}}^{\infty}\frac{\hat{f}(n)}{n}$ converges, which is not true by our assumption. ∎

To illustrate the above corollary, we may take

$$c_n=\begin{cases}1/\log(n), & n\geq 2,\\ 0, & n\leq 1,\end{cases}$$

Note that $(c_n)\in c_0(\mathbb{Z})$ and $\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{c_n}{n}=\sum_{n=2}^{\infty}\frac{1}{n\log n}$ diverges.

2.8 Divergence of Fourier series

We will show:

Theorem 2.8.1 *There exists $f\in C(\mathbb{T})$ such that $\{S_N(f,0)\}$ is unbounded; in particular, the Fourier series of f does not converge to f at 0.*

We shall also show:

Theorem 2.8.2 *There exists a dense subset $\mathcal{D}$ of $C(\mathbb{T})$ such that for each $f\in\mathcal{D}$, the Fourier series of f diverges at 0.*

For proving Theorem 2.8.1, we shall make use of an important theorem in Functional Analysis, namely, *Uniform Boundedness Principle*, which is stated as follows[7].

Theorem 2.8.3 (Uniform Boundedness Principle) *Let (F_n) be a sequence of continuous linear transformations from a Banach space X to a normed linear space Y. If for each $u \in X$, the set $\{\|F_n u\| : n \in \mathbb{N}\}$ is bounded, then there exists $M > 0$ such that*

$$\sup_{\|u\| \le 1} \|F_n u\| \le M \quad \forall\, n \in \mathbb{N}.$$

For $f \in C(\mathbb{T})$, let

$$\varphi_N(f) := S_N(f, 0), \quad f \in C(\mathbb{T}).$$

Recall that

$$S_N(f,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-t) D_N(t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(t)dt.$$

Hence,

$$|\varphi_N(f)| \le \|f\|_\infty \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(t)|dt \right) = \|f\|_\infty \|D_N\|_1.$$

Hence, each $\varphi_N : C(\mathbb{T}) \to \mathbb{C}$ is a bounded linear operator, and

$$\sup_{\|u\|_\infty \le 1} |\varphi_N(f)| \le \|D_N\|_1.$$

We shall prove:

Theorem 2.8.4

$$\sup_{\|u\|_\infty \le 1} |\varphi_N(f)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(t)|dt.$$

Proof. It is enough to find a sequence (f_n) in $C(\mathbb{T})$ such that

$$\|f_n\|_\infty \le 1 \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad |\varphi_N(f_n)| \to \|D_N\|_1 \quad \text{as} \quad n \to \infty.$$

[7] For its proof, see M.T. Nair, *Functional Analysis: A First Course*, PHI-Learning, New Delhi, 2002 (Fourth Print: 2014).

Note that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(t)|dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(t)g_N(t)dt,$$
where
$$g_N(t) = \begin{cases} \frac{|D_N(t)|}{D_N(t)}, & D_N(t)\neq 0,\\ 1, & D_N(t)=0 \end{cases} = \begin{cases} 1, & D_N(t)\geq 0,\\ -1, & D_N(t)=0. \end{cases}$$
Since D_N cn vanish only at a finite number of points in $[-\pi,\pi]$, g_N is a step function, and hence it can be approximated pointwise by a sequence (f_n) in $C(\mathbb{T})$ such that $\|f_n\|_\infty \leq 1$. Therefore, by DCT,
$$\varphi_N(f_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(t)f_n(t)dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(t)g_N(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(t)|dt.$$
Thus, the proof is complete. ∎

Proof of Theorem 2.8.1. Recall from Theorem 2.8.4 and Lemma 2.7.1 that
$$\sup_{\|u\|_\infty\leq 1}|\varphi_N(f)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(t)|dt \geq \frac{4}{\pi^2}\sum_{k=0}^{2N}\frac{1}{k+1}.$$
Hence, by Theorem 2.8.4, there does not exist $M > 0$ such that $\sup_{\|u\|_\infty\leq 1}|\varphi_N(f)| \leq M$. Hence, by UBP (Theorem 2.8.3), there exists $f \in C(\mathbb{T})$ such that $\{|\varphi_n(f)| : n \in \mathbb{N}\}$ is unbounded. Thus, there exists $f \in C(\mathbb{T})$ such that Fourier series of f diverges at 0. ∎

For proving Theorem 2.8.2, we shall use the following result from Functional Analysis:

Proposition 2.8.5 *A proper subspace of a normed linear space cannot contain any open ball.*

Proof of Theorem 2.8.2. Let
$$X_0 := \{f \in C(\mathbb{T}) : \{S_N(f,0)\} \text{ converges}\}.$$
Then we see that X_0 is a subspace of $C(\mathbb{T})$, and by Theorem 2.8.1, it is a proper subspace. By Proposition 2.8.5, X_0 contain any open ball. Hence, its compliment,
$$\mathcal{D} := C(\mathbb{T})\setminus X_0 = \{f \in C(\mathbb{T}) : \{S_N(f,0)\} \text{ does not converge}\}$$
is dense in $C(\mathbb{T})$. Thus, for every f in the dense subset $\mathcal{D}$ of $C(\mathbb{T})$, the Fourier series of f diverges at 0. ∎

2.9 Convolution

Definition 2.9.1 The convolution of $f, g \in L^1(\mathbb{T})$ is defined by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y)dy, \quad \text{a.a.}\, x \in [-\pi, \pi]. \qquad \Diamond$$

It is to be observed that the integral in the above definition is well defined for almost all $x \in [-\pi, \pi]$: Given $f, g \in L^1(\mathbb{T})$, it can be shown that

$$(x, y) \mapsto f(x-y)g(y)$$

is measurable on $\mathbb{R} \times \mathbb{R}$. Further,

$$\begin{aligned}
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x-y)g(y)|dydx &= \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} |f(x-y)|\, dx \right] |g(y)|dy \\
&= \int_{-\pi}^{\pi} 2\pi \|f\|_1 \, |g(y)|dy \\
&= (2\pi)^2 \|f\|_1 \|g\|_1.
\end{aligned}$$

Hence, $\int_{-\pi}^{\pi} |f(x-y)g(y)|dy < \infty$ for a.a. $x \in [-\pi, \pi]$ so that the integral

$$\int_{-\pi}^{\pi} f(x-y)g(y)dy$$

is well-defined almost everywhere.

Theorem 2.9.2 *For $f, g \in L^1(\mathbb{T})$,*

(i) $f * g \in L^1(\mathbb{T})$,

(ii) $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$,

(iii) $f * g = g * f$,

(iv) $\widehat{f * g}(n) = \hat{g}(n\hat{f}(n)$ *for all* $n \in \mathbb{Z}$.

Also, for $f, g, h \in L^1(\mathbb{T})$,

$$(f * g) * h = f * (g * h).$$

Proof. (i) We have already observed that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x-y)|\,|g(y)|dydx = (2\pi)^2\|f\|_1\|g\|_1.$$

Thus, $f * g \in L^1(\mathbb{T})$.

(ii) From the above,

$$\|f*g\|_1 := \frac{1}{2\pi}\int_{-\pi}^{\pi} |(f*g)(x)|dx \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)g(y)|dy\Big)dx \le \|f\|_1\|g\|_1.$$

(iii) Note that

$$\int_{-\pi}^{\pi} f(x-y)g(y)dy = \int_{x-\pi}^{x+\pi} f(\tau)g(x-\tau)dy = \int_{-\pi}^{\pi} f(\tau)g(x-\tau)dy.$$

(iv) Note that

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f*g)(x)e^{-inx}dx,$$

$$(f*g)(x)e^{-inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)e^{-in(x-y)}e^{-iny}dy,$$

$$\begin{aligned}
\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}dx &= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} f(x-y)g(y)e^{-in(x-y)}e^{-iny}dy\right]dx \\
&= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} f(x-y)e^{-in(x-y)}dx\right]g(y)e^{-iny}dy \\
&= \int_{-\pi}^{\pi} \hat{f}(n)g(y)e^{-iny}dy \\
&= (2\pi)\hat{f}(n)\hat{g}(n).
\end{aligned}$$

Thus,

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}dx = \hat{f}(n)\hat{g}(n).$$

Next, let $f, g, h \in L^1(\mathbb{T})$. Then,

$$\begin{aligned}
[(f*g)*h](x) &= \int_{-\pi}^{\pi}(f*g)(x-y)h(y)dy \\
&= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} f(x-y-t)g(t)dt\right]h(y)dy \\
&= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} f(x-\tau)g(\tau-y)d\tau\right]h(y)dy \\
&= \int_{-\pi}^{\pi} f(x-\tau)\left[\int_{-\pi}^{\pi} g(\tau-y)h(y)dy\right]d\tau \\
&= 2\pi\int_{-\pi}^{\pi} f(x-\tau)(g*h)(\tau)d\tau \\
&= (2\pi)^2[f*(g*h)](x).
\end{aligned}$$

Thus, the proof is complete. ∎

From the above theorem, we see that

Theorem 2.9.3 *With respect to convolution as multiplication, $L^1(\mathbb{T})$ is a commutative Banach algebra.*

Theorem 2.9.4 *The Banach algebra $L^1(\mathbb{T})$ does not have a multiplicative identity:*

Proof. Suppose there exists $\varphi \in L^1(\mathbb{T})$ such that $f * \varphi = f$ for all $f \in L^1(\mathbb{T})$. Then $\hat{f}(n)\hat{\varphi}(n) = \hat{f}(n)$ for all $f \in L^1(\mathbb{T})$. Hence, $\hat{\varphi}(n) = 1$ whenever $\hat{\varphi}(n) \neq 0$. But, $\hat{\varphi}(n) \to 0$ as $|n| \to \infty$. Hence, there exists $N \in \mathbb{N}$ such that $\hat{\varphi}(n) = 0$ for all $n \geq N$. Let $f \in L^1(\mathbb{T})$ be such that $\hat{f}(n) \neq 0$ for some $n \geq N$. Then for such n, we obtain

$$0 = \hat{f}(n)\hat{\varphi}(n) = \hat{f}(n) \neq 0,$$

which is a contradiction. ∎

- There exists (φ_n) in $L^1(\mathbb{T})$ such that $\|f * \varphi_n - f\|_1 \to 0$.

In fact, we have the following.

Theorem 2.9.5 *Let K_n be the Fejér kernel. Then, for every $f \in L^1(\mathbb{T})$,*

$$\|f * K_n - f\|_1 \to 0 \quad \textit{as} \quad N \to \infty.$$

Proof. Recall

$$\|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)|dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)|dt = 1$$

and if $g \in C(\mathbb{T})$, then $(g * K_n)$ converges to g uniformly on $[-\pi, \pi]$. In particular, $\|g * K_n - g\|_1 \to 0$. Let $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$ be given. Let $g \in C(\mathbb{T})$ be such that $\|f - g\|_1 < \varepsilon$, and let $N \in \mathbb{N}$ be such that $\|g * \varphi_n - g\|_1 < \varepsilon$ for all $n \geq N$. Then, for $n \geq N$, we have

$$\begin{aligned}
\|f * K_n - f\|_1 &\leq \|f * K_n - g * K_n\|_1 + \|g * K_n - g\|_1 + \|g - f\|_1 \\
&\leq \|(f - g) * K_n\|_1 + \varepsilon + \varepsilon \\
&\leq \|(f - g) * K_n\|_1 + 2\varepsilon \\
&\leq \|f - g\|_1 \|K_n\|_1 + 2\varepsilon \\
&\leq 3\varepsilon.
\end{aligned}$$

Thus, $\|f * K_n - f\|_1 \to 0$ as $N \to \infty$. ∎

Definition 2.9.6 A sequence (φ_n) in $L^1(\mathbb{T})$ is said to be an **approximate identity** in $L^1(\mathbb{T})$ if $\|f * \varphi_n - f\|_1 \to 0$. ♢

By Theorem 2.9.5, (K_n) is an approximate identity in $L^1(\mathbb{T})$.

2.10 L^2-Theory

Recall that $L^2(\mathbb{T})$ is a Hilbert space w.r.t. the inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx, \quad f, g \in L^2(\mathbb{T}),$$

and the corresponding norm is given by

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{1/2}, \quad f \in L^2(\mathbb{T}).$$

Recall that $E := \{u_n : n \in \mathbb{Z}\}$ with $u_n(t) := e^{int}$ is an orthonormal set in $L^2(\mathbb{T})$. That is, for $n, m \in \mathbb{Z}$,

$$\langle u_n, u_m\rangle = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Using this observation,, the results in the following theorem be proved easily (Exercise).

Theorem 2.10.1 *Let $f \in L^2(\mathbb{T})$. Then the following are true.*

(i) $S_N(f) = \sum_{n=-N}^{N} \langle f, u_n \rangle u_n$.

(ii) $\|S_N(f)\|_2^2 = \sum_{n=-N}^{N} |\hat{f}(n)|^2$.

(iii) $\|f - S_N(f)\|_2^2 = \|f\|_2^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2$.

(iv) **(Bessel's inequality)**: $\sum_{n=-N}^{N} |\hat{f}(n)|^2 \leq \|f\|_2^2 \quad \forall N \in \mathbb{N}$.

(v) $\hat{f}(n) \to 0$ *as* $|n| \to \infty$.

(vi) $\langle f - S_N(f), u_n \rangle = 0 \quad \forall |n| \leq N\}$.

Theorem 2.10.2 *For every $f \in L^2(\mathbb{T})$,*

$$\|f - S_N(f)\|_2 \leq \|f - g\| \quad \forall g \in \mathcal{X}_N := \mathit{span}\{u_n : n \in \mathbb{Z},\ |n| \leq N\}.$$

Proof. Let $g \in \mathcal{X}_N := \text{span}\{u_n : n \in \mathbb{Z},\ |n| \leq N\}$. Then $S_N(f) - g \in \mathcal{T}_N$. Hence, by Theorem 2.10.1(vi), $\langle f - S_N(f), S_N(f) - g \rangle = 0$ so that

$$\|f - g\|_2^2 = \|f - S_N(f)\|_2^2 + \|S_N(f) - g\|_2^2.$$

Thus,

$$\|f - g\|_2^2 \geq \|f - S_N(f)\|_2^2$$

which proves the result. ∎

Remark 2.10.3 In view of Theorem 2.10.2,

$$\|f - S_N(f)\|_2 = \inf\{\|f - g\|_2 : g \in \mathcal{X}_N\},$$

where $\mathcal{X}_N := \text{span}\{u_n : n \in \mathbb{Z},\ |n| \leq N\}$. In other words, $S_N(f)$ is the (unique!) *best approximation* of f from $\mathcal{X}_N$. Uniqueness is due to the following: Suppose φ be in $\mathcal{X}_N$ such that

$$\|f - \varphi\|_2 = \inf\{\|f - g\|_2 : g \in \mathcal{X}_N\}.$$

Then,

$$\|f - \varphi\|_2^2 = \|f - S_N(f)\|_2^2 + \|S_N(f) - \varphi\|_2^2$$

since $\langle f - S_N(f),\ S_N(f) - \varphi \rangle = 0$ so that we obtain $\|S_N(f) - \varphi\|_2 = 0$. $\Diamond$

We know that if $f \in C^1(\mathbb{T})$, then the Fourier series of f converges absolutely, and uniformly to f. Now, we give another proof for the same with some additional information on the rate of convergence of the series w.r.t. the L^2-norm, using Bessel's inequality.

Theorem 2.10.4 *If $f \in C^1(\mathbb{T})$, then the Fourier series of f converges absolutely, and uniformly to f. Further,*

$$\|f - S_N(f, \cdot)\|_\infty \le \frac{\|f'\|_\infty}{\sqrt{N}}.$$

Proof. Let $f \in C^1(\mathbb{T})$. Recall that $\hat{f'}(n) = in\hat{f}(n)$. Hence,

$$\sum_{n\neq 0} |\hat{f}(n)| = \sum_{n\neq 0} \frac{1}{n} |in\hat{f}(n)| = \sum_{n\neq 0} \frac{1}{n} |\hat{f'}(n)| \le \Big(\sum_{n\neq 0} \frac{1}{n^2}\Big)^{1/2} \|\hat{f'}\|_2 = \frac{\pi}{\sqrt{3}} \|\hat{f'}\|_2.$$

Hence the Fourier series of f converges absolutely and uniformly to a continuous function, say $g \in C(\mathbb{T})$. Since $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$, we obtain $g = f$. Also, using Bessel's inequality, for all $x \in \mathbb{R}$, we have

$$|f(x) - S_N(f, x)| \le \sum_{|n|>N} |\hat{f}(n)| = \sum_{|n|\ge N} \frac{1}{n} | \, |\hat{f'}(n)| \le \Big(\sum_{|n|>N} \frac{1}{n^2}\Big)^{1/2} \|\hat{f'}\|_2 \le \frac{\|\hat{f'}\|_2}{\sqrt{N}}.$$

In particular, $\|f - S_N(f, \cdot)\|_\infty \le \frac{\|f'\|_\infty}{\sqrt{N}}$. ∎

Recall the following definition from linear algebra[8]:

Definition 2.10.5 A subset E of an inner product space V is an orthonormal basis, if E is an orthonormal set[9], and if E is a maximal with respect to the set inclusion. ◊

We observe the following result.

Theorem 2.10.6 *Let E be orthonormal set E in an inner product space. Then E is an orthonormal basis iff $E^\perp = \{0\}$.*

[8] M.T. Nair and A. Singh, *Linear Algebra*, Springer, 2018

[9] i.e., $\langle v, v\rangle = 1$ for every $v \in V$, and $\langle u, v\rangle = 0$ for every $u, v \in V$ with $u \neq v$

Proof. Suppose E is an orthonormal basis. If $E^\perp \neq \{0\}$ and if $u \in E^\perp$, then $v = u/\|u\|$ also belongs to $E^\perp$, and $E \cup \{v\}$ would be an orthonromal set, contrdicting the maximality of E.

Conversely, if $E^\perp = \{0\}$, and if E is not an orthonormal basis, then there would be v with $\|v\| = 1$ and $E \cup \{v\}$ would be an orthonormal set; but then, $v \perp u$ for every $u \in E$, which contrdicts $R^\perp$. ∎

Theorem 2.10.7 *The set* $E := \{u_n : n \in \mathbb{Z}\}$ *with* $u_n(t) := e^{int}$ *is an orthonormal basis of* $L^2(\mathbb{T})$.

Proof. Clearly, E is an orthonormal set. By Corollary 2.5.9, $E^\perp = \{0\}$ so that E is an orthonormal basis of $L^2(\mathbb{T})$. ∎

Theorem 2.10.8 *Let* $f \in L^2(\mathbb{T})$. *Then we have the following:*

(i) $\text{span}\{u_n : n \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{T})$, *i.e., a maximal orthonormal set in* $L^2(\mathbb{T})$.

(ii) **(Fourier expansion)** $f = \sum_{n\in\mathbb{Z}} \hat{f}(n) u_n$ *in* $L^2(T)$.

(iii) **(Parseval's formula)** $\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.

Proof. (i) It can be seen that $\langle f, u_n\rangle = 0$ for all $n \in \mathbb{Z}$ implies $f = 0$ in $L^2(T)$. Hence, $\text{span}\{u_n : n \in \mathbb{Z}\}$ is a maximal orthonormal set in $L^2(T)$.

(ii) We observe that, for $n > m$,

$$\|S_n(f) - S_m(f)\|^2 \le \sum_{n\le|k|\le m} |\hat{f}(n)|^2.$$

Hence, $\{S_n(f)\}$ is a Cauchy sequence in $L^2(T)$. Therefore, it converges to some $g \in L^2(\mathbb{T})$. It can be seen that $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. Therefore, $g = f$ in $L^2(\mathbb{T})$.

(iii) Follows from (ii). ∎

Remark 2.10.9 We may recall that $\ell^2(\mathbb{Z})$ is the space of all sequences $(a_n)_{n\in\mathbb{Z}})$ of complex numbers such that $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$. That is,

$$\ell^2(\mathbb{Z}) := \Big\{(a_n) : a_n \in \mathbb{C},\, n \in \mathbb{Z},\, \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\Big\}.$$

It is also known that $\ell^2(\mathbb{Z})$ is a Hilbert space with inner product,

$$\langle (a_n),(b_n)\rangle_{\ell^2(\mathbb{Z})} := \sum_{n=-\infty}^{\infty} a_n \bar{b}_n,$$

with corresponding norm

$$\|(a_n)\|_{\ell^2(\mathbb{Z})} := \sqrt{\langle (a_n),(a_n)\rangle}.$$

Thus, Parseval's formula tells that for $f \in L^2(\mathbb{T})$,

$$\|f\|_{L^2(\mathbb{T})} = \|(\hat{f}(n))\|_{\ell^2(\mathbb{Z})}.$$

In fact, the map $f \mapsto (\hat{f}(n))$ is a surjective linear isometry from $L^2(\mathbb{Z})$ onto $\ell^2(\mathbb{Z})$. Indeed, this map is injective. To see that it is surjective, for any $(a_n) \in \ell^2(\mathbb{Z})$, tt can shown (Exercise) that this series converges in $L^2(\mathbb{T})$, say to a function f so that $f := \sum_{-\infty}^{\infty} a_n u_n$ and $\hat{f}(n) = \langle f, u_n\rangle = a_n$. $\Diamond$

2.11 Appendix: Proof of Jordan theorem

Lemma 2.11.1 *If $\varphi : [a,b] \to \mathbb{R}$ is of bounded variation and $g : [a,b] \to \mathbb{R}$ is continuous, then*

$$\begin{aligned}\int_a^b f(x)g(x)dx &= \left[f(x)\int_a^x g(s)ds\right]_a^b - \int_a^b \Big(\int_a^x g(s)ds\Big)dg(x)\\ &= \left[f(b-)\int_a^b g(s)ds\right] - \int_a^b \Big(\int_a^x g(s)ds\Big)dg(x)\end{aligned}$$

Proof of Jordan's theorem. Let $f \in L^1(\mathbb{T})$ be of bounded variation and let $L := (f(x+) + f(x-))/2$. Then we have

$$S_N(f,x) - L = \frac{1}{\pi}\int_0^\pi \Big[\frac{f(x+t)+f(x-t)}{2} - L\Big]\frac{\sin(N+1/2)t}{\sin t/2}dt.$$

It can be shown that $\lim_{N\to\infty}[S_N(f,x) - L] = 0$ iff for some $0 < \delta < \pi$,

$$\lim_{N\to\infty}\frac{1}{\pi}\int_0^\delta \Big[\frac{f(x+t)+f(x-t)}{2} - L\Big]\frac{\sin(N+1/2)t}{t/2}dt = 0$$

$\iff$

$$\lim_{N\to\infty}\frac{1}{\pi}\int_0^{\delta}\Big[\frac{f(x+t)+f(x-t)}{2}-L\Big]\frac{\sin(N+1/2)t}{t}dt=0$$

Let

$$F(t):=\frac{f(x+t)+f(x-t)}{2}-L$$

Hence, $\lim_{N\to\infty}[S_N(f,x)-L]=0$ iff

$$\lim_{N\to\infty}\frac{1}{\pi}\int_0^{\delta}F(t)\frac{\sin(N+1/2)t}{t}dt=0$$

Since F is of bounded variation we have

$$\begin{aligned}\int_0^{\delta}F(t)\frac{\sin(N+1/2)t}{t}dt &= \left[F(t)\int_0^{t}\frac{\sin(N+1/2)s}{t}ds\right]_0^{\delta}\\ &\quad -\int_0^{\delta}\Big(\int_0^{t}\frac{\sin(N+1/2)s}{t}ds\Big)dF(t)\\ &= F(\delta-)\int_0^{\delta}\frac{\sin(N+1/2)s}{t}ds\\ &\quad -\int_0^{\delta}\Big(\int_0^{t}\frac{\sin(N+1/2)s}{t}ds\Big)dF(t)\end{aligned}$$

It can be seen that $\lim_{N\to\infty}\int_0^{\delta}\frac{\sin(N+1/2)s}{t}ds=\frac{\pi}{2}$. Hence,

$$\lim_{N\to\infty}\int_0^{\delta}F(t)\frac{\sin(N+1/2)t}{t}dt=F(\delta-)\frac{\pi}{2}-\int_0^{\delta}\frac{\pi}{2}dF(t).$$

Since $F(0+)=0$, we obtain

$$\lim_{N\to\infty}\int_0^{\delta}F(t)\frac{\sin(N+1/2)t}{t}dt=\frac{\pi}{2}\Big[F(\delta-)-(F(\delta-)-F(0+))\Big]=0.$$

Therefore, $\lim_{N\to\infty}[S_N(f,x)-L]=0$. ∎

3

Fourier transform

3.1 Basic properties

Recall that $1 \leq p < \infty$, $L^p(\mathbb{R}^d)$ is the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $\int_{\mathbb{R}^d} |f(x)|^p dx < \infty$, and the $L^p(\mathbb{R}^d)$ with norm

$$\|f\|_p := \Big(\int_{\mathbb{R}^d} |f(x)|^p dx \Big)^{1/p}, \quad f \in L^1(\mathbb{R}^d),$$

is a Banach space.

Definition 3.1.1 Let $d \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^d)$. The **Fourier transform** of f is the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx, \quad \xi \in \mathbb{R}^d,$$

where $x.\xi$ denotes the *dot product* of $x, \xi \in \mathbb{R}^d$.

$\Diamond$

Proof of the following theorem is easy, and hence left as an exercise.

Theorem 3.1.2 *The map $f \mapsto \hat{f}$ is a bounded linear operator*[1] *from $L^1(\mathbb{R}^d)$ to $B(\mathbb{R}^d)$.*

Example 3.1.3 Let $d = 1$ and $f = \chi_{(0,1)}$. Then

$$\hat{f}(\xi) = \int_{-1}^{1} e^{-ix\xi} dx = \begin{cases} 2, & \xi = 0, \\ 2\frac{\sin \xi}{\xi}, & \xi \neq 0. \end{cases}$$

$\Diamond$

[1] Here, $B(\mathbb{R}^d)$ denotes the space of all bounded complex valued functions with sup-norm. This space is a Banach space.

Example 3.1.4 Let $d = 1$ and $f(x) = e^{-x^2/2}$. Then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-(x^2/2+ix\xi)}dx = e^{-\xi^2/2}\int_{-\infty}^{\infty} e^{-(x+i\xi)^2/2}dx = e^{-\xi^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}\, e^{-\xi^2/2}.$$

More generally, if $f(x) = e^{-|x|^2/2}$, $x \in \mathbb{R}^d$, then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-(|x|^2/2+ix\cdot\xi)}dx = (2\pi)^{d/2}\, e^{-|\xi|^2/2}.$$

$\Diamond$

In the above examples we see that $\hat{f}$ is uniformly continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. This is, in fact, true for any $f \in L^1(\mathbb{R}^d)$.

Theorem 3.1.5 *Let $f \in L^1(\mathbb{R}^d)$. Then $\hat{f}$ is uniformly continuous.*

Proof. For $\xi, h \in \mathbb{R}^d$, we have

$$\hat{f}(\xi+h) - \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)[e^{-ix\cdot(\xi+h)} - e^{-ix.\xi}]dx = \int_{\mathbb{R}^d} f(x)e^{-ix.\xi}[e^{-ix\cdot h} - 1]dx$$

Thus,

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| \le \int_{\mathbb{R}^d} |f(x)|\,|e^{-ix\cdot h} - 1|dx.$$

Note that

$$|f(x)|\,|e^{-ix\cdot h} - 1| \to 0 \quad \text{as} \quad |h| \to 0,$$
$$|f(x)|\,|e^{-ix\cdot h} - 1| \le 2|f(x)| \quad \forall\, h \in \mathbb{R}^d$$

with $f \in L^1(\mathbb{R}^d)$. Hence, by DCT,

$$\int_{\mathbb{R}^d} |f(x)|\,|e^{-ix\cdot h} - 1|dx \to 0$$

as $|h| \to 0$. Hence, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| \le \int_{\mathbb{R}^d} |f(x)|\,|e^{-ix\cdot h} - 1|dx < \varepsilon$$

for every h with $|h| < \delta$. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| < \varepsilon$$

for every h with $|h| < \delta$ and for every $\xi \in \mathbb{R}^d$. Thus, $\hat{f}$ is uniformly continuous. ∎

Combining Theorems 3.1.2 and 3.1.5, we have

Theorem 3.1.6 *The map $f \mapsto \hat{f}$ is a bounded linear operator*[2] *from $L^1(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$ with norm atmost* 1.

In fact, we have

Theorem 3.1.7 (Riemann Lebesgue lemma) *Let $f \in L^1(\mathbb{R}^d)$. Then*

$$\hat{f}(\xi) \to 0 \quad as \quad |\xi| \to \infty.$$

In particular[3]*, $\hat{f} \in C_0(\mathbb{R}^d)$ for every $f \in L^1(\mathbb{R}^d)$.*

For its proof we shall make use of the following lemma.

Lemma 3.1.8 *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$. Then*

$$\lim_{y \to 0} \int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx = 0.$$

In particular, if we define $f_y(x) = f(x-y)$ for $x \in \mathbb{R}^d$, then the map $y \mapsto f_y$ from $\mathbb{R}^d$ to $L^p(\mathbb{R}^d)$ is continuous.

Proof. For $y \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{C}$, let $f_y(x) := f(x-y)$, $x \in \mathbb{R}^d$. Thus, we have to prove that $\|f - f_y\|_p \to 0$ as $y \to 0$ for $f \in L^p(\mathbb{R}^d)$. So, let $f \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$. By denseness of $C_c(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$, there exists $g \in C_c(\mathbb{R}^d)$ such that $\|f - g\|_p < \varepsilon$. Then, we also have $\|f_y - g_y\|_p < \varepsilon$. Hence,

$$\|f - f_y\|_p \leq \|f - g\|_p + \|g - g_y\|_p + \|g_y - f_y\|_p < 2\varepsilon + \|g - g_y\|_p.$$

Now, by the uniform continuity of g, there exists $\delta > 0$ such that

$$|g(x) - g(x-y)| < \varepsilon$$

for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with $|y| < \delta$. Also, there exists a parallelepiped $K \subset \mathbb{R}^d$ such that $g(x) = 0$, $g(x-y) = 0$ for every $x \notin K$ and $|y| < \delta$. Hence,

$$\|g - g_y\|_p^p = \int_K |g(x) - g(x-y)|^p dx \leq \varepsilon^p m(K)$$

[2] $C_b(\mathbb{R}^d)$ denotes the space of all bounded, continuous complex valued functions on $\mathbb{R}$ with sup-norm. This space is a Banach space.

[3] $C_0(\mathbb{R}^d)$ denotes the space of all continuous functions $h : \mathbb{R}^d \to \mathbb{C}$ such that $|h(x)| \to 0$ as $|x| \to \infty$. This space is a Banach space.

for every y with $|y| < \delta$. Thus,

$$\|f - f_y\|_p < 2\varepsilon + [m(K)]^{1/p}\varepsilon$$

so that $\lim\limits_{y\to 0} \|f - f_y\|_p = 0$. Also, for $y, y_0 \in \mathbb{R}^d$,

$$\|f_y - f_{y_0}\|_p = \|f - f_{y-y_0}\|_p \to 0 \quad \text{as} \quad y \to y_0.$$

Hence, $y \mapsto f_y$ is continuous on $\mathbb{R}^d$. ▮

Proof of Theorem 3.1.7. We observe that for every $\xi \neq 0$,

$$\begin{aligned} \hat{f}(\xi) &= -\int_{\mathbb{R}^d} f(x)e^{-ix.\xi}e^{-i\pi}dx \\ &= -\int_{\mathbb{R}^d} f(x)e^{-i(x+\pi\xi/|\xi|^2).\xi}dx \\ &= -\int_{\mathbb{R}^d} f\Big(y - \frac{\pi\xi}{|\xi|^2}\Big)e^{-iy.\xi}dy. \end{aligned}$$

Hence,

$$2\hat{f}(\xi) = \int_{\mathbb{R}^d} [f(x) - f\Big(x - \frac{\pi\xi}{|\xi|^2}\Big)]e^{-ix.\xi}dx$$

so that

$$2|\hat{f}(\xi)| \le \int_{\mathbb{R}^d} |f(x) - f\Big(x - \frac{\pi\xi}{|\xi|^2}\Big)|dx = \|f - f_{\pi\xi/|\xi|^2}\|_1.$$

By Lemma 3.1.8, $\|f - f_{\pi\xi/|\xi|^2}\|_1 \to 0$ as $|\xi| \to \infty$. Thus, $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$. ▮

Combining Theorems 3.1.2 and 3.1.7 we obtain:

Theorem 3.1.9 *The map $f \mapsto \hat{f}$ is a bounded linear operator from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$.*

Theorem 3.1.10 *Suppose $f \in L^1(\mathbb{R})$ is differentiable with $f' \in L^1(\mathbb{R})$. Then*

$$\widehat{f'}(\xi) = (i\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

For its proof, we shall make use of the following lemma.

Lemma 3.1.11 *If $f \in L^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$, then $|f(x)| \to 0$ as $|x| \to \infty$.*

Proof. Observe that for any $x > 0$,

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

Since $f' \in L^1(\mathbb{R})$, $\lim_{x\to\infty} \int_0^x f'(t)dt$ exists so that $\lim\limits_{x\to\infty} f(x)$ exists.

Now, suppose that $\lim\limits_{a\to\infty} f(a) = \beta \neq 0$. Then $\lim\limits_{a\to\infty} |f(a)| = |\beta| > 0$. Then there exists $\alpha > 0$ such that $|f(x)| > |\beta|/2$ for all $x \geq \alpha$. Hence $\displaystyle\int_\alpha^a |f(x)| \geq (a - \alpha)|\beta|/2$ for all $a \geq \alpha$. This contradicts the fact that $f \in L^1(\mathbb{R})$. Thus, we have proved that $f(x) \to 0$ as $x \to \infty$. Similarly, we can prove that $f(x) \to 0$ as $x \to -\infty$. ∎

Proof of Theorem 3.1.10. We have

$$\widehat{f'}(\xi) = \int_{\mathbb{R}} f'(x)e^{-ix\xi}dx.$$

Since $f \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} f'(x)e^{-ix\xi}dx = \lim_{a\to\infty} \int_{-a}^a f'(x)e^{-ix\xi}dx.$$

By integration by parts, for $a > 0$,

$$\int_{-a}^a f'(x)e^{-ix\xi}dx = \left[e^{-ix\xi}f(x)\right]_{-a}^a + (i\xi)\int_{-a}^a f(x)e^{-ix\xi}dx.$$

By Lemma 3.1.11, $\lim_{a\to\pm\infty} \left[e^{-ix\xi}f(x)\right]_{-a}^a = 0$. Hence,

$$\lim_{a\to\infty} \int_{-a}^a f'(x)e^{-ix\xi}dx = (i\xi)\int_{\mathbb{R}} f(x)e^{-ix\xi}dx.$$

Thus, we have proved that $\widehat{f'}(\xi) = (i\xi)\hat{f}(\xi)$. ∎

Let

$$(Df)(x) := f'(x), \qquad (Mf)(x) := ixf(x).$$

Then the conclusion in Theorem 3.1.10 can be written as

$$\widehat{Df}(\xi) = (M\hat{f})(\xi).$$

Theorem 3.1.10 has multi-variable analogue as well. For stating it we introduce a few notations:

For $z = (z_1, \dots z_d) \in \mathbb{C}^d$ and $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, define $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and

$$\partial^\alpha \varphi := \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \Big(\frac{\partial}{\partial x_2}\Big)^{\alpha_2} \cdots \Big(\frac{\partial}{\partial x_d}\Big)^{\alpha_d}.$$

For $k \in \mathbb{N}$, we write $f \in C^k(\mathbb{R}^d)$ iff $\partial^\alpha f$ exists and continuous for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$.

Theorem 3.1.12 *If $f \in L^1(\mathbb{R}^d)$ such that $\partial^\alpha f$ exists a.e. and $\partial^\alpha f \in L^1(\mathbb{R}^d)$ for some multi-index $\alpha \in \mathbb{N}_0^d$, then*

$$\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi) \quad \textit{for every} \quad \xi \in \mathbb{R}^d.$$

Using the notations:

$$(D_\alpha f)(x) = \partial^\alpha f(x), \qquad (M_\alpha f)(x) = (ix)^\alpha f(x),$$

the conclusion in Theorem 3.1.10 can be written as

$$\widehat{D_\alpha f}(\xi) = (M_\alpha \hat{f})(\xi).$$

Theorem 3.1.13 *Suppose $f \in L^1(\mathbb{R})$ such that $x \mapsto g(x) := xf(x)$ belongs to $L^1(\mathbb{R})$. Then $\hat{f}$ is differentiable and*

$$(\hat{f})'(\xi) = i\hat{g}(\xi), \quad \xi \in \mathbb{R}.$$

This fact is sometimes written as

$$\widehat{f(x)}'(\xi) = \widehat{ixf(x)}(\xi).$$

Proof. For $\xi, h \in \mathbb{R}$, we have

$$\hat{f}(\xi + h) - \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)[e^{-ix\cdot(\xi+h)} - e^{-ix.\xi}]dx = \int_{\mathbb{R}} f(x)e^{-ix.\xi}[e^{-ix\cdot h} - 1]dx$$

Hence,

$$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{ih} = \int_{\mathbb{R}} xf(x)e^{-ix.\xi}\left(\frac{e^{-ix\cdot h}-1}{ix.h}\right)dx$$

Note that

$$\frac{e^{-ix\cdot h}-1}{-ix.h} \to 1 \quad \text{as} \quad |h| \to 0$$

and, since $|\frac{e^{-ix\cdot h}-1}{ix.h}| \leq 1$,

$$\left|xf(x)e^{-ix.\xi}\left(\frac{e^{-ix\cdot h}-1}{ix.h}\right)\right| \leq |xf(x)|.$$

Hence, by DCT,

$$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{ih} = \int_{\mathbb{R}} xf(x)e^{-ix.\xi}\left(\frac{e^{-ix\cdot h}-1}{ix.h}\right)dx \to \int_{\mathbb{R}}(-x)f(x)e^{-ix.\xi}dx$$

as $|h| \to 0$. ∎

Using the operators D and M, we can write the conclusion in Theorem 3.1.13 as

$$D\hat{f}(\xi) = \widehat{Mf}(\xi).$$

More generally,

Theorem 3.1.14 *If $f \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|x|^k|f(x)|dx < \infty$ for some $k \in \mathbb{N}$, then $\hat{f} \in C^k(\mathbb{R}^d)$ and*

$$\partial^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^d}(-ix)^\alpha f(x)e^{-ix.\xi}dx$$

for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$.

Using the operators D and M, we can write the conclusion in the above theorem as

$$D_\alpha \hat{f}(\xi) = \widehat{M_\alpha f}(\xi).$$

$$\widehat{Df}(\xi) = M\hat{f}(\xi), \qquad D\hat{f}(\xi) = \widehat{Mf}(\xi).$$
$$\widehat{D_\alpha f}(\xi) = M_\alpha\hat{f}(\xi), \qquad D_\alpha\hat{f}(\xi) = \widehat{M_\alpha f}(\xi).$$

Let us introduce the following operators: For $f \in L^1(\mathbb{R}^d)$,

$$(e_h f)(x) = e^{ih\cdot x} f(x), \qquad (\tau_h f)(x) = f(x-h),$$
$$(\mathcal{R}f)(x) = f(-x), \qquad (D_t f)(x) = f(tx).$$

Theorem 3.1.15 *The following results hold: For $f \in L^1(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, $0 \neq t \in \mathbb{R}$,*

1. $\widehat{e_h f} = \tau_h \hat{f}$,
2. $\widehat{\tau_h f} = e_{-h} \hat{f}$,
3. $\widehat{\mathcal{R}f} = \mathcal{R}\hat{f}$,
4. $\widehat{D_t f} = |t|^{-d} D_{1/t} \hat{f},\ 0 \neq t \in \mathbb{R}$.

Proof. (1) For $\xi, h \in \mathbb{R}^d$, we have

$$\widehat{e_h f}(\xi) = \int_{\mathbb{R}^d} e_h f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} e^{-ih.x} f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(x) e^{-ix(\xi - h)} dx.$$

Thus, $\widehat{e_h f}(\xi) = \hat{f}(\xi - h) = (\tau_h \hat{f})(\xi)$.

(2) For $\xi, h \in \mathbb{R}^d$, we have

$$\widehat{\tau_h f}(\xi) = \int_{\mathbb{R}^d} \tau_h f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(x-h) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(y) e^{-i(y+h).\xi} dx.$$

Thus,

$$\widehat{\tau_h f}(\xi) = \int_{\mathbb{R}^d} f(y) e^{-i(y+h).\xi} dx = e^{-ih.\xi} \int_{\mathbb{R}^d} f(y) e^{-iy.\xi} dx = e^{-ih.\xi} \hat{f}(\xi) = (e_{-h}\hat{f})(\xi).$$

(3) For $\xi \in \mathbb{R}^d$, we have

$$\widehat{\mathcal{R}f}(\xi) = \int_{\mathbb{R}^d} (\mathcal{R}) f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(-x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(y) e^{iy.\xi} dx = \hat{f}(-\xi).$$

Hence, $\widehat{(\mathcal{R}f)}(\xi) = \hat{f}(-\xi) = (\mathcal{R}\hat{f})(\xi)$.

(4) For $\xi \in \mathbb{R}^d$ and $0 \neq t \in \mathbb{R}$, we have

$$(\widehat{D_t f})(\xi) = \int_{\mathbb{R}^d} (D_t f)(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(tx) e^{-ix.\xi} dx = \frac{1}{|t|^d} \int_{\mathbb{R}^d} f(y) e^{-iy.\xi/t}.$$

Thus, $(\widehat{D_t f})(\xi) = \frac{1}{|t|^d} \hat{f}(\frac{\xi}{t}) = \frac{1}{|t|^d} (D_{1/t}\hat{f})(\xi)$. ∎

3.2 On surjectivity

We have seen (see Theorem 3.1.9) the map $f \mapsto \hat{f}$ is a bounded linear operator from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$. A natural question is whether this operator is onto. We show that the answer is negative by using the following proposition.

Proposition 3.2.1 *If $f \in L^1(\mathbb{R})$ is an odd function, then $\hat{f}$ is an odd function and there exists $M > 0$ such that*

$$\left| \int_r^R \frac{\hat{f}(\xi)}{\xi} d\xi \right| \leq M$$

for all r, R with $0 < r < R$.

Proof. Let $f \in L^1(\mathbb{R})$ is an odd function. It can be easily seen that[4],

$$\hat{f}(\xi) = 2i \int_0^\infty f(x) \sin(\xi x) dx.$$

Thus, $\hat{f}$ is odd. Let $R \geq r > 0$. Then, we have

$$\begin{aligned}
\int_r^R \frac{\hat{f}(\xi)}{\xi} d\xi &= 2i \int_r^R \frac{1}{\xi} \left(\int_0^\infty f(x) \sin(\xi x) dx \right) d\xi \\
&= 2i \int_0^\infty f(x) \left(\int_r^R \frac{\sin(\xi x)}{\xi} d\xi \right) dx \\
&= 2i \int_0^\infty f(x) \left(\int_{rx}^{Rx} \frac{\sin(s)}{s} ds \right) dx.
\end{aligned}$$

We know that there exists $M_0 > 0$ such that $\left| \int_a^b \frac{\sin x}{x} dx \right| \leq M_0$ for all $(a, b) \subseteq \mathbb{R}$. Thus,

$$\left| \int_r^R \frac{\hat{f}(\xi)}{\xi} d\xi \right| \leq 2 \int_0^\infty |f(x)| \left| \int_{rx}^{Rx} \frac{\sin(s)}{s} ds \right| dx \leq 2M_0 \|f\|_1.$$

∎

Theorem 3.2.2 *The bounded linear operator $f \mapsto \hat{f}$ is a bounded linear operator from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$ is not onto.*

[4] If f is a real valued even function then $\hat{f}(\xi) = 2 \int_0^\infty f(x) \cos(\xi x) dx$.

Proof. By Proposition 3.2.1, it is enough to construct an odd function $g \in C_0(\mathbb{R})$ such that

$$\left| \int_r^R \frac{g(t)}{t} dt \right| \to \infty \quad \text{as} \quad R \to \infty.$$

A candidate[5] for such a function is the odd extension of g defined by

$$g(t) := \begin{cases} t/e, & 0 \le t \le e, \\ 1/\ln(t), & t > e. \end{cases}$$

Note that

$$\int_e^R \frac{g(t)}{t} dt = \ln(\ln(R)) \to \infty \quad \text{as} \quad R \to \infty.$$

∎

3.3 Inversion theorem

Another question one may ask is whether f can be recovered from $\hat{f}$. Here is an answer for this.

Theorem 3.3.1 (Inversion theorem) *Suppose $f \in L^1(\mathbb{R})$. If $\hat{f} \in L^1(\mathbb{R})$ and if*

$$g(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt, \quad x \in \mathbb{R},$$

then $g \in C_0(\mathbb{R})$ and $f = g$ a.e., i.e.,

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt, \quad a.a.\, x \in \mathbb{R}.$$

In particular if $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt \quad \forall\, x \in \mathbb{R}.$$

An immediate corollary to the above theorem is the following.

Corollary 3.3.2 *The bounded linear operator $f \mapsto \hat{f}$ from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$ is injective.*

---

[5]This example is taken from the book *Classical Fourier Transforms*, Springer–Verlag, 1989 by K. Chandrasekharan.

Proof. By the above theorem, we can infer that if $f \in L^1(\mathbb{R})$ such that $\hat{f} = 0$, then f satisfies the assumption that $\hat{f} \in L^1(\mathbb{R})$ so that $f = 0$ a.e. ∎

Now we give two examples dealing with the cases in which the conditions of inverse theorem are satisfied/not satisfied.

Example 3.3.3 Let $f(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$ Then we see that

$$\hat{f}(t) = \int_{-1}^{1} (1 - |x|) e^{-itx} dx = 2 \int_0^1 (1 - x) \cos(tx) dx = \frac{\sin^2(t/2)}{(t/2)^2}.$$

Thus both f and $\hat{f}$ belong to $L^1(\mathbb{R})$.

Example 3.3.4 Let $f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$ Then we see that

$$\hat{f}(t) = \int_{-1}^{1} e^{-itx} dx = 2 \int_0^1 \cos(tx) dx = 2 \left(\frac{\sin t}{t} \right).$$

Thus $f \in L^1(\mathbb{R})$ but $\hat{f} \notin L^1(\mathbb{R})$.

3.4 Proof of inversion theorem

The idea of the proof is to have a family of appropriate funcctions $\varphi_\lambda \in L^1(\mathbb{R})$ for $\lambda > 0$ such that the functions f_λ, $\lambda > 0$, defined by

$$f_\lambda(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_\lambda(t) \hat{f}(t) e^{ixt} dt, \quad x \in \mathbb{R},$$

are such that

1. $f_\lambda(x) \to \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt \quad \text{as} \quad \lambda \to 0$ a.e. and
2. $\exists \, (\lambda_n)$ such that $f_{\lambda_n} \to f$ a.e. as $n \to \infty$

so that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt, \quad \text{for a.a.} \quad x \in \mathbb{R}.$$

In this regard, we are also going to make use of an integrable function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $0 < \varphi(x) \leq 1$ for all $x \in \mathbb{R}$;

(ii) For $\lambda > 0$, $\varphi_\lambda(x) := \varphi(\lambda x) \to 1$ as $\lambda \to 0$;

(iii) The function ψ defined by $\psi(x) = \dfrac{1}{2\pi}\int_{\mathbb{R}} \varphi(t)e^{itx}dt$, $x \in \mathbb{R}$ is non-negative and satisfies $\int_{\mathbb{R}} \psi(x)dx = 1$.

For example, one may take $\varphi(x) = e^{-|x|}$. Clearly, this function satisfies properties (i) and (ii) above. To see (iii), we note that

$$2\pi\psi(x) = \int_{\mathbb{R}} e^{-|t|}e^{itx}dt = \int_{-\infty}^{0} e^{t}e^{itx}dt + \int_{0}^{\infty} e^{-t}e^{itx}dt = \frac{2}{1+x^2},$$

and hence

$$\int_{\mathbb{R}} \psi(x)\,dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{dx}{1+x^2} = 1.$$

Thus, (iii) is also satisfied.

Proof of inversion theorem. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function satisfying the conditions (i)-(iii) above. For $\lambda > 0$, let

$$f_\lambda(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_\lambda(t)\hat{f}(t)e^{itx}dt, \quad x \in \mathbb{R}.$$

We show that

$$\|f_\lambda - f\|_1 \to 0 \quad \text{as} \quad \lambda \to 0. \tag{1}$$

Then[6] there exists a sequence (λ_n) of positive reals such that $\lambda_n \to 0$ and

$$f_{\lambda_n} \to f \quad \text{a.e..} \tag{2}$$

Since $0 \leq \varphi_{\lambda_n}(x) \leq 1$ and $\varphi_{\lambda_n}(x) \to 1$ for every $x \in \mathbb{R}$, we have

$$\varphi_{\lambda_n}(t)\hat{f}(t)e^{itx} \to \hat{f}(t)e^{itx}$$

and

$$|\varphi_{\lambda_n}(t)\hat{f}(t)e^{itx}| \leq |\hat{f}(t)| \quad \text{for all} \quad n \in \mathbb{N}$$

[6]Recall from measure theory (cf. [4]) that if $1 \leq p < \infty$, then for every Cauchy sequence in $L^p(\mathbb{R}^d)$ has a subsequence which converges almost every where to some function in $L^p(\mathbb{R}^d)$.

so that by DCT,

$$f_{\lambda_n}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_{\lambda_n}(t)\hat{f}(t)e^{itx}dt \to \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)e^{itx}dt. \qquad (3)$$

By (2) and (3),

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)e^{itx}dt \quad \text{for almost all} \quad x \in \mathbb{R}.$$

Now, for proving (1), we first observe, applying Fubini's theorem, that

$$\begin{aligned}
f_\lambda(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_\lambda(t)\hat{f}(t)e^{itx}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(\lambda t)e^{itx}\left(\int_{\mathbb{R}} f(y)e^{-iyt}dy\right)dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\lambda}\varphi(u)e^{iux/\lambda}\left(\int_{\mathbb{R}} f(y)e^{-iyu/\lambda}dy\right)du \\
&= \int_{\mathbb{R}} f(y)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\lambda}\varphi(u)e^{iu(x-y)/\lambda}du\right)dy \\
&= \int_{\mathbb{R}} f(y)\frac{1}{\lambda}\psi\Big(\frac{x-y}{\lambda}\Big)dy \\
&= \int_{\mathbb{R}} f(x-\tau)\frac{1}{\lambda}\psi\Big(\frac{\tau}{\lambda}\Big)d\tau \\
&= \int_{\mathbb{R}} f(x-\lambda s)\psi(s)ds.
\end{aligned}$$

Since $\int_{\mathbb{R}} \psi(s)ds = 1$, we have

$$f_\lambda(x) - f(x) = \int_{\mathbb{R}} [f(x-\lambda s) - f(x)]\psi(s)ds.$$

Hence, appealing again to Fubini,

$$\begin{aligned}
\int_{\mathbb{R}} |f_\lambda(x) - f(x)|dx &\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x-\lambda s) - f(x)|\psi(s)ds\right)dx \\
&\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x-\lambda s) - f(x)|dx\right)\psi(s)ds
\end{aligned}$$

Let

$$g_\lambda(s) := \left(\int_{\mathbb{R}} |f(x - \lambda s) - f(x)| dx \right) \psi(s).$$

By Lemma 3.1.8, $g_\lambda(s) \to 0$ as $\lambda \to 0$. Also, $|g_\lambda(s)| \le 2\|f\|_1 \psi(s)$ with $2\|f\|_1 \psi \in L^1(\mathbb{R})$. Hence, by DCT,

$$\|f_\lambda - f\|_1 := \int_{\mathbb{R}} |f_\lambda(x) - f(x)| dx \le \int_{\mathbb{R}} g_\lambda(s) ds \to 0 \quad \text{as} \quad \lambda \to 0.$$

This completes the proof. ∎

3.5 The Banach algebra $L^1(\mathbb{R})$

Now, we define a multiplicative structure on $L^1(\mathbb{R})$. For f, g in $L^1(\mathbb{R}^d)$, we shall show that the integral

$$\int_{\mathbb{R}^d} f(x - y) g(y)\, dy$$

is well-defined for almost all $x \in \mathbb{R}$ and define the *convolution* of f and g, denoted by $f * g$ as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y) g(y)\, dy,$$

for almost all $x \in \mathbb{R}^d$.

It can be seen that, for each $x \in \mathbb{R}$, $y \mapsto f(x - y)g(y)$ is a Lebesgue measurable function (Exercise). But, to prove that $\int_{\mathbb{R}^d} |f(x - y)g(y)|\, dy$ is finite for a.a. $x \in \mathbb{R}$, some work is involved.

We may recall:

- If Y is a topological space and X is a measurable space, then a function $f : X \to Y$ is measurable if and only if for every Borel set B in Y, $f^{-1}(B)$ is measurable. This follows by showing that

$$\mathcal{S} := \{B \subseteq Y : f^{-1}(B) \text{ measrable } \}$$

  is a σ-algebra on Y, and Borel sets belong to $\mathcal{S}$.

- If X, Y, Z are topological spaces and $\varphi : X \to Y$ and $\psi : Y \to Z$ are Borel measurable functions, then $\psi \circ \varphi : X \to Y$ is Borel measurable.

- If f is Lebesgue measurable, then there exists a Borel measurable function f_0 such that $f = f_0$ a.e. (Exercise)

Also, we know:

> Let X be a measurable space, and Y and Z be topological spaces. If $\varphi : X \to Y$ is measurable and $\psi : Y \to Z$ is continuous, then $\psi \circ \varphi : X \to Z$ is measurable.

However:

> If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and $\psi : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable, then $\psi \circ \varphi : X \to Y$ **need not be** Lebesgue measurable.

Thus, knowing that $(x, y) \mapsto x - y$ is continuous from $\mathbb{R}$ to $\mathbb{R}$ and f is Lebesgue measurable, **we cannot** assert that $(x, y) \mapsto f(x - y)$ is Lebesgue measurable.

Theorem 3.5.1 *Let f and g belong to $L^1(\mathbb{R}^d)$. Then*

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y)g(y)\, dy$$

is well-defined for almost all $x \in \mathbb{R}^d$, $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. We prove the result for $d = 1$. The same arguments work for any $d \in \mathbb{N}$. First let us assume that f and g are Borel measurable. Note that the function $(x, y) \to x - y$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$. Therefore, it can be seen that the functions $(x, y) \mapsto f(x - y)$ and $(x, y) \mapsto g(y)$ are Borel measurable whenever $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Borel measurable. Hence, $(x, y) \mapsto f(x - y)g(y)$ is Borel measurable. Now, using the fact that f and g belong to $L^1(\mathbb{R})$ and by Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x - y)g(y)| dy \right) dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x - y)g(y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f(x - y)| dx \right) dy \\
&\leq \|f\|_1 \|g\|_1.
\end{aligned}$$

Thus, $\int_{\mathbb{R}} |f(x - y)g(y)| dy < \infty$ for all most all $x \in \mathbb{R}$, $f * g$ is well defined for almost all $x \in \mathbb{R}$ and $f * g \in L^1(\mathbb{R})$. We also have

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Now, let us assume that f and g are Lebesgue measurable. Then we know that there exist Borel measurable functions f_0 and g_0 such that $f = g_0$ and $g = g_0$ a.e. Hence, we obtain the conclusion of the theorem by applying the first part to f_0 and g_0. ∎

Definition 3.5.2 For f, g in $L^1(\mathbb{R}^d)$, we define the **convolution** of f and g, denoted by $f * g$, is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy,$$

for almost all $x \in \mathbb{R}^d$. ◊

Remark 3.5.3 *In fact, $f * g$ can be defined for any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, where $1 \leq p < \infty$.* ◊

Theorem 3.5.4 *The following hold.*

1. *$f * g = g * f$ for all $f, g \in L^1(\mathbb{R}^d)$.*
2. *$f * (g * h) = (f * g) * h$ for all $f, g, h \in L^1(\mathbb{R}^d)$.*

Theorem 3.5.5 *If $f, g \in L^1(\mathbb{R}^d)$, then*

$$\widehat{f * g)} = \hat{f}\hat{g}.$$

Proof. Note that

$$\begin{aligned}
\widehat{f * g)}(\xi) &= \int_{\mathbb{R}^d} (f * g)(x) e^{-ix.\xi} dx \\
&= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}} f(x-y)g(y)dy \Big) e^{-ix.\xi} dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y) e^{-i(x-y).\xi} dx \right) g(y) e^{-iy.\xi} dy \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

Thus, $\widehat{f * g)} = \hat{f}\hat{g}$. ∎

We already know that $L^1(\mathbb{R}^d)$ is a Banach space. With respect to the multiplication $(f, g) \mapsto f * g$, $L^1(\mathbb{R}^d)$ is a commutative Banach algebra.

Theorem 3.5.6 *The Banach algebra $L^1(\mathbb{R}^d)$ does not have a multiplicative identity with respect to convolution.*

Proof. Suppose there exists $\varphi \in L^1(\mathbb{R}^d)$ such that $f * \varphi = f$ for all $f \in L^1(\mathbb{R}^d)$. Then we have

$$\hat{f}(\xi)\hat{\varphi}(\xi) = \hat{f}(\xi) \quad \forall\, f \in L^1(\mathbb{R}^d)\, \forall\, \xi \in \mathbb{R}^d.$$

Thus, if f is such that $\hat{f}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$, then $\hat{f}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$, which is a contradiction to the fact that $\hat{f} \in C_0(\mathbb{R}^d)$. Note that if $f(x) := e^{-|x|^2/2}$, then $\hat{f}(\xi) = (2\pi)^{d/2}e^{-|\xi|^2/2} \neq 0$ for all $\xi \in \mathbb{R}$. Thus, we have proved that there is no $\varphi \in \mathbb{R}^d$ such that $f * \varphi = f$ for all $f \in L^1(\mathbb{R}^d)$. ∎

However, it has an *approximate identity.*.

Definition 3.5.7 By an **approximate identity** we mean a family $\{e_\lambda : \lambda > 0\}$ in $L^1(\mathbb{R}^d)$ such that $\|f * e_\lambda - f\|_1 \to 0$ as $\lambda \to 0$. ♢

Now, we specify an approximate identity for $L^1(\mathbb{R})$. The same can be extended to the case of $L^1(\mathbb{R}^d)$ for appropriate changes.

Here onwards, we consider a non-negative measurable function ψ which satisfies $\int_{\mathbb{R}} \psi(x)dx = 1$, and let

$$\psi_\lambda(x) := \frac{1}{\lambda}\psi\left(\frac{x}{\lambda}\right), \quad x \in \mathbb{R},\, \lambda > 0.$$

For example, the function

$$\psi(x) := \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad x \in \mathbb{R},$$

satisfies the above requirements. Then we have

$$(f * \psi_\lambda)(x) = \int_{\mathbb{R}} f(x-y)\psi_\lambda(y)dy = \int_{\mathbb{R}} f(x-\lambda s)\psi(s)ds. \qquad (*)$$

Indeed,

$$\begin{aligned}(f * \psi_\lambda)(x) &= \int_{\mathbb{R}} f(x-y)\psi_\lambda(y)dy \\ &= \int_{\mathbb{R}} f(y)\psi_\lambda(x-y)dy \\ &= \int_{\mathbb{R}} f(y)\frac{1}{\lambda}\psi\Big(\frac{x-y}{\lambda}\Big)dy \\ &= \int_{\mathbb{R}} f(x-\lambda s)\psi(s)ds.\end{aligned}$$

Theorem 3.5.8 *The set $\{\psi_\lambda : \lambda > 0\}$ is an approximate identity for $L^1(\mathbb{R})$.*

Proof. Recall from (∗) that

$$(f * \psi_\lambda)(x) = \int_{\mathbb{R}} f(x-y)\psi_\lambda(y)dy = \int_{\mathbb{R}} f(x-\lambda s)\psi(s)ds.$$

Since $\int_{\mathbb{R}} \psi(x)dx = 1$, we have

$$(f * \psi_\lambda)(x) - f(x) == \int_{\mathbb{R}} [f(x-\lambda s) - f(x)]\psi(s)ds$$

and hence,

$$\begin{aligned} \|f * \psi_\lambda - f\|_1 &= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} [f(x-\lambda s) - f(x)]\psi(s)ds \Big| dx \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x-\lambda s) - f(x)|\psi(s)ds \Big) dx. \end{aligned}$$

Now, $g_\lambda(s) := \displaystyle\int_{\mathbb{R}} |f(x-\lambda s) - f(s)|dx\, \psi(s) \to 0$ as $\lambda \to 0$, by Lemma 3.1.8, and $g_\lambda(s) \le 2\|f\|_1 \psi(s)$. Hence, by DCT,

$$\int_{\mathbb{R}} |(f * \psi_\lambda)(x) - f(x)|dx \to 0 \quad \text{as} \quad \lambda \to 0.$$

That is, $\|f * \psi_\lambda - f\|_1 \to 0$ as $\lambda \to 0$. ∎

In fact, something more is true.

Lemma 3.5.9 *If $f \in L^p(\mathbb{R})$ for $1 \le p < \infty$, then $f * \psi_\lambda \in L^p(\mathbb{R})$ and*

$$\lim_{\lambda \to 0} \|f * \psi_\lambda - f\|_p = 0.$$

Proof. Let $f \in L^p(\mathbb{R})$ for $1 \le p < \infty$. We have already proved for $p = 1$. So, assume that $1 < p < \infty$. By (∗),

$$(f * \psi_\lambda)(x) - f(x) = \int_{\mathbb{R}} [f(x-\lambda s) - f(x)]\psi(s)ds.$$

Applying Hölder's inequality,

$$\begin{aligned} |(f*\psi_\lambda)(x)-f(x)| &\leq \int_{\mathbb{R}} |f(x-\lambda s)-f(x)|\psi(s)ds \\ &\leq \int_{\mathbb{R}} |f(x-\lambda s)-f(x)|\psi(s)^{1/p}\psi(s)^{1/q}ds \\ &\leq \left(\int_{\mathbb{R}} |f(x-\lambda s)-f(x)|^p\psi(s)ds\right)^{1/p}\left(\int_{\mathbb{R}}\psi(s)ds\right)^{1/q} \\ &\leq \left(\int_{\mathbb{R}} |f(x-\lambda s)-f(x)|^p\psi(s)ds\right)^{1/p}. \end{aligned}$$

Hence, applying Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}} |(f*\psi_\lambda)(x)-f(x)|^p dx &\leq \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x-\lambda s)-f(x)|^p\psi(s)ds\right)dx \\ &= \int_{\mathbb{R}}\psi(s)\left(\int_{\mathbb{R}} |f(x-\lambda s)-f(x)|^p dx\right)ds. \end{aligned}$$

Since $\int_{\mathbb{R}}\psi(s)ds = 1$, we have

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x-\lambda s)-f(x)|^p dx\right)\psi(s)ds \leq (2\|f\|_p)^p$$

so that $f*\psi_\lambda \in L^p$. Further, $\psi(s)\int_{\mathbb{R}}|f(x-\lambda s)-f(x)|^p dx \leq (2\|f\|_p)^p\psi(s)$ and by Lemma 3.1.8,

$$\psi(s)\int_{\mathbb{R}}|f(x-\lambda s)-f(x)|^p dx \to 0 \quad \text{as} \quad \lambda\to 0.$$

Hence, by DCT,

$$\|f*\psi_\lambda - f\|_p = \int_{\mathbb{R}}|(f*\psi_\lambda)(x)-f(x)|^p dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}}\psi(s)|f(x-\lambda s)-f(x)|^p dx ds \to 0$$

as $\lambda \to 0$. ∎

We observe that for each $t\in\mathbb{R}$, the linear functional $\varphi_t : L^1(\mathbb{R})\to\mathbb{R}$ defined by $\varphi_t(f)=\hat{f}(t)$, $f\in L^1(\mathbb{R})$ is a linear functional which also satisfies the multiplicative property:

$$\varphi_t(f*g)=\varphi_t(f)\varphi_t(g), \quad f,g\in L^1(\mathbb{R}).$$

In other words, φ_t is a complex homomorphism on Banach algebra $L^1(\mathbb{R})$. This linear functional is continuous: For each $t \in \mathbb{R}$,

$$|\varphi_t(f)| = |\hat{f}(t)| \leq \|f\|_1 \quad \text{for all} \quad f \in L^1(\mathbb{R}).$$

It also shows that $\|\varphi_t\| \leq 1$ for all $t \in \mathbb{R}$. In fact, this is true for every complex homomorphism on any Banach algebra as the following theorem shows.

Theorem 3.5.10 *Let $\mathcal{A}$ be a complex Banach algebra and let $\varphi : \mathcal{A} \to \mathbb{C}$ be an algebra homomorphism. Then φ is continuous and $\|\varphi\| \leq 1$.*

Proof. We prove that $\|f(a)| \leq \|a\|$ for every $a \in \mathcal{A}$: Suppose this is not true. then there exists $a_0 \in \mathcal{A}$ such that $|\varphi(a_0)| > \|a_0\|$. Let $a = a_0/|\varphi(a_0)|$. Then $\|a\| < 1$ and $\varphi(a) = 1$. Since $a^n \to 0$, the sequence (b_n) defined by $b_n = -(a + a^2 + \cdots + a^n)$ is convergent, say $b_n \to b$. Note that

$$ab_n = -(a^2 + a^3 + \cdots + a^{n+1}) = a + b_n - a^{n+1}.$$

Taking limits, $ab = a + b$. Hence,

$$\varphi(a) + f(b) = \varphi(ab) = \varphi(a)\varphi(b).$$

Since $\varphi(a) = 1$, we obtain $1 + \varphi(b) = \varphi(b)$. This is impossible. ∎

The following theorem shows that Fourier transform is the only complex linear functional on $L^1(\mathbb{R})$ having the multiplicative property.

Theorem 3.5.11 *If $\varphi : L^1(\mathbb{R}) \to \mathbb{C}$ is a multiplicative linear functional, that is, φ is a linear functional satisfying*

$$\varphi(f * g) = \varphi(f)\varphi(g) \quad \forall\, f, g \in L^1(\mathbb{R}),$$

then there exists $t \in R$ such that $\varphi(f) = \hat{f}(t)$ for all $f \in L^1(\mathbb{R})$.

Proof. Let φ be a nonzero continuous linear functional on $L^1(\mathbb{R})$. Then by Riesz representation theorem, there exists a unique $h \in L^\infty(\mathbb{R})$ such that

$$\varphi(f) = \int_{\mathbb{R}} f(x)h(x)dx \quad \text{for every} \quad f \in L^1(\mathbb{R}).$$

Using the fact that $\varphi(f * g) = \varphi(f)\varphi(g)$, we obtain the relation

$$\int_{\mathbb{R}} g(y)\varphi(f_y)dy = \varphi(f)\int_{\mathbb{R}} g(y)h(y)dy$$

where $f_y(x) := f(x-y)$. From this, $\varphi(f_y) = \varphi(f)h(y)$. Since $y \mapsto f_y$ is continuous, we may assume, without loss of generality, that h is continuous. We also have

$$\varphi(f)h(x+y) = \varphi(f_{x+y}) = \varphi((f_x)_y) = \varphi(f_x)h(y) = \varphi(f)h(x)h(y).$$

Hence, $h(x+y) = h(x)h(y)$ and hence, $h(0) = 1$, as $h \neq 0$. Continuity of h implies the existence of $\delta > 0$ such that $\int_0^\delta h(y) \neq 0$. Now, from $h(x+y) = h(x)h(y)$ we have

$$h(x)\int_0^\delta h(y)dy = \int_0^\delta h(x+y)dy = \int_x^{x+\delta} h(t)dt.$$

Hence, h is differentiable so that from $h(x+y) = h(x)h(y)$, $h'(x) = h(x)h'(0)$, i.e.,

$$h'(x) = \alpha h(x), \quad \alpha := h'(0).$$

Thus,

$$h(x) = e^{\alpha x}.$$

Since h is bounded, $\alpha := -i\tau$ for some $\tau \in \mathbb{R}$. Thus, $h(x) = e^{-i\tau x}$ for some $\tau \in \mathbb{R}$. ▮

Remark 3.5.12 *The above proof is adopted from Rudin[6].* ◊

3.6 Fourier-Plancheral Transform on $L^2(\mathbb{R})$

We have already observed that $f \mapsto \hat{f}$ is an injective bounded linear operator from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$. We have also given an example to show that this map is not surjective. Next theorem introduce the definition of Fourier transform of functions in $L^2(\mathbb{R})$ by extending the known definition from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to all of $L^2(\mathbb{R})$.

We shall make use of the following theorem from Functional Analysis:

Theorem 3.6.1 *Let* X *be a normed linear space,* Y *be a Banach space and* X_0 *be a dense subspace of* X*. Let* $T_0 : X_0 \to Y$ *be a bounded linear operator. Then there exists a unique bounded linear operator* $T : X \to Y$ *such that*

$$T\Big|_{X_0} = T_0 \quad \textit{and} \quad \|T\| = \|T_0\|.$$

Lemma 3.6.2 *Let* $f \in C_b(\mathbb{R})$ *and let* $\psi \in \Lambda^1(\mathbb{R})$ *be such that* $\int_{\mathbb{R}} \psi(x)dx = 1$*. Let* $\psi_\lambda(x) := \frac{1}{\lambda}\psi\left(\frac{x}{\lambda}\right)$, $\lambda > 0$*. Then*

$$\lim_{\lambda \to 0}(f * \psi_\lambda)(x) = f(x) \quad \textit{for every} \quad x \in \mathbb{R},$$

where ψ_λ *is as in Theorem 3.5.8.*

Proof. Note that for every $x \in \mathbb{R}$,

$$(f * \psi_\lambda)(x) - f(x) = \int_{\mathbb{R}} [f(x - \lambda s) - f(x)]\psi(s)ds.$$

Note that $|f(x - \lambda s) - f(x)|\psi(s) \leq 2\|f\|_\infty \psi(s)$ with $\psi \in L^1(\mathbb{R})$ and

$$[f(x - \lambda s) - f(x)]\psi(s) \to 0 \quad \text{as} \quad \lambda \to 0$$

as f is continuous at x. Hence, the result follows by DCT. ∎

Lemma 3.6.3 *For* $f \in L^1(\mathbb{R})$,

$$\bar{\hat{f}} = \hat{\tilde{f}},$$

where $\tilde{f}(x) := \overline{f(-x)}$, $x \in \mathbb{R}$.

Proof. Note that, for every $\xi \in \mathbb{R}$,

$$\overline{\hat{f}(\xi)} = \overline{\int_{\mathbb{R}} f(x)e^{-ix\xi}dx} = \int_{\mathbb{R}} \overline{f(x)}e^{ix\xi}dx = \int_{\mathbb{R}} \overline{f(-x)}e^{-ix\xi}dx = \widehat{\tilde{f}}(\xi).$$

The lemma is proved. ∎

Theorem 3.6.4 *If* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*, then* $\hat{f} \in L^2(\mathbb{R})$ *and* $\|\hat{f}\|_2 = \sqrt{2\pi}\|f\|_2$.

Proof. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. To show that $|\hat{f}|^2 \in L^1(\mathbb{R})$. Note that

$$|\hat{f}|^2 = \hat{f}\bar{\hat{f}} = \hat{f}\hat{\tilde{f}} = \widehat{f * \tilde{f}} = \hat{g},$$

where $g := f * \tilde{f}$. Note that

$$g(x) = \int_{\mathbb{R}} f(x-y)\tilde{f}(y)dy = \int_{\mathbb{R}} f(x-y)\overline{f(-y)}dy = \int_{\mathbb{R}} f(x+u)\overline{f(u)}du = \langle f_{-x}, f\rangle,$$

where $f_\tau(x) := f(x-\tau)$, and hence

$$|g(x)| = |\langle f_{-x}, f\rangle| \leq \|f\|_2^2.$$

Thus, the above integral is well defined for every $x \in \mathbb{R}$ and g is a bounded function. By Lemma 3.1.8, $\tau \mapsto f_\tau$ from $\mathbb{R}$ to $L^2(\mathbb{R})$ is continuous so that g is continuous. Hence, by Lemma 3.6.2,

$$(g * \psi_\lambda)(x) \to g(x) \quad \text{as} \quad \lambda \to 0^+$$

for every $x \in \mathbb{R}$. In particular,

$$(g * \psi_\lambda)(0) \to g(0) = \|f\|_2^2 \quad \text{as} \quad \lambda \to 0^+.$$

Let us take φ and ψ as in the proof of the inversion theorem. For example, one may take $\varphi(x) = e^{-|x|}$. Then,

$$\begin{aligned}
(g * \psi_\lambda)(x) &= \int_{\mathbb{R}} g(y)\frac{1}{\lambda}\psi\Big(\frac{x-y}{\lambda}\Big)dy \\
&= \int_{\mathbb{R}} g(y)\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{\lambda}\varphi(u)e^{iu(x-y)/\lambda}du\right)dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{\lambda}\varphi(u)e^{iux/\lambda}\left(\int_{\mathbb{R}} g(y)e^{-iyu/\lambda}dy\right)du \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(\lambda t)e^{itx}\left(\int_{\mathbb{R}} g(y)e^{-iyt}dy\right)dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(\lambda t)\hat{g}(t)e^{itx}dt.
\end{aligned}$$

In particular,

$$(g * \psi_\lambda)(0) = \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(\lambda t)\hat{g}(t)dt.$$

Recall that $\hat{g}(t) = |\hat{f}|^2 \geq 0$. Hence, assuming that $\varphi(\lambda t)$ increases to 1 as $\lambda \to 0^+$ (e.g., $\varphi(x) = e^{-|x|}$), we obtain $\varphi(\lambda t)\hat{g}(t)$ increases to $\hat{g}(t)$ as $\lambda \to 0^+$. Hence, by MCT

$$(g*\psi_\lambda)(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(\lambda t)\hat{g}(t)dt \to \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(t)dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(t)|^2 dt \quad \text{as} \quad \lambda \to 0^+,$$

We already have $(g * \psi_\lambda)(0) \to \|f\|^2$ as $\lambda \to 0^+$. Thus,

$$\frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(t)|^2 dt = \|f\|_2^2$$

so that $\hat{f} \in L^2(\mathbb{R})$ and $\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2$.

∎

Theorem 3.6.5 (Fourier-Plancheral Theorem) *There exists a unique surjective continuous linear operator Φ from $L^2(\mathbb{R})$ onto itself such that*

$$\Phi(f) = \hat{f} \quad \textit{for every} \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

and $\|\Phi(f)\|_2 = \sqrt{2\pi}\|f\|_2$ for every $f \in L^2(\mathbb{R})$.

Proof. By Theorem 3.6.4, the map $\Phi_0 : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\Phi_0(f) = \hat{f}, \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

is a continuous linear operator with its domain $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ dense in $L^2(\mathbb{R})$. Recall also that $\|\Phi_0(f)\| = \|\hat{f}\|_2 = \sqrt{2\pi}\|f\|_2$. Hence, it has a unique continuous linear extension $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $\|\Phi\| = \|\Phi_0\|$.

Also, for $f \in L^2(\mathbb{R})$, if (f_n) in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is such that $\|f - f_n\|_2 \to 0$, then we have

$$\begin{aligned} \|\Phi(f)\|_2 &= \|\Phi(\lim_{n\to\infty} f_n)\|_2 = \lim_{n\to\infty} \|\Phi(f_n)\|_2 \\ &= \lim_{n\to\infty} \|\hat{f}_n\|_2 = \lim_{n\to\infty} \sqrt{2\pi}\|f_n\|_2 = \sqrt{2\pi}\|f\|_2. \end{aligned}$$

This also shows that Φ is one-one and the range of Φ is closed.

Now, to show that Φ is onto, it is enough to show that range of Φ is dense. We in fact show that the range of Φ_0 is dense in $L^2(\mathbb{R})$: Let

$$Y := \{\hat{f} \in L^2(\mathbb{R}) : f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}.$$

By projection theorem, it is enough to show that $Y^\perp = \{0\}$, i.e., to prove that if $g \in L^2(\mathbb{R})$ such that $\langle g, \hat{f}\rangle = 0$ for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $g = 0$.

So, let $g \in L^2(\mathbb{R})$ be such that $\langle g, \hat{f}\rangle = 0$ for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in L^2, there exists (g_n) in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\|g_n - g\|_2 \to 0$. Thus,

$$0 = \langle g, \hat{f}\rangle = \lim_{n\to\infty} \langle g_n, \hat{f}\rangle = \lim_{n\to\infty} \langle \hat{g}_n, \tilde{f}\rangle = \lim_{n\to\infty} \langle \Phi(g_n), \hat{f}\rangle = \langle \Phi(g), \tilde{f}\rangle.$$

Now, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ iff $\tilde{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence, we obtain that $\langle \Phi(g), f\rangle = 0$ for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Consequently, $\Phi(g) = 0$ and hence $g = 0$. Thus, range of Φ_0 is dense in $L^2(\mathbb{R})$.

Since range of Φ_0 is dense in $L^2(\mathbb{R})$, and range of Φ is closed, Φ is onto. ∎

Definition 3.6.6 The operator $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ in Theorem 3.6.5 is called the **Fourier-Plancheral transform**. ♢

Notation 3.6.7 *Abusing the notation, for $f \in L^2(\mathbb{R})$, $\Phi(f)$ is also denoted by $\hat{f}$. This is somewhat justified by Theorem 3.6.11 below.*

Now, let us observe the following.

Theorem 3.6.8 *Let $f \in L^1(\mathbb{R}^d)$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \le n\}$. Then*

$$\hat{f}(\xi) = \lim_{n\to\infty} \int_{E_n} f(x) e^{-ix.\xi} dx$$

for every $\xi \in \mathbb{R}^d$.

Proof. Let $\xi \in \mathbb{R}^d$. Since $\bigcup_{n=1}^\infty E_n = \mathbb{R}^d$ and $E_n \subseteq E_{n+1}$ for every $n \in \mathbb{N}$, we have $(\chi_{E_n} f)(x) e^{-ix.\xi} \to f(x) e^{-ix.\xi}$ as $n \to \infty$. Also,

$$|(\chi_{E_n} f)(x) e^{-ix.\xi}| \le |f(x)|, \quad f \in L^1(\mathbb{R}^d).$$

Hence, by DCT,

$$\int_{E_n} f(x) e^{-ix.\xi} dx \to \int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx = \hat{f}(\xi) \quad \text{as} \quad n \to \infty$$

for every $\xi \in \mathbb{R}^d$. ∎

What about for $f \in L^2(\mathbb{R}^d)$? Here is the answer.

Theorem 3.6.9 *Let $f \in L^2(\mathbb{R})$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \le n\}$. The following hold.*

1. *$f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $n \in \mathbb{N}$.*
2. *$\|f - f_n\|_2 \to 0$ as $n \to \infty$.*
3. *$(\hat{f}_n)$ is a Cauchy sequence in $L^2(\mathbb{R})$.*
4. *$\|\hat{f}_n - \Phi(f)\|_2 \to 0$ as $n \to \infty$, where Φ is the Fourier-Plancheral transform.*
5. *There exists a subsequence $(\hat{f}_{k_n})$ for $(\hat{f}_n)$ such that $\hat{f}_{k_n} \to \Phi(f)$ a.e. on $\mathbb{R}$.*

Proof. (1) We have

$$\int_{\mathbb{R}} |f_n(x)|dx = \int_{\mathbb{R}} \chi_{E_n}(x)|f(x)|dx \le \|f\|_2[\mu(E_n)]^{1/2},$$

$$\int_{\mathbb{R}} |f_n(x)|^2 dx \le \|f\|_2^2.$$

Hence, $f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $n \in \mathbb{N}$.

(2) We have

$$\int_{\mathbb{R}} |f - f_n(x)|^2 dx = \int_{\mathbb{R}} [1 - \chi_{E_n}(x)] f(x)|^2 dx = \int_{\mathbb{R}} \chi_{E_n^c}(x)|f(x)|^2 dx,$$

$\chi_{E_n^c}(x)|f(x)|^2 \to 0$ as $n \to \infty$ and $\chi_{E_n^c}(x)|f(x)|^2 \le |f(x)|^2$ with $|f|^2 \in L^1(\mathbb{R})$ so that by DCT, $\|f - f_n\|_2^2 = \int_{\mathbb{R}} |f - f_n(x)|^2 dx \to 0$ as $n \to \infty$.

(3) We have

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|\widehat{f_n - f_m}\|_2 = \sqrt{2\pi}\|f_n - f_m\|_2.$$

Hence, by (2), $\|\hat{f}_n - \hat{f}_m\|_2 = \sqrt{2\pi}\|f_n - f_m\|_2 \to 0$ as $n, m \to \infty$.

(4) By (2) and Fourier-Plancheral theorem, we have

$$\|\hat{f}_n - \Phi(f)\|_2 = \|\Phi(f_n) - \Phi(f)\|_2 = \sqrt{2\pi}\|f_n - f\|_2 \to 0 \quad \text{as} \quad n \to \infty.$$

(5) This follows from (4). ∎

Remark 3.6.10 For $f \in L^2(\mathbb{R})$, let

$$P_r f = f\chi_{[-r,r]}, \quad r > 0.$$

Then $P_r f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $P_r : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an orthogonal projection satisfying

$$\|P_r f - f\|_2 \to 0 \quad \text{and} \quad \|\widehat{P_r f} - \Phi(f)\|_2 \to 0 \quad \text{as} \quad r \to \infty.$$

$\Diamond$

Theorem 3.6.11 *For $f \in L^2(\mathbb{R})$ and $r > 0$, let*

$$g_r(t) := \int_{-r}^{r} f(x)e^{-itx}dx, \quad h_r(x) := \int_{-r}^{r} \Phi(f)(t)e^{itx}dt$$

for all $x, t \in \mathbb{R}$. Then $g_r \in L^2(\mathbb{R})$, $h_r \in L^2(\mathbb{R})$, and

$$\|g_r - \Phi(f)\|_2 \to 0, \quad \|h_r - f\|_2 \to 0 \quad \textit{as} \quad r \to \infty.$$

Proof. Exercise. ∎

3.7 Problems

1. Let $f \in L^1(\mathbb{R}^d)$. Prove the following:

   (a) $\hat{f}(\xi)$ is well-defined for every $\xi \in \mathbb{R}^d$.

   (b) $\sup_{\xi \in \mathbb{R}^d)} |\hat{f}(\xi)| \le \|f\|_1$.

   (c) $\xi \to \hat{f}(\xi)$ is uniformly continuous on $\mathbb{R}^d$.

   (d) The map $f \mapsto \hat{f}$ is a linear operator from $L^1(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$ with norm at most 1.

   (e) $|\hat{f}(\xi)| \to 0$ as $|\xi| \to 0$.

2. Let $1 \le p < \infty$. For $f \in L^p(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, let

   $$(\tau_y f)(x) := f(x - y), \quad x \in \mathbb{R}^d.$$

   For each $f \in L^p(\mathbb{R}^d)$, prove the following:

(a) $\tau_y f \in L^p(\mathbb{R}^d)$ for every $y \in \mathbb{R}^d$.

(b) The map $y \mapsto \tau_y f$ from $\mathbb{R}^d$ to $L^p(\mathbb{R}^d)$ is continuous.

3. Prove that, if $f \in L^1(\mathbb{R})$ is differentiable with $f' \in L^1(\mathbb{R})$, then

$$\widehat{f'}(\xi) = (i\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

4. Suppose $f \in L^1(\mathbb{R})$ such that $x \mapsto g(x) := xf(x)$ belongs to $L^1(\mathbb{R})$. Prove that $\hat{f}$ is differentiable and

$$(\hat{f})'(\xi) = i\hat{g}(\xi), \quad \xi \in \mathbb{R}.$$

5. Let

$$\begin{aligned} (e_h f)(x) &= e^{ih\cdot x} f(x), \qquad & (\tau_h f)(x) &= f(x-h), \\ (\mathcal{R}f)(x) &= f(-x), \qquad & (D_t f)(x) &= f(tx). \end{aligned}$$

For $f \in L^1(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, $0 \neq t \in \mathbb{R}$, prove the following.

(a) $\widehat{e_h f} = \tau_h \hat{f}$,

(b) $\widehat{\tau_h f} = e_{-h} \hat{f}$,

(c) $\widehat{\mathcal{R}f} = \mathcal{R}\hat{f}$,

(d) $\widehat{D_t f} = |t|^{-d} D_{1/t} \hat{f}$, $0 \neq t \in \mathbb{R}$.

6. Prove that the operator $f \mapsto \hat{f}$ from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$ is not onto.

7. Suppose $f \in L^1(\mathbb{R})$. If $\hat{f} \in L^1(\mathbb{R})$ and if

$$g(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt, \quad x \in \mathbb{R},$$

then prove that $f = g$ a.e. Also deduce the following.

(a) If $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt \quad \forall\, x \in \mathbb{R}.$$

(b) $f \mapsto \hat{f}$ is an injective operator.

8. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $0 < \varphi(x) \leq 1$ for all $x \in \mathbb{R}$ and $\varphi(\lambda x) := \varphi(\lambda x) \to 1$ as $\lambda \to 0+$. Prove that, for every $g \in L^1(\mathbb{R})$,
$$\int_{\mathbb{R}} \varphi(\lambda t) g(t) e^{itx} dt \to \int_{\mathbb{R}} g(t) e^{itx} dt.$$

9. Let φ be as in Problem(8) and let ψ be defined by $\psi(x) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} \varphi(t) e^{itx} dt$ for $x \in \mathbb{R}$. If $f \in L^1(\mathbb{R})$ is such that $\hat{f} \in L^1(\mathbb{R})$ and if $\int_{\mathbb{R}} \psi(x) dx = 1$, then prove that
$$\frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) \hat{f}(t) e^{itx} dt = \int_{\mathbb{R}} f(x - \lambda s) \psi(s) ds.$$

10. Let $\psi \in L^1(\mathbb{R})$ be a non-negative function such that $\int_{\mathbb{R}} \psi(x) dx = 1$ and for $f \in L^1(\mathbb{R})$ and $\lambda > 0$, let
$$h_\lambda(x) := \int_{\mathbb{R}} f(x - \lambda s) \psi(s) ds.$$
Prove that $\|h_\lambda - f\|_1 \to 0$ as $\lambda \to 0$. Deduce the following.
    (a) There exists a sequence (λ_n) of positive real numbers such that $h_{\lambda_n} \to f$ a.e.
    (b) If φ is as in Problem(8), $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, and
    $$f_\lambda(x) := \int_{\mathbb{R}} \varphi(\lambda t) \hat{f}(t) e^{itx} dt,$$
    then $\|f_\lambda - f\|_1 \to 0$ as $\lambda \to 0$.

11. Let $\varphi(x) = e^{-|x|}$. Verify the following.
    (a) φ satisfies the conditions in Problem(8).
    (b) $\psi(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t) e^{itx} dt$, $x \in \mathbb{R}$ satisfies $\int_{\mathbb{R}} \psi(x) dx = 1$.

12. Let ψ be as in Problem(9). If $f \in L^p(\mathbb{R})$ for $1 \leq p < \infty$, then prove that $f * \psi_\lambda \in L^p(\mathbb{R})$ and $\lim\limits_{\lambda \to 0} \|f * \psi_\lambda - f\|_p = 0$.

13. Let ψ be as in Problem(9) and $f \in L^\infty(\mathbb{R})$. Prove that if f is continuous at a point $x \in \mathbb{R}$, then $\lim_{\lambda \to 0}(f * \psi_\lambda)(x) = f(x)$, where ψ_λ is as in Problem(9).

14. Prove the following.

    (a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = 2\pi\|f\|_2$.

    (b) The set $Y := \{\hat{f} : f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$.

15. Prove that there exists a unique surjective continuous linear operator Φ from $L^2(\mathbb{R})$ onto itself such that $\Phi(f) = \hat{f}$ for every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Show also that $\|\Phi(f)\|_2 = 2\pi\|f\|_2$ for every $f \in L^2(\mathbb{R})$.

16. Let $f \in L^1(\mathbb{R}^d)$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \le n\}$. Prove that

$$\int_{E_n} f(x)e^{-ix.\xi}dx \to \int_{\mathbb{R}^d} f(x)e^{-ix.\xi}dx = \hat{f}(\xi) \quad \text{as} \quad n \to \infty$$

    for every $\xi \in \mathbb{R}^d$.

17. Let $f \in L^2(\mathbb{R})$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \le n\}$. Prove the following.

    (a) $f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $n \in \mathbb{N}$.

    (b) $\|f - f_n\|_2 \to 0$ as $n \to \infty$.

    (c) $(\hat{f}_n)$ is a Cauchy sequence in $L^2(\mathbb{R})$.

    (d) $\|\hat{f}_n - \Phi(f)\|_2 \to 0$ as $n \to \infty$, where Φ is the Fourier-Plancheral transform.

    (e) There exists a subsequence $(\hat{f}_{k_n})$ for $(\hat{f}_n)$ such that $\hat{f}_{k_n} \to \Phi(f)$ a.e. on $\mathbb{R}$.

18. For $f \in L^2(\mathbb{R})$ and $r > 0$, let

$$g_r(\xi) := \int_{-r}^{r} f(x)e^{-ix\xi}dx, \quad h_r(x) := \int_{-r}^{r} \Phi(f)(\xi)e^{ix\xi}dt$$

    for all $x, \xi \in \mathbb{R}$. Then $g_r \in L^2(\mathbb{R})$, $h_r \in L^2(\mathbb{R})$, and

$$\|g_r - \Phi(f)\|_2 \to 0, \quad \|h_r - f\|_2 \to 0 \quad \text{as} \quad r \to \infty.$$

4

Elements of distribution theory

4.1 Test functions and distributions

Let Ω be an open subset of $\mathbb{R}^d$. We shall define a sense of convergence in the vector space $C_c^\infty(\Omega)$ as follows:

Notation 4.1.1 We use the following notations: For $\alpha := (\alpha_1, \ldots, \alpha_d) \in N_0 := \mathbb{N} \cup \{0\}$,

$$\partial^\alpha f := \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \Big(\frac{\partial}{\partial x_d}\Big)^{\alpha_2} \cdots \Big(\frac{\partial}{\partial x_d}\Big)^{\alpha_d} f.$$

For $f : \Omega \to \mathbb{C}$, support of f is the closure of $\{x \in \Omega : f(x) \neq 0\}$, and it is denoted by supp(f). Thus,

$$\operatorname{supp}(f) := \operatorname{cl} \{x \in \Omega : f(x) \neq 0\}.$$

Definition 4.1.2 A sequence (φ_n) in $C_c^\infty(\Omega)$ is said to converge to $\varphi \in C_c^\infty(\Omega)$ if there exists a compact set $K \subset \Omega$ such that supp(φ_n) and supp(φ) are contained in K for all $n \in \mathbb{N}$ and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on Ω for every $\alpha \in \mathbb{N}_0^d$.

The space $C_c^\infty(\Omega)$ together with the above sense of convergence is denoted by $\mathcal{D}(\Omega)$, and it is called the **space of test functions**. $\Diamond$

Let us give an example of a function in $\mathcal{D}(\mathbb{R}^d)$:

Example 4.1.3 Let

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then $\psi \in \mathcal{D}(\mathbb{R}^d)$ with supp$(\psi) = \overline{B_1(0)}$. For $\varepsilon > 0$, let $\psi_\varepsilon(x) := \psi\left(\frac{x}{\varepsilon}\right)$. Then $\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ with supp$(\psi_\varepsilon) = \overline{B_e(0)}$. $\Diamond$

Definition 4.1.4 A **distribution** on Ω is a linear functional u on $\mathcal{D}(\Omega)$ such that for every (φ_n) in $\mathcal{D}(\Omega)$, $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ implies $u(\varphi_n) \to u(\varphi)$.

A sequence (u_n) of distributions on Ω is said to **converge** to a distribution u on Ω if

$$u_n(\varphi) \to u(\varphi) \quad \text{for every} \quad \varphi \in \mathcal{D}(\Omega).$$

The set of all distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. ♢

Notation 4.1.5 We denote by $L^1_{\text{loc}}(\Omega)$, the space of all complex valued measurable functions f on Ω such that

$$\int_K |f(x)|dx < \infty \quad \text{for all compact } K \subseteq \Omega.$$

- For $1 \le p \le \infty$, $L^p(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$ for every p with $1 \le p < \infty$: Indeed, if $f \in L^p(\Omega)$ and compact $K \subseteq \Omega$ implies,

$$\int_K |f(x)|dx \le \|f\|_p \|\chi_K\|_q.$$

Example 4.1.6 Corresponding to $f \in L^1_{\text{loc}}(\Omega)$, let

$$u_f(\varphi) := \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega).$$

Then u_f is a distribution on Ω:

First observe that, for $\varphi \in \mathcal{D}(\Omega)$ if $K := \text{supp}(\varphi)$, then

$$\int_\Omega |f(x)\varphi(x)|dx = \int_K |f(x)\varphi(x)|dx \le \|f\|_\infty \int_K |f(x)|dx.$$

Hence, the integral in the definition of u_f is well-defined. Next, let (φ_n) in $\mathcal{D}(\Omega)$ be such that $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Let K be a compact set containing $\text{supp}(\varphi)$ and $\text{supp}(\varphi_n)$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} |u_f(\varphi_n) - u_f(\varphi)| &= |u_f(\varphi_n - \varphi)| \\ &\le \int_K |f(x)|\,|(\varphi_n - \varphi)(x)|dx \\ &\le \|\varphi_n - \varphi\|_\infty \int_K |f(x)|dx. \end{aligned}$$

Hence, $u(\varphi_n) \to u(\varphi)$. ♢

4.1.1 Regular distributions

Definition 4.1.7 A distribution u on Ω is called a **regular distributions** if $u = u_f$ for some $f \in L^1_{\text{loc}}(\Omega)$, and in that case u_f is said to be the distribution[1] generated by f. ♢

There are distributions that are not regular.

Example 4.1.8 For $a \in \Omega$, let

$$\delta_a(\varphi) := \varphi(a), \quad \varphi \in \mathcal{D}(\Omega).$$

It is easily seen that δ_a is a distribution on Ω. This distribution is not a regular distribution: To see this, suppose there exists $f \in L^1_{\text{loc}}(\Omega)$ such that $\delta_a = u_f$, i.e.,

$$\varphi(a) = \int_\Omega f(x)\varphi(x)dx \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega).$$

Let ψ be as in Example 4.1.3, that is,

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Let $\varepsilon > 0$ small enough such that $B_\varepsilon(a) \subseteq \Omega$, and let

$$\psi_\varepsilon(x) := \psi\Big(\frac{x-a}{\varepsilon}\Big).$$

Then $\psi_\varepsilon \in \mathcal{D}(\Omega)$ and $\text{supp}(\psi_\varepsilon) = \overline{B_\varepsilon(a)}$ and we have

$$\psi_\varepsilon(a) = \int_\Omega f(x)\psi_\varepsilon(x)dx = \int_{|x-a|<\varepsilon} f(x)\psi_\varepsilon(x)dx.$$

Note that

$$\left|\int_{|x-a|<\varepsilon} f(x)\psi_\varepsilon(x)dx\right| \leq \int_{|x-a|<\varepsilon} |f(x)|dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus,

$$|\psi_\varepsilon(a)| \leq \int_{|x-a|<\varepsilon} |f(x)|dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This is a contradiction, since $\psi_\varepsilon(a) \neq 0$. ♢

[1] We shall prove that a regular distribution can be generated by only one function in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Definition 4.1.9 The distribution δ_a in Example 4.1.8 is called the **Dirac delta distribution** at a. $\Diamond$

Theorem 4.1.10 *There exists a sequence u_n of regular distributions which converge to a delta distribution.*

Proof. We consider the case of $d = 1$. Let $f_n := \frac{n}{2}\chi_{E_n}$, where $E_n := \{x \in \Omega : |x - a| < 1/n\}$, and let $u_n := u_{f_n}$. Let $\varphi \in \mathcal{D}(\Omega)$. Then

$$u_n(\varphi) = \frac{n}{2}\int_{|x-a|<1/n} \varphi(x)dx.$$

Note that

$$u_n(\varphi) = \frac{n}{2}\int_{|x-a|<1/n} \varphi(x)dx = \frac{n}{2}\int_{|x-a|<1/n} [\varphi(x) - \varphi(a)]dx + \varphi(a)$$

and

$$\frac{n}{2}\int_{|x-a|<1/n} |\varphi(x) - \varphi(a)|dx \leq \max_{|x-a|<1/n} |\varphi(x) - \varphi(a)| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $u_n(\varphi) \to \varphi(a)$ as $n \to \infty$. ∎

Example 4.1.11 For $n \in \mathbb{N}$, let

$$u_n(\varphi) := \int_{\mathbb{R}} \varphi(x)e^{inx}dx, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Note that, defining $f_n(x) := e^{inx}$, $x \in \mathbb{R}$, $u_n = u_{f_n}$. Thus u_n is a regular distribution for every $n \in \mathbb{N}$. Further, by Riemann-Lebesgue lemma,

$$u_n(\varphi) = \int_{\mathbb{R}} \varphi(x)e^{inx}dx \to 0 \quad \text{as} \quad n \to \infty$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$. Thus, (u_n) converges to the zero distribution. $\Diamond$

Remark 4.1.12 In the books on *signals and systems* one comes across a function called **implulse function**.

It is defined as a function $\delta : \mathbb{R} \to [0, \infty]$ such that

1. $\displaystyle\int_{-\infty}^{\infty} \delta(x)dx = 1,$

2. $\delta(x) = 0 \quad \text{for} \quad x \neq 0$, and

3. $\delta(0) = \infty$.

Unfortunately, there is no function having the above two properties! $\Diamond$

Even though we can define a function $\delta : \mathbb{R} \to [0, \infty]$ satisfying

1. $\delta(x) = 0 \quad \text{for} \quad x \neq 0$, and

2. $\delta(0) = \infty$,

such a function cannot satisfy the requirement $\displaystyle\int_{-\infty}^{\infty} \delta(x)dx = 1$.

Then what does one have?

We can only have an ε-*impulse function* which can be defined as follows:

Definition 4.1.13 For $\varepsilon > 0$, an **ε-impulse function** is a non-negative function $\delta_\varepsilon(x)$ defined for $-\infty < x < \infty$ such that

1. $\displaystyle\int_{-\infty}^{\infty} \delta_\varepsilon(x)dx = 1$,

2. $\delta_\varepsilon(x) = 0 \quad \text{for} \quad |x| > \varepsilon$.

3. $\delta_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$.

$\Diamond$

Example 4.1.14 (i) Define $\delta_\varepsilon(x)$ to be a function whose graph is an isosceles triangle with base $[-\varepsilon, \varepsilon]$ and hight $1/\varepsilon$. Then δ_ε is an ε-impulse function.

(ii) Define $\delta_\varepsilon(x)$ to be $1/2\varepsilon$ in the interval $[-\varepsilon, \varepsilon]$ and 0 elsewhere. Then δ_ε is an ε-impulse function.

(iii) Let ψ be as in Example 4.1.3 and $\varphi(x) := C_0\psi(x)$ with $C_0 := 1/\int_{\mathbb{R}^d} \psi(x)dx$. Let

$$\varphi_{a,\varepsilon}(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x-a}{\varepsilon}\Big), \quad x \in \mathbb{R}^d.$$

Then $\varphi_{0,\varepsilon}$ is an ε-impulse function. $\Diamond$

Theorem 4.1.15 *For $\varepsilon > 0$, if δ_ε is an ε-impulse function, then $u_{\delta_\varepsilon} \to \delta_0$ as $\varepsilon \to 0$, where δ_0 is the delta-distribution at 0.*

Proof. The proof is along the same line as that of Theorem 4.1.10:

Let φ be a continuous function defined on $\mathbb{R}$ and $\delta_\varepsilon(x)$ is an ε-impulse function. Then we have

$$\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx = \int_{-\varepsilon}^{\varepsilon} \varphi(x)\delta_\varepsilon(x)dx.$$

Hence,

$$\left|\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx - \varphi(0)\right| = \left|\int_{-\varepsilon}^{\varepsilon} \varphi(x)\delta_\varepsilon(x)dx - \int_{\varepsilon}^{\varepsilon} \varphi(0)\delta_\varepsilon(x)dx\right|.$$

Thus we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx - \varphi(0)\right| &\leq \int_{-\varepsilon}^{\varepsilon} |\varphi(x) - \varphi(0)|\delta_\varepsilon(x)dx \\ &\leq \sup_{|x|<\varepsilon} |\varphi(x) - \varphi(0)| \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x)dx \\ &\leq \sup_{|x|<\varepsilon} |\varphi(x) - \varphi(0)|.\end{aligned}$$

Since φ is uniformly continuous, $\sup_{|x|<\varepsilon} |\varphi(x) - \varphi(0)| \to 0$ as $\varepsilon \to 0$. Thus,

$$\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx \to \varphi(0) \quad \text{as} \quad \varepsilon \to 0.$$

Hence, $u_{\delta_\varepsilon}(\varphi) \to \delta_0(\varphi)$ as $\varepsilon \to 0$. where δ_0 is the *delta distribution* at 0. ∎

In view of the following theorem, regular distributions can be identified with the functions that correspond to them. That is, regular distributions are uniquely defined by functions in $L^1_{\text{loc}}(\Omega)$.

Theorem 4.1.16 (Uniqueness theorem) *For* $f, g \in L^1_{\text{loc}}(\Omega)$,

$$u_f = u_g \quad \Rightarrow f = g \quad a.e.$$

For proving the above theorem we shall make use of some definitions and results.

4.1.2 Mollifiers

Throughout, we shall make use of a special type of functions in $C_c^\infty(\Omega)$, called *mollifiers*. In the due course it will be clear why such functions are called mollifiers.

Definition 4.1.17 A non-negative function φ on $\mathbb{R}^d$ is called a **mollifier** if

$$\varphi \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\varphi) \subseteq \overline{B_1(0)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi(x)dx = 1.$$

$\Diamond$

Given a mollifier φ and $\varepsilon > 0$, let

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x}{\varepsilon}\Big).$$

Then

$$\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\varphi_\varepsilon) \subset \overline{B_\varepsilon(0)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi_\varepsilon(x)dx = 1.$$

Also, for any $a \in \mathbb{R}^d$ and $\varepsilon > 0$, the function $\varphi_{\varepsilon,a}$ defined by

$$\varphi_{\varepsilon,a}(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x-a}{\varepsilon}\Big)$$

satisfies

$$\varphi_{\varepsilon,a} \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\varphi_{\varepsilon,a}) \subset \overline{B_\varepsilon(a)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi_{\varepsilon,a}(x)dx = 1.$$

Observe that

$$\varphi_{\varepsilon,a}(a) := \frac{\varphi(0)}{\varepsilon^d}.$$

In particular, $\varphi_{\varepsilon,a}(a) \to \infty$ as $\varepsilon \to 0$.

Here is an example of a mollifier.

Example 4.1.18 The function φ in Example 4.1.14 (iii) is a mollifier. $\Diamond$

4.1.3 Convolution revisited

The **convolution** of measurable functions $f, g : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy, \quad x \in \mathbb{R}^d,$$

whenever the above integral exists, and in that case it is also equal to

$$\int_{\mathbb{R}^d} f(y)g(x-y)dy, \quad x \in \mathbb{R}^d.$$

We shall also define **convolution** of measurable functions $f, g : \Omega \to \mathbb{C}$ by

$$(f * g)(x) = \int_{\Omega} f(x-y)g(y)dy, \quad x \in \mathbb{R}^d,$$

by extending f and g to all of $\mathbb{R}^d$ by assigning the value 0 on Ω^c.

Throughout we shall use the notation φ_ε for the function defined as Section 4.1.2, that is,

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x}{\varepsilon}\Big), \quad x \in \mathbb{R}^d,$$

where φ is a mollifier.

Theorem 4.1.19 *Let $1 \leq p < \infty$. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, then*

$$f * g \in L^p(\mathbb{R}^d) \quad \textit{and} \quad \|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Proof. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$. We already know that if $p = 1$, then $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Next, let $1 < p < \infty$ and let q such that $(1/p) + (1/q) = 1$. Then

$$\begin{aligned}
|f * g)(x)| &\leq \int |f(x-y)g(y)|dy \\
&\leq \int |f(x-y)|\,|g(y)|^{1/p}|g(y)|^{1/q}dy \\
&\leq \left(\int |f(x-y)|)|^p|g(y)|)dy\right)^{1/p} \left(\int |g(y)|dy\right)^{1/q} \\
&= \left(\int |f(x-y)|)|^p|g(y)|dy\right)^{1/p} \|g\|_1^{1/q}.
\end{aligned}$$

Hence,

$$\begin{aligned}\int |(f*g)(x)|^p dx &= \|g\|_1^{p/q} \int \left(\int |f(x-y)|)|^p |g(y)| dy \right) dx \\ &= \|g\|_1^{p/q} \int \left(\int |f(x-y)|)|^p dx \right) |g(y)| dy \\ &= \|g\|_1^{1+\frac{p}{q}} \|f\|_p^p\end{aligned}$$

so that

$$\left(\int |(f*g)(x)|^p dx \right)^{1/p} = \|g\|_1^{\frac{1}{p}+\frac{1}{q}} \|f\|_p = \|g\|_1 \|f\|_p.$$

Thus, $f*g \in L^p(\mathbb{R}^d)$ and $\|f*g\|_p \le \|f\|_p \|g\|_1$. ∎

The proof of the following theorem is easy, and hence we omit its proof.

Theorem 4.1.20 *If $f, g \in L^p(\mathbb{R}^d)$ with compact support, then*

$$\text{supp}(f*g) \subseteq \text{supp}(f) + \text{supp}(g).$$

Theorem 4.1.21 *Suppose $f \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$ and $g \in C^k(\mathbb{R}^d)$ with $\partial^\alpha g \in L^q(\mathbb{R}^d)$ for $|\alpha| \le k$. Then*

$$f*g \in C^k(\mathbb{R}^d) \quad \textit{and} \quad \partial^\alpha (f*g) = f*\partial^\alpha g \quad \textit{for} \quad |\alpha| \le k.$$

In particular, if $f \in L^p(\mathbb{R}^d)$ is with compact support and $g \in C_c^\infty(\mathbb{R}^d)$, then $f*g \in C_c^\infty(\mathbb{R}^d)$ and and $\partial^\alpha(f*g) = f*\partial^\alpha g$ for all $\alpha \in \mathbb{N}_0^d$.

Proof. We prove the case for $p = 1$ and $k = 1$, i.e., $|\alpha| = 1$. Proof of the case of $k > 1$ will follow similarly. The case of $p > 1$ involves more calculations, and it is left to the reader to workout the details.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ and let j be such that $\alpha_j = 1$ and $\alpha_i = 0$ for $i \ne j$. We have to show that

$$\lim_{h\to 0} \frac{(f*g)(x+he_j) - (f*g)(x)}{h} \quad \text{exists}$$

and it is equal to $(f*\partial_j g)(x)$. Note that

$$\frac{(f*g)(x+he_j) - (f*g)(x)}{h} = \int_{\mathbb{R}^d} f(y) \frac{g(x-y+he_j) - g(x-y)}{h} dy.$$

Since

$$\frac{g(x-y+he_j)-g(x-y)}{h} \to \partial_j g(x-y) \quad \text{as} \quad h \to 0 \quad \text{and} \quad \partial_j g \in L^\infty(\Omega),$$

there exists $\alpha > 0$ such that for all h with $|h| \leq \alpha$,

$$|f(y)| \left| \frac{g(x-y+he_j)-g(x-y)}{h} \right| \leq |f(y)|(|\partial_j g(x-y)|+1).$$

Since $y \mapsto |f(y)|(|\partial_j g(x-y)|+1)$ belongs to $L^1(\Omega)$, by DCT, we have

$$\int_{\mathbb{R}^d} f(y) \frac{g(x-y+he_j)-g(x-y)}{h} dy \to \int_{\mathbb{R}^d} f(y) \partial_j g(x-y) dy.$$

Thus, $\partial_j(f * g)$ exists and $\partial_j(f * g) = f * \partial_j g$. The particular case follows from Theorem 4.1.20. ∎

Theorem 4.1.22 *Let $L^p(\Omega)$ for $1 \leq p < \infty$, and for $\varepsilon > 0$, let φ_ε be as in Section 4.1.2. Then $f * \varphi_\varepsilon \in C^\infty(\Omega) \cap L^p(\mathbb{R}^d)$ and*

$$\|f * \varphi_\varepsilon - f\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proof. By Theorem 4.1.21, $f * \varphi_\varepsilon \in C^\infty(\Omega)$. If $\Omega \neq \mathbb{R}^d$, then we extend f to all of $\mathbb{R}^d$ by defining it to be zero on Ω^c. First let $p = 1$. Then we have

$$\begin{aligned}
\int |f(x) - (f * \varphi_\varepsilon)(x)| dx &\leq \int \left| \int [f(x) - f(x-y)] \varphi_\varepsilon(y) dy \right| dx \\
&= \int \int |f(x) - f(x-y)| \varphi_\varepsilon(y) dy dx \\
&\leq \int \left(\int |f(x) - f(x-y)| dx \right) \varphi_\varepsilon(y) dy \\
&= \int \|f - \tau_y f\|_1 \varphi_\varepsilon(y) dy.
\end{aligned}$$

Next let $1 < p < \infty$. Then we have

$$\begin{aligned}
|f(x) - (f * \varphi_\varepsilon)(x)| &\leq \int |f(x) - f(x-y)| \varphi_\varepsilon(y) dy \\
&\leq \int |f(x) - f(x-y)| [\varphi_\varepsilon(y)]^{1/p} [\varphi_\varepsilon(y)]^{1/q} dy \\
&\leq \left(\int |f(x) - f(x-y)|)|^p \varphi_\varepsilon(y) dy \right)^{1/p} \left(\int \varphi_\varepsilon(y) dy \right)^{1/q} \\
&= \left(\int |f(x) - f(x-y)|)|^p \varphi_\varepsilon(y) dy \right)^{1/p}.
\end{aligned}$$

Hence,

$$\begin{aligned}\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx &\le \int\left(\int |f(x)-f(x-y)|)|^p\varphi_\varepsilon(y)dy\right)dx\\ &= \int\left(\int |f(x)-f(x-y)|)|^p dx\right)\varphi_\varepsilon(y)dy\\ &= \int \|f-\tau_y f\|_p^p\varphi_\varepsilon(y)dy.\end{aligned}$$

Thus, for $1 \le p < \infty$, we have

$$\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx \le \int \|f-\tau_y f\|_p^p\varphi_\varepsilon(y)dy.$$

Now, recall that $\|f-\tau_y f\|_p^p \to 0$ as $y \to 0$. Therefore, for any given $\eta > 0$, there exists $\delta > 0$ such that

$$\|f-\tau_y f\|_p^p < \eta \quad \text{whenever} \quad |y| < \delta.$$

Hence,

$$\begin{aligned}\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx &\le \int \|f-\tau_y f\|_p^p\varphi_\varepsilon(y)dy\\ &= \int_{|y|<\delta} \|f-\tau_y f\|_p^p\varphi_\varepsilon(y)dy + \int_{|y|\ge\delta} \|f-\tau_y f\|_p^p\varphi_\varepsilon(y)dy\\ &\le \eta\int_{|y|<\delta}\varphi_\varepsilon(y)dy + (2\|f\|_p)^p\int_{|y|\ge\delta}\varphi_\varepsilon(y)dy.\end{aligned}$$

Since $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, $\int_{|y|\ge\delta}\varphi_\varepsilon(y)dy \to 0$ as $\varepsilon \to 0$. Hence, there exists $\varepsilon_0 > 0$ such that

$$\int_{|y|\ge\delta}\varphi_\varepsilon(y)dy < \eta \quad \text{whenever} \quad 0 < \varepsilon < \varepsilon_0.$$

Thus, we obtain

$$\begin{aligned}\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx &\le \eta\int_{|y|<\delta}\varphi_\varepsilon(y)dy + (2\|f\|_p)^p\int_{|y|\ge\delta}\varphi_\varepsilon(y)dy\\ &\le (1+(2\|f\|_p)^p)\eta\end{aligned}$$

whenever $\varepsilon < \varepsilon_0$. Thus, $f*\varphi_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f*\varphi_\varepsilon - f\|_p \to 0$ as $\varepsilon \to 0$. ∎

Theorem 4.1.23 *$C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. The proof involves the following two steps:

1. For every $f \in L^p(\Omega)$ and $\varepsilon > 0$, there exists $g \in L^p(\Omega)$ with compact support such that $\|f - g\| < \varepsilon$.

2. For every $g \in L^p(\Omega)$ with compact support, $g * \varphi_\varepsilon \in C_c^\infty(\Omega)$ and $\|g - g * \varphi_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Proof of Step (1): Let $f \in L^p(\Omega)$. For $n \in \mathbb{N}$, let

$$K_n = \{x \in \Omega : |x| \leq n, \operatorname{dist}(x, \Omega^c) \geq 1/n\}.$$

Then each K_n is a compact subset of Ω. Taking $f_n := f\chi_{K_n}$, we see that $f_n \in L^p(\Omega)$ with $\operatorname{supp}(f_n) \subseteq K_n$ and

$$\|f - f_n\|_p \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, given $\varepsilon > 0$, there exists $g := f_N$ such that $\|f - g\|_p < \varepsilon$.

Proof of Step (2): Let $g \in L^p(\Omega)$ with compact support. Let φ be a mollifier and $\varepsilon > 0$ be given. By Theorem 4.1.21, $g * \varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$. We may take ε small enough such that $\operatorname{supp}(g * \varphi_\varepsilon) \subseteq \Omega$. Also, by Theorem 4.1.22,

$$\|g - (g * \varphi_\varepsilon)\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Now, let $f \in L^p(\Omega)$ and $\varepsilon > 0$. Then by Step (1), there exists $g \in L^p(\Omega)$ with compact support such that $\|f-g\|_p < \varepsilon$ and by Step (2), $g*\varphi_\varepsilon \in C_c^\infty(\Omega)$ and $\|g - g * \varphi_\varepsilon\| \to 0$ as $\varepsilon \to 0$. Thus,

$$\|f - g * \varphi_\varepsilon\|_p \leq \|f - g\|_p + \|g - g * \varphi_\varepsilon\|_p \to 0$$

as $\varepsilon \to 0$. This completes the proof. ∎

We have proved in Theorem 4.1.22 that $\|f - f * \varphi_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$ for every $f \in L^p(\Omega)$ with $1 \leq p < \infty$. The next theorem show that the convergence can be stronger if $f \in C_c(\Omega)$.

Theorem 4.1.24 *Suppose $f \in C_c(\Omega)$. Then $f * \varphi_\varepsilon \to f$ uniformly on Ω.*

Proof. For $x \in \Omega$, we have

$$\begin{aligned} |f(x) - (f * \varphi_\varepsilon)(x)| &\leq \int |f(x) - f(x-y)|\varphi_\varepsilon(y)dy \\ &\leq \sup_{|y|<\varepsilon} |f(x) - f(x-y)| \int_{|y|<\varepsilon} \varphi_\varepsilon(y)dy \\ &\leq \sup_{|y|<\varepsilon} |f(x) - f(x-y)|. \end{aligned}$$

Since f is uniformly continuous on supp(f), given $\eta > 0$, there exists $\varepsilon > 0$ such that

$$\sup_{|x-u|<\varepsilon} |f(x) - f(u)| < \eta.$$

Thus, $f * \varphi_\varepsilon \to f$ uniformly on Ω. ∎

4.1.4 Unique identifiability of regular distributions

We shall make use of the following proposition.

Proposition 4.1.25 *Let K is a compact subset of Ω. Then there exists $\psi \in \mathcal{D}(\Omega)$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on K.*

Proof. Let $\alpha > 0$ be such that the α-neighbourhood of K, namely

$$G_\alpha := \{x \in \Omega : \text{dist}(x, \Omega) < \alpha\}$$

is contained in Ω. Let φ be a mollifier and for $\varepsilon > 0$, let $\psi_\varepsilon := \varphi_\varepsilon * \chi_\alpha$, where $\chi_\alpha := \chi_{G_\alpha}$ and $\varphi_\varepsilon := (1/\varepsilon^d)\varphi(x/\varepsilon)$. Since $\chi_\alpha \in L^1(\mathbb{R}^d)$ is with compact support, by Theorem 4.1.21, $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$. Note that

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)\chi_\alpha(y)dy \leq \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)dy = 1.$$

Further, if $x \in K$ and $\varepsilon \leq \alpha$, then

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\chi_\alpha(x-y)dy = \int_{B_\varepsilon(0)} \varphi_\varepsilon(y)\chi_\alpha(x-y)dy = 1,$$

since

$$x \in K, \quad y \in B_\varepsilon(0) \quad \text{implies} \quad x - y \in G_\alpha.$$

Thus, $0 \leq \psi_\alpha \leq 1$ and $\psi_\alpha = 1$ on K.

Also,

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)\chi_\alpha(y)dy = 0$$

whenever x is not in the ε-neighbourhood of G_α. Since α-neighbourhood of G_α is contained in the 2α-neighbourhood of K, taking α small enough such that $G_{2\alpha} \subseteq \Omega$ and $\varepsilon < \alpha$, we have $\operatorname{supp}(\psi_\varepsilon) \subseteq G_{2\alpha} \subseteq \Omega$ so that $\psi_\varepsilon \in \mathcal{D}(\Omega)$. ∎

Theorem 4.1.26 (Uniqueness theorem) *For $f, g \in L^1_{loc}(\Omega)$, $u_f = u_g$ implies $f = g$ a.e.*

Proof. It is enough to proof that

$$f \in L^1_{\text{loc}}(\Omega), \quad u_f = 0 \quad \Rightarrow f = 0 \quad \text{a.e.}$$

So, let $f \in L^1_{\text{loc}}(\Omega)$ such that $u_f = 0$, i.e., $\int_\Omega f(x)\varphi(x)dx = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Let K be a compact subset of Ω and ψ be as in Proposition 4.1.25. Then $f\psi \in L^1(\mathbb{R}^d)$. This is seen as follows: Let $K_\psi := \operatorname{supp}(\psi)$. Then

$$\int_{\mathbb{R}^d} |f\psi| = \int_{K_\psi} |f\psi| \le \|\psi\|_\infty \int_{K_\psi} |f| < \infty.$$

Let φ be a mollifier on $\mathbb{R}^d$ and $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$. Then we have

$$(\varphi_\varepsilon * f\psi)(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)f(y)\psi(y)dy = 0$$

for every $x \in \mathbb{R}^d$ since $y \mapsto \varphi_\varepsilon(x-y)\psi(y)$ belongs to $\mathcal{D}(\Omega)$. Also, by Theorem 4.1.22, we have

$$\|\varphi_\varepsilon * f\psi - f\psi\|_1 \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence, $f\psi = 0$ in $L^1(\mathbb{R}^d)$ so that $f = 0$ a.e. on K. Since Ω can be written as a countable union of compact subsets it follows that $f = 0$ a.e. on Ω. ∎

Example 4.1.27 For each $k \in \mathbb{N}$, let

$$f_k(x) := \sum_{n=-k}^{k} e^{inx}, \quad x \in \mathbb{R}.$$

Then, we have

$$u_{f_k}(\varphi) = \int_{\mathbb{R}} f_k(x)\varphi(x) = \sum_{n=-k}^{k} \int_{\mathbb{R}} \varphi(x)e^{inx}dx = 2\pi \sum_{n=-k}^{k} \hat{\varphi}(-n).$$

Hence, for every $\varphi \in \mathcal{D}(\mathbb{R})$,

$$u_{f_k}(\varphi) \to 2\pi \sum_{n\in\mathbb{N}} \hat{\varphi}(n) = 2\pi\varphi(0) = 2\pi\delta_0(\varphi).$$

Thus, $u_{f_k} \to 2\pi\delta_0$ as $k \to \infty$. Identifying u_{f_k} with f_k, we may write the above fact as

$$\sum_{n\in\mathbb{Z}} e_n = 2\pi\delta_0,$$

where $e_n(x) := e^{inx}$. ♢

4.2 Properties of distributions

4.2.1 Differentiation of distributions

Let $f \in C^1(0,1) \cap C[0,1]$. Then for every $\varphi \in C_c^\infty(0,1)$, we have

$$\int_0^1 f'(x)\varphi(x)dx = [\varphi(x)f(x)]_0^1 - \int_0^1 \varphi'(x)f(x)dx = -\int_0^1 \varphi'(x)f(x)dx.$$

Thus,

$$u_{f'}(\varphi) = -u_f(\varphi').$$

More generally, it can be seen that:

> If $f \in C^1(\Omega) \cap C(\overline{\Omega})$, then for every $\varphi \in C_c^\infty(\Omega)$ and for every $\alpha \in \mathbb{N}_0^d$,

$$\int_\Omega (\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|} \int_0^1 f(x)(\partial^\alpha\varphi)(x)dx.$$

$$u_{\partial^\alpha f}(\varphi) = (-1)^{|\alpha|} u_f(\partial^\alpha\varphi).$$

Thus,

$$\boxed{u_{\partial^\alpha f}(\varphi) = (-1)^{|\alpha|} u_f(\partial^\alpha\varphi).}$$

Note that the r.h.s. of the above is well-defined for every $f \in L^1_{\rm loc}(\Omega)$, and it is easily seen that the map

$$\varphi \mapsto (-1)^{|\alpha|} u_f(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is a distribution.

Definition 4.2.1 For $f \in L^1_{\rm loc}(\Omega)$, the distribution

$$\varphi \mapsto (-1)^{|\alpha|} u_f(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is called the α**-th distributional derivative** of f, denoted by $\partial^\alpha f$. $\Diamond$

Thus for $f \in L^1_{\rm loc}(\Omega)$,

$$\boxed{\partial^\alpha f(\varphi) := (-1)^{|\alpha|} u_f(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega).}$$

In fact,

Theorem 4.2.2 *For any $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, the map*

$$\varphi \mapsto (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is a distribution.

Definition 4.2.3 For $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, the distribution

$$\varphi \mapsto (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is called the α**-th derivative** of u, and it is dented by $\partial^\alpha u$. $\Diamond$

Thus for $u \in \mathcal{D}'(\Omega)$

$$\boxed{(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega).}$$

In view of the above, for $f \in L^1_{\rm loc}(\Omega)$,

$$(\partial^\alpha f)(\varphi) = (\partial^\alpha u_f)(\varphi) = (-1)^{|\alpha|} u_f(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega).$$

Example 4.2.4 (i) Consider the *Heaveside function*:

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Then

$$\int_{\mathbb{R}} H(x)\varphi'(x)dx = \int_0^\infty \varphi'(x)dx = -\varphi(0) = -\delta_0(\varphi).$$

Thus, $H' = \delta_0$.

(ii) We have

$$\delta_0(\varphi') = \varphi'(0) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Hence,

$$\delta_0'(\varphi) = -\delta_0(\varphi') = -\varphi'(0), \quad \varphi \in \mathcal{D}(\Omega).$$

◊

Definition 4.2.5 Suppose $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. We say that $\partial^\alpha u$ **belongs to** $L^1_{\mathrm{loc}}(\Omega)$, and write as $\partial^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$, if there exists a function $f \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$(\partial^\alpha u)(\varphi) = u_f(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Similarly, we say that $\partial^\alpha u \in L^p(\Omega)$ iff there exists a function $f \in L^p(\Omega)$ such that

$$(\partial^\alpha u)(\varphi) = u_f(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

◊

Suppose $f \in L^1_{\mathrm{loc}}(\Omega)$.

1. We say that $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ iff there exists $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$(\partial^\alpha u_f)(\varphi) = u_g(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$

i.e., iff

$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx = \int_\Omega g(x)\varphi(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$

and this fact is also written as

$$\int_\Omega (\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

2. We say that $\partial^\alpha f \in L^p(\Omega)$ iff there exists $g \in L^p(\Omega)$ such that
$$(\partial^\alpha u_f)(\varphi) = u_g(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$
i.e., iff
$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx = \int_\Omega g(x)\varphi(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$
and this fact is also written as
$$\int_\Omega (\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

In view of the above, we have the following definition.

Definition 4.2.6 (Sobolev spaces) For $r \in \mathbb{N}_0$ and $1 \leq p \leq \infty$, the **Sobolev space** $W^{r,p}(\Omega)$ is the vector space
$$W^{r,p}(\Omega) := \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega)\ \forall\, |\alpha| \leq r\}.$$
$\Diamond$

Thus, if $f \in L^p(\Omega)$, then

- $f \in W^{r,p}(\Omega)$ iff there exists $g \in L^p(\Omega)$ such that
$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx = \int_\Omega g(x)\varphi(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

It is known that:

- For $1 \leq p < \infty$
$$\|f\|_{r,p} := \Big(\sum_{|\alpha| \leq r} \|\partial^\alpha f\|_p^p \Big)^{1/p}$$
defines a norm on $W^{r,p}(\Omega)$, which makes it a Banach space, and for $p = 2$,
- $H^r(\Omega) := W^{r,2}(\Omega)$ is a Hilbert space w.r.t. the inner product
$$\langle f, g\rangle_r := \sum_{|\alpha| \leq r} \langle \partial^\alpha f, \partial^\alpha g\rangle.$$

Theorem 4.2.7 *For every multi-index α, $u \mapsto \partial^\alpha u$ is continuous on $\mathcal{D}'(\Omega)$, i.e.,*
$$u_n \to u \ \text{in}\ \mathcal{D}'(\Omega) \ \Rightarrow\ \partial^\alpha u_n \to \partial^\alpha u \ \text{in}\ \mathcal{D}'(\Omega).$$

Proof. Follows from the definitions. ∎

4.2.2 A characterization of distributions

Theorem 4.2.8 *Let u be a linear functional on $\mathcal{D}(\Omega)$. Then u is a distribution if and only if for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that*

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty \tag{1}$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq K$.

Proof. Suppose u is a distribution. Assume for a moment that there exists a compact $K \subseteq \Omega$ such that (1) is not satisfied for any $C > 0$ and for any $N \in \mathbb{N}$. Then for every $N \in \mathbb{N}$ and for every $C > 0$, there exists φ, depending on (N, C), such that $\operatorname{supp}(\varphi) \subseteq K$ and

$$|u(\varphi)| > C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty.$$

In particular, for every $N \in \mathbb{N}$, there exists φ_N such that $\operatorname{supp}(\varphi_N) \subseteq K$ and

$$|u(\varphi_N)| > N \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi_N\|_\infty.$$

Let $\tilde{\varphi}_N := \varphi_N / |u(\varphi_N)|$, $N \in \mathbb{N}$. Then we have

$$1 = |u(\tilde{\varphi}_N)| > N \sum_{|\alpha| \leq N} \|\partial^\alpha \tilde{\varphi}_N\|_\infty \geq N \|\partial^\alpha \tilde{\varphi}_N\|_\infty$$

for all $N \in \mathbb{N}$. Hence, $\tilde{\varphi}_N \to 0$ in $\mathcal{D}(\Omega)$ as $N \to \infty$. But, $u(\tilde{\varphi}_N) = 1$ for all $N \in \mathbb{N}$. Thus, we arrived at a contradiction to the fact that u is a distribution.

Conversely, assume that u is a linear functional on $\mathcal{D}(\Omega)$ and for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that (1) is satisfied. Let (φ_n) in $\mathcal{D}(\Omega)$ such that $\varphi_n \to 0$ in $\mathcal{D}(\Omega)$. Let $K \subseteq \Omega$ be a compact set with $\operatorname{supp}(\varphi_n) \subseteq K$ for all $n \in \mathbb{N}$. Let $C > 0$ and $N \in \mathbb{N}_0$ be such that

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq K$. Then we have

$$|u(\varphi_n)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi_n\|_\infty.$$

By the assumption on (φ_n), $u(\varphi_n) \to 0$. ∎

4.2.3 Order of a distribution

Let u be a distribution on Ω. By Theorem 4.2.8, we know that for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that (1) in the theorem is satisfied.

Does there exist and $N \in \mathbb{N}_0$ such that (1) in the theorem is satisfied for every compact set $K \subseteq \Omega$? If such an N exists, then the least such N is called the *order of the distribution* u. More precisely, we have:

Definition 4.2.9 Let u be a distribution on Ω. Then u is said to be of **finite order**, if there exists $N \in \mathbb{N}_0$ such that for every compact $K \subseteq \Omega$ there is a $C_K > 0$ satisfying

$$|u(\varphi)| \leq C_K \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty \qquad (1)$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq K$, and in that case, the least such N is called the **order** of u. If the distribution u is not of finite order, then it is said to be of **infinite order**. $\Diamond$

Example 4.2.10 Every regular distribution is of finite order: To see this, let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$|u_f(\varphi)| \leq \int_\Omega |f(x)|\,|\varphi(x)| dx \leq \|\varphi\|_\infty \int_\Omega |f(x)| dx.$$

Thus, (1) in Theorem 4.2.8 is satisfied with $N = 0$ and $C = \int_\Omega |f(x)| dx$. $\Diamond$

Example 4.2.11 Define

$$u(\varphi) := \sum_{j=0}^{\infty} \varphi^{(j)}(j), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Note that, since φ is with compact support, the above is a finite sum for each φ. More precisely, if $\mathrm{supp}(\varphi) \subseteq [-k, k]$ for some $k \in \mathbb{N}$, then

$$u(\varphi) = \sum_{j=0}^{k-1} \varphi^{(j)}(j).$$

Further, if K is a compact set and if $K \subseteq [-k, k]$ for some $k \in \mathbb{N}$, then we have

$$|u(\varphi)| \leq \sum_{j=0}^{k-1} \|\varphi^{(j)}\|_\infty$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subseteq K$. Hence, by Theorem 4.2.8, $u \in \mathcal{D}'(\mathbb{R})$. This distribution is of infinite order (Why?). ◊

Exercise 4.2.12 *Show that the delta-distribution is of* 0 *order.*

Exercise 4.2.13 *Show that the distribution in Example 4.2.11 is of infinite order.*

4.2.4 Restriction and support of distributions

Let Ω_0 be an open set in $\mathbb{R}^\delta$ such that $\Omega_0 \subseteq \Omega$. Then $\mathcal{D}(\Omega_0)$ can be imbedded in $\mathcal{D}(\Omega)$ using the map

$$\varphi \mapsto \tilde{\varphi}, \qquad \varphi \in \mathcal{D}(\Omega_0),$$

where $\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in \Omega_0, \\ 0, & x \in \Omega \setminus \Omega_0. \end{cases}$

Note that, since $\varphi \in C_c(\Omega_0)$ implies that $\tilde{\varphi} \in \mathbb{C}_c(\Omega)$. With the above imbedding in mind, that is, denoting $\tilde{f}$ by φ, we may consider $\mathcal{D}(\Omega_0) \subseteq \mathcal{D}(\Omega)$.

Definition 4.2.14 Let u be a distribution on Ω and Ω_0 be an open subset of Ω. Then **restriction** of u to Ω_0, denoted by u_{Ω_0}, is the distribution on Ω_0 defined by

$$u_{\Omega_0}(\varphi) := u(\varphi) \quad \text{for every} \quad \varphi \in \mathcal{D}(\Omega_0).$$

◊

Definition 4.2.15 Let u be a distribution on Ω. Then the **support** of u is the set

$$\operatorname{supp}(u) := \{x \in \Omega : u_G \neq 0 \text{ for every open set } G \subset \Omega \text{ with } x \in G\}.$$

◊

Note that, for $u \in \mathcal{D}'(\Omega)$ and $x \in \Omega$,

$$x \notin \operatorname{supp}(u) \iff \exists \text{ open set } G \subset \Omega \text{ with } x \in G \text{ such that } u_G = 0.$$

Hence,

$$\text{supp}(u) = \Omega \setminus \bigcup \{G : u_G = 0\}.$$

Thus, supp(u) is a closed subset of Ω.

Exercise 4.2.16 supp$(\delta_a) = \{a\}$.

Exercise 4.2.17 *For* $f \in L^1_{\text{loc}}(\Omega)$, supp$(u_f) = $ supp(f).

4.2.5 Multiplication by C^∞ functions

We observe:

- If $f \in C^\infty(\Omega)$, then $f\varphi \in \mathcal{D}(\Omega)$ for every $\varphi \in \mathcal{D}(\Omega)$.
- If $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, then $f\varphi_n \to f\varphi$ in $\mathcal{D}(\Omega)$. Hence, $u(f\varphi_n) \to u(f\varphi)$.

Theorem 4.2.18 *For each* $f \in C^\infty(\Omega)$ *and* $u \in \mathcal{D}'(\Omega)$, $\varphi \mapsto u(f\varphi)$ *is a distribution on* Ω.

Definition 4.2.19 For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, the **multiplication** of u by f is the distribution fu defined by $(fu)(\varphi) = u(f\varphi)$, $\varphi \in \mathcal{D}(\Omega)$. ◊

Example 4.2.20 $f \in C^\infty(\Omega)$ and $a \in \Omega$, we have

$$(f\delta_a)(\varphi) = \delta_a(f\varphi) = f(a)\varphi(a) = f(a)\delta_a(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Hence, $f\delta_a = f(a)\delta$. ◊

Example 4.2.21 If $f \in C^\infty(\Omega)$ and $g \in L^1_{\text{loc}}(\Omega)$, we have

$$(fu_g)(\varphi) = u_g(f\varphi) = \int g(x)f(x)\varphi(x)dx = u_{fg}(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Hence, $fu_g = u_{fg}$.

Also, if $f, g \in C^\infty(\Omega)$, then $fu_g = u_{fg} = gu_f$. ◊

Theorem 4.2.22 *Let* $f \in C^\infty(\Omega)$. *Then the map* $u \mapsto fu$ *is continuous in the sense that*

$$u_n \to u \text{ in } \mathcal{D}'(\Omega) \quad \Rightarrow \quad fu_n \to fu \text{ in } \mathcal{D}'(\Omega).$$

4.2.6 Translation of distributions

We observe that if $f \in L^1_{\rm loc}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$,

$$u_{\tau_h f}(\varphi) = \int (\tau_h f)(x)\varphi(x)dx = \int f(x-h)\varphi(x)dx = \int f(x)\varphi(x+h)dx = u_f(\tau_{-h}\varphi).$$

Identifying $L^1_{\rm loc}$-functions with the corresponding distributions, we may write the above as

$$(\tau_h f)(\varphi) = f(\tau_{-h}\varphi) \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Motivated by this, for $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, we may define

$$\boxed{(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).}$$

Theorem 4.2.23 *If $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, then $\tau_h u$ defined by*

$$(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

is distribution.

Definition 4.2.24 For $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, the distribution $\tau_h u$ defined by

$$(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

is called the **translation** of u by h. ♢

Example 4.2.25 Observe that

$$(\tau_h \delta_a)(\varphi) = \delta_a(\tau_{-h}\varphi) = (\tau_{-h}\varphi)(a) = \varphi(a+h) = \delta_{a+h}(\varphi).$$

Hence, $\tau_h \delta_a = \delta_{a+h}$. ♢

Theorem 4.2.26 *For each $h \in \mathbb{R}^d$, the map $u \mapsto \tau_h u$ is continuous on $\mathcal{D}'(\mathbb{R}^d)$ in the sense that $u_n \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ implies $\tau_h u_n \to \tau_h u$ in $\mathcal{D}'(\mathbb{R}^d)$.*

4.2.7 Convolution involving distributions

Convolution of a distribution and a function

Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then we have

$$(f * \varphi)(x) = \int_{\mathbb{R}^d} f(y)\varphi(x-y), \quad x \in \mathbb{R}^d.$$

Let us introduce the notation:

$$\tilde{\varphi}(x) = \varphi(-x), \quad \varphi \in C(\mathbb{R}^d),\, x \in \mathbb{R}^d.$$

Then

$$\varphi(x-y) = \tilde{\varphi}(y-x) = (\tau_x\tilde{\varphi})(y).$$

Thus, we have

$$(f * \varphi)(x) = u_f(\tau_x\tilde{\varphi}), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),\, x \in \mathbb{R}^d.$$

Motivated by this, we have the following definition.

Definition 4.2.27 The convolution of $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is a function, denoted by $u * \varphi$, is defined by

$$(u * \varphi)(x) = u(\tau_x\tilde{\varphi}), \quad x \in \mathbb{R}^d,$$

where $\tilde{\varphi}(s) = \varphi(-s)$. ♢

Example 4.2.28

$$(\delta_a * \varphi)(x) = \delta_a(\tau_x\tilde{\varphi}) = (\tau_x\tilde{\varphi})(a) = \tilde{\varphi}(a-x) = \varphi(x-a).$$

$$(\delta_0 * \varphi)(x) = \delta_0(\tau_x\tilde{\varphi}) = (\tau_x\tilde{\varphi})(0) = \tilde{\varphi}(0-x) = \varphi(x).$$

Thus, $\delta_0 * \varphi = \varphi \,\forall\, \varphi \in \mathcal{D}(\mathbb{R}^d)$. ♢

The above example shows that if we look at δ_0 as a function, then it must satisfy the relation:

$$\int_{\mathbb{R}^d} \delta_0(x-y)\varphi(y)dy = \varphi(x), \quad x \in \mathbb{R}^d.$$

Theorem 4.2.29 *For $u \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$,*

$$\tau_x(u * \varphi) = \tau_x u * \varphi = u * \tau_x\varphi.$$

Proof. For every $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
\tau_x(u * \varphi)(y) &= (u * \varphi)(y - x) = u(\tau_{y-x}\tilde{\varphi}) \\
((\tau_x u) * \varphi)(y) &= (\tau_x u)(\tau_y \tilde{\varphi}) = u(\tau_{-x}\tau_y\tilde{\varphi}) = u(\tau_{y-x}\tilde{\varphi}), \\
(u * \tau_x\varphi)(y) &= u(\tau_y \widetilde{\tau_x\varphi}) = u(\tau_{y-x}\tilde{\varphi}).
\end{aligned}$$

Thus, $\tau_x(u * \varphi) = \tau_x u * \varphi = u * \tau_x\varphi$. ∎

Theorem 4.2.30 *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then $u * \varphi \in C^\infty(\mathbb{R}^d)$, and*

$$\partial^\alpha(u * \varphi) = u * \partial^\alpha\varphi = (\partial^\alpha u) * \varphi.$$

Proof. For $j \in \{1, \dots, d\}$ and for $h \in \mathbb{R}$ with $|h|$ small enough,

$$\begin{aligned}
\frac{1}{h}\{(u * \varphi)(x + he_j) - (u * \varphi)(x)\} &= \frac{1}{h}\{\tau_{he_j}(u * \varphi)(x) - (u * \varphi)(x)\} \\
&= \frac{1}{h}\{(u * \tau_{he_j}\varphi)(x) - (u * \varphi)(x)\} \\
&= \frac{1}{h}\{(u * [(\tau_{he_j}\varphi)(x) - \varphi(x)]\} \\
&= (u * \frac{1}{h}[\varphi(x + he_j) - \varphi(x)].
\end{aligned}$$

Since $(u * \frac{1}{h}[\varphi(x + he_j) - \varphi(x)] \to u * (\partial_j\varphi)(x)$ as $|h| \to 0$, it follows that $\partial_j(u * \varphi)(x)$ exists and

$$\partial_j(u * \varphi)(x) = (u * \partial_j\varphi)(x) = (\partial_j u * \varphi)(x).$$

Iterating on that we obtain the result. ∎

Exercise 4.2.31 *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$. Then*

1. $\operatorname{supp}(u * \varphi) \subseteq \operatorname{supp}(u) + \operatorname{supp}(\varphi)$,
2. $u * (\varphi * \psi) = (u * \varphi) * \psi$.

Let φ be a mollifier, that is, $\varphi \in \mathcal{D}$ is called a such that $\varphi \geq 0$ and $\int \varphi = 1$, and for $\varepsilon > 0$ if $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(x\varepsilon)$. We know that $\varphi_\varepsilon \in \mathcal{D}$ and

1. $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies $f * \varphi_\varepsilon \in C^\infty(\mathbb{R}^d)$.
2. $f \in C_c(\mathbb{R}^d)$ implies $f * \varphi_\varepsilon \to f$ uniformly as $\varepsilon \to 0$.

3. f continuous at x implies $(f * \varphi_\varepsilon)(x) \to f(x)$ as $\varepsilon \to 0$.

4. $f \in L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ implies $f * \varphi_\varepsilon \to f$ in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$.

In the following we use the notation φ_ε for an approximate identity, and also use the convention that for a sequence $(f_n) \in L^1_{loc}(\Omega)$ and $u \in \mathcal{D}(\Omega)$,

$$f_n \to u \iff u_{f_n} \to u.$$

Theorem 4.2.32 (Regularization of distributions) *Let $u \in \mathcal{D}'(\mathbb{R}^d)$. Then*

$$u * \varphi_\varepsilon \to u \quad in \quad \mathcal{D}'(\mathbb{R}^d).$$

Proof. For $\psi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\begin{aligned} (u * \varphi_\varepsilon)(\psi) &= \int (u * \varphi_\varepsilon)(y)\psi(y)dy = \int (u * \varphi_\varepsilon)(y)\tilde{\psi}(0 - y)dy = [(u * \varphi_\varepsilon) * \tilde{\psi}](0) \\ &= [u * (\varphi_\varepsilon * \tilde{\psi})](0) \to (u * \tilde{\psi})(0) \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

But,

$$(u * \tilde{\psi})(0) = u(\tau_0 \psi) = u(\psi).$$

Thus, $u * \varphi_\varepsilon \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ as $\varepsilon \to 0$. ∎

By the above theorem,

- $\delta_a * \varphi_\varepsilon \to \delta_a$ as $\varepsilon \to 0$.

Corollary 4.2.33 *Let $u \in \mathcal{D}'(\mathbb{R})$ such that $u' = 0$. Then u is a constant.*

Proof. Let $u_\varepsilon := u * \varphi_\varepsilon$. Then $u'_\varepsilon = u' * \varphi_\varepsilon = 0$. Hence, $u_\varepsilon = C_\varepsilon$, constants. But, $u_\varepsilon \to u$. Therefore, there exists a constant C such that $u_\varepsilon \to C$ and hence $u = C$. ∎

Convolution of distributions

Now, suppose $f, g \in L^1(\mathbb{R}^d)$. Then for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\begin{aligned} u_{f*g}(\varphi) &= \int (f * g)(x)\varphi(x)dx = \int \left(\int f(y)g(x - y)dy \right) \varphi(x)dx \\ &= \int f(y) \left(\int g(x - y)\varphi(x)dx \right) dy = \int f(y) \left(\int g(s)\varphi(s + y)ds \right) dy \\ &= \int f(y) \left(\int g(s)(\tau_{-y}\varphi)(s)ds \right) dy \\ &= u_f(\varphi_g) \end{aligned}$$

where

$$\varphi_g(y) := g(\tau_{-y}\varphi).$$

Thus,

$$u_{f*g}(\varphi) = u_f(\varphi_g), \quad \varphi_g(y) := u_g(\tau_{-y}\varphi).$$

Identifying u_h with h, we may write the above relation as

$$(f * g)(\varphi) = f(\varphi_g), \quad \varphi_g(y) := g(\tau_{-y}\varphi).$$

Motivated by this we have the following definition.

Definition 4.2.34 For $u, v \in \mathcal{D}'(\mathbb{R}^d)$, and if v is of compact support, then

$$(u * v)(\varphi) := u(\varphi_v)$$

where

$$\varphi_v(y) := v(\tau_{-y}\varphi).$$

$\Diamond$

Exercise 4.2.35 *1. Show that*

$$(u * v)(\varphi) = u * \widetilde{(v * \tilde{\varphi})} = [u * (v * \tilde{\varphi})(0)].$$

Here, $\tilde{\varphi}(x) := \varphi(-x)$.

2. Show that if $u \in \mathcal{D}'(\mathbb{R}^d)$ *and* $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$u * \varphi = u * u_\varphi.$$

4.3 Schwarz space and tempered distributions

Definition 4.3.1 A function $f \in C^\infty(\mathbb{R}^d)$ is said to be **rapidly decreasing** if $x \mapsto x^\alpha \partial^\beta f(x)$ is bounded for each $\alpha, \beta \in \mathbb{N}_0^d$. The space of all rapidly decreasing functions on $\mathbb{R}^d$ is called **Schwarz space**. $\Diamond$

Thus, if $f \in C^\infty(\mathbb{R}^d)$, then

$$f \in \mathcal{S}(\mathbb{R}^d) \iff \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty$$

for every $\alpha, \beta \in \mathbb{N}_0^d$.

We observe that for each $\alpha, \beta \in \mathbb{N}_0^d$,

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

defines a norm on $\mathcal{S}(\mathbb{R}^d)$.

Clearly,

$$C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d).$$

Note also that if $f \in C_b^\infty(\mathbb{R}^d)$, $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if for every $\alpha, \beta \in \mathbb{N}_0^d$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial^\beta f(x)| \leq \frac{C_{\alpha,\beta}}{|x^\alpha|} \quad \forall\, z \in \mathbb{R}^d.$$

In fact,

$$|\partial^\beta f(x)| \leq \frac{\|f\|_{\alpha,\beta}}{|x^\alpha|} \quad \forall\, 0 \neq x \in \mathbb{R}^d.$$

Thus,

$$\mathcal{S}(\mathbb{R}^d) \subseteq C_0^\infty(\mathbb{R}^d).$$

Theorem 4.3.2 *For $1 \leq p \leq \infty$, $\mathcal{S}(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$, and*

$$\|f\|_p \leq C_p \sum_{|\alpha|\leq 2d} \|f\|_{\alpha,0} \quad \forall\, f \in \mathcal{S}(\mathbb{R}^d),$$

where $C_p := \left(\int \frac{dx}{(1+|x|^2)^p}\right)^{1/p}$ for $1 \leq p < \infty$ and $C_\infty = 1$. Further, $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$. The result is trivially true if $p = \infty$. So, let $1 \leq p < \infty$. Then

$$\int |f|^p = \int \frac{(1+|x|^2)^p |f|^p}{(1+|x|^2)^p} \leq C \sup_{x\in\mathbb{R}^d} (1+|x|^2)^p |f|^p,$$

where $C := \int \frac{dx}{(1+|x|^2)^p}$. But,

$$(1+|x|^2)|f| = \left(1 + \sum_{j=1}^d x_j^2\right)|f| = |f| + \sum_{j=1}^d |x_j^2 f| \leq \sum_{|\alpha|\leq 2d} \|f\|_{\alpha,0}.$$

Thus, we obtain $f \in L^p(\mathbb{R}^d)$, and $\|f\|_p \leq C^{1/p} \sum_{|\alpha|\leq 2d} \|f\|_{\alpha,0}$. The last part follows, because, $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. ∎

Definition 4.3.3 A sequence (f_n) in $\mathcal{S}(\mathbb{R}^d)$ is said to **converge** to $f \in \mathcal{S}(R^d)$ if

$$\|f_n - f\|_{\alpha,\beta} \to 0 \quad \text{as} \quad n \to \infty$$

for each $\alpha, \beta \in \mathbb{N}_0^d$, and in that case we write $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$. $\Diamond$

Theorem 4.3.4 *For* $\varphi_n, \varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\varphi_n \to \varphi \text{ in } \mathcal{D}(\mathbb{R}^d) \quad \Rightarrow \quad \varphi_n \to \varphi \text{ in } \mathcal{S}(\mathbb{R}^d).$$

Proof. Let $\varphi_n \in \mathcal{D}$ such that $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$. Let K be a compact set in $\mathbb{R}^d$ such that $\operatorname{supp}(\varphi_n) \cup \operatorname{supp}(\varphi) \subseteq K$ for all $n \in \mathbb{N}$. Then for every $\alpha, \beta \in \mathbb{N}_0^d$,

$$\begin{aligned} \|\varphi_n - \varphi\|_{\alpha,\beta} &= \sup_{x \in K} |x^\alpha \partial^\beta (\varphi_n - \varphi)(x)| \\ &\leq C_\alpha \sup_{x \in K} |\partial^\beta (\varphi_n - \varphi)(x)| \end{aligned}$$

for some $C_\alpha > 0$. Since $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, $\sup_{x \in K} |\partial^\beta (\varphi_n - \varphi)(x)| \to 0$ so that $\varphi_n \to \varphi$ in the space $\mathcal{S}(\mathbb{R}^d)$. ∎

The above theorem may be stated as follows:

Theorem 4.3.5 *The space* $\mathcal{D}(\mathbb{R}^d)$ *is a dense subspace of* $\mathcal{S}(\mathbb{R}^d)$.

Proof. Let $f \in \mathcal{S}$ and let $\psi \in \mathcal{D}$ such that $\psi(x) = 1$ for $|x| \leq 1$. For $r > 0$, let

$$f_r(x) = f(x)\psi(rx), \quad x \in \mathbb{R}^d.$$

Then $f \in \mathcal{D}$ and for every $\alpha, \beta \in \mathbb{N}_0^d$,

$$x^\alpha \partial^\beta (f - f_r)(x) = x^\alpha \sum_{\gamma \leq \beta} c_{\alpha,\beta} \partial^{\beta-\gamma} f(x) \partial^\gamma [1 - \psi(rx)] r^{|\gamma|}.$$

Since $1 - \psi(rx) = 0$ for $|x| \leq 1/r$, it follows that $x^\alpha \partial^\beta (f - f_r)(x) \to 0$ uniformly on $\mathbb{R}^d$. ∎

Theorem 4.3.6 *The space* $\mathcal{S}(\mathbb{R}^d)$ *is complete, in the sense that, if* (f_n) *in* $\mathcal{S}(\mathbb{R}^d)$ *is a Cauchy sequence with respect to* $\|\cdot\|_{\alpha,\beta}$ *for every* $\alpha, \beta \in \mathbb{N}_0^d$*, then it converges to a function in* $\mathcal{S}(\mathbb{R}^d)$.

Proof. Exercise. ∎

Theorem 4.3.7 *For every $f \in \mathcal{S}(\mathbb{R}^d)$, $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$ and the following relations hold:*

(i) $\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$.

(ii) $(\partial^\alpha \hat{f})(\xi) = \widehat{(-x)^\alpha f}(\xi)$.

Proof. (i) Follows from the fact that $f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \partial^\alpha f \in L^1(\mathbb{R}^d)$.
(ii) Follows from the fact that $f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow (ix)^\alpha f \in L^1(\mathbb{R}^d)$. ∎

Theorem 4.3.8 *The map $f \mapsto \hat{f}$ from $\mathcal{S}(\mathbb{R}^d)$ into itself is bijective, linear and satisfies $\|\hat{f}\| = \sqrt{2\pi}\|f\|_2$ for every $f \in \mathcal{S}(\mathbb{R}^d)$.*

We shall make us of the following lemma.

Lemma 4.3.9 *For any $h \in \mathcal{S}(\mathbb{R}^d)$,*

$$\check{h}(x) = \frac{1}{2\pi}\hat{\hat{h}}(x) = \frac{1}{2\pi}\tilde{\hat{h}}(x).$$

Proof. For any $h \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\check{h}(x) := \frac{1}{2\pi}\int_{\mathbb{R}^\delta} h(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}^\delta} h(-\xi)e^{-ix\xi}d\xi = \frac{1}{2\pi}\hat{\hat{h}}(x).$$

$$\check{h}(x) := \frac{1}{2\pi}\int_{\mathbb{R}^\delta} h(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}^\delta} h(\xi)e^{-i(-x)\xi}d\xi = \frac{1}{2\pi}\hat{h}(-x) = \frac{1}{2\pi}\tilde{\hat{h}}(x).$$

Thus, $\frac{1}{2\pi}\hat{\hat{h}}(x) = \check{h}(x) = \frac{1}{2\pi}\tilde{\hat{h}}(x)$. ∎

Proof of Theorem 4.3.8. Since $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$ for every $f \in \mathcal{S}(\mathbb{R}^d)$, by Fourier inversion theorem, $f \mapsto \hat{f}$ from $\mathcal{S}(\mathbb{R}^d)$ into itself is injective, linear and satisfies $\|\hat{f}\| = \sqrt{2\pi}\|f\|_2$ for every $f \in \mathcal{S}(\mathbb{R}^d)$. So, it remains to show that this map is onto. So, let $g \in \mathcal{S}(\mathbb{R}^d)$. Take $f := \check{g}$. We show that $\hat{f} = g$: Taking $h = \hat{g}$, by the above lemma, and by Fourier inversion theorem,

$$\hat{f}(\xi) = \frac{1}{2\pi}\hat{\hat{g}}(\xi) = \check{\hat{h}}(\xi) = \check{\hat{g}}(\xi) = g(\xi).$$

Thus, $f \mapsto \hat{f}$ is onto. ∎

Theorem 4.3.10 (Fourier-Plancheral theorem) *There exists a unique bijective linear transformation* $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *such that*

$$(Tf)(\xi) = \hat{f}(\xi) \quad \forall\, f \in \mathcal{S}(\mathbb{R}^d),\, \xi \in \mathbb{R}^d$$

and

$$\|Tf\|_2 = \sqrt{2\pi} f\|_2 \quad \forall\, f \in L^2(\mathbb{R}^d).$$

Proof. By Theorem 4.3.8 , the map $f \mapsto \hat{f}$ is an injective linear transformation from $\mathcal{S}(\mathbb{R}^d)$ into itself and $\|\hat{f}\| = \sqrt{2\pi}\|f\|_2$ for every $f \in \mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^d)$, there exists a unique continuous linear transformation $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $Tf = \hat{f}$ for all $f \in L^2(\mathbb{R}^d)$. It remains to show that T is onto. For this, let $g \in L^2(\mathbb{R}^d)$. Then there exists (g_n) in $\mathcal{S}(\mathbb{R}^d)$ such that $\|g - g_n\|_2 \to 0$. Let $f_n \in \mathcal{S}(\mathbb{R}^d)$ be such that $\hat{f}_n = g_n$. Since $\|f_n - f_m\|_2 = \frac{1}{\sqrt{2\pi}}\|\hat{f}_n - \hat{f}_m\|_2 = \frac{1}{\sqrt{2\pi}}\|g_n - g_m\|_2$, $(\hat{f}_n)$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$. Let $f \in L^2(\mathbb{R}^d)$ be such that $\|f_n - f\| - 2 \to 0$. Then we have

$$\|T(f) - g_n\| = \|T(f) - T(f_n)\|_2 = \frac{1}{\sqrt{2\pi}}\|f - f_n\|_2 \to 0.$$

Therefore, $g = T(f)$. ∎

Definition 4.3.11 A linear functional u on $\mathcal{S}(\mathbb{R}^d)$ is called a **tempered distribution** if for every sequence (f_n) in $\mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$,

$$f_n \to f \text{ in } \mathcal{S}(\mathbb{R}^d) \quad \Rightarrow \quad u(f_n) \to u(f).$$

The space of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. ◊

Definition 4.3.12 A sequence (u_n) in $\mathcal{S}'(\mathbb{R}^d)$ is said to converge to $u \in \mathcal{S}'(\mathbb{R}^d)$ if

$$u_n(f) \to u(f)$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. ◊

Theorem 4.3.13 *If* $u \in \mathcal{S}'(\mathbb{R}^d)$, *then* $u|_{\mathcal{D}(\mathbb{R}^d)} \in \mathcal{D}'(\mathbb{R}^d)$. *Further, the map* $u \mapsto u|_{\mathcal{D}(\mathbb{R}^d)}$ *is a continuous embedding of* $\mathcal{S}'(\mathbb{R}^d)$ *into* $\mathcal{D}'(\mathbb{R}^d)$, *in the sense that if* $u_n \to u$ *in* $\mathcal{S}'(\mathbb{R}^d)$, *then* $u_n|_{\mathcal{D}(\mathbb{R}^d)} \to u|_{\mathcal{D}(\mathbb{R}^d)}$ *in* $\mathcal{D}'(\mathbb{R}^d)$.

Proof. Let $u \in \mathcal{S}'$. Let $\varphi_n \in \mathcal{D}$ be such that $\varphi_n \to \varphi$ in $\mathcal{D}$. Then by Theorem 4.3.4, $\varphi_n \to \varphi$ in $\mathcal{S}$. Hence, $u(\varphi_n) \to u(\varphi)$. Thus, $u|_{\mathcal{D}} \in \mathcal{D}'$. Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, $u|_{\mathcal{D}} = 0$ implies $u = 0$. Clearly, for a sequence (u_n) in $\mathcal{S}'(\mathbb{R}^d)$, $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^d)$ implies that $u_n|_{\mathcal{D}} \to u|_{\mathcal{D}}$ in $\mathcal{D}'(\mathbb{R}^d)$. ∎

We state the following characterization theorem without proof.

Theorem 4.3.14 *Let u be a linear functional on $\mathcal{S}(\mathbb{R}^d)$. Then $u \in \mathcal{S}'(\mathbb{R}^d)$ if and only if there is a constant $C > 0$ and $m \in \mathbb{N}_0$ such that*

$$|u(f)| \leq C \sum_{|\alpha|,|\beta| \leq m} \|f\|_{\alpha,\beta}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

For $f : \mathbb{R}^d \to \mathbb{C}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we define

$$u_f(\varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x)\, dx$$

whenever the above integral exists.

Theorem 4.3.15 *Let $1 \leq p \leq \infty$ and let $f \in L^p(\mathbb{R}^d)$. Then $u_f \in \mathcal{S}'(\mathbb{R}^d)$. Further, $f_n \to f$ in $L^p(\mathbb{R}^d)$ implies $u_{f_n} \to u_f$ in $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. Let $f \in L^p(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$|u_f(\varphi)| \leq \int |f|\,|\varphi| \leq \|f\|_p \|\varphi\|_q.$$

By Theorem 4.3.2, $\|\varphi\|_q \leq C \sum_{|\alpha| \leq 2d} \|\varphi\|_{\alpha,0}$ for some $C > 0$. Hence,

$$|u_f(\varphi)| \leq C \|f\|_p \sum_{|\alpha| \leq 2d} \|\varphi\|_{\alpha,0}$$

for some $C > 0$. The above inequality also shows that, if $\varphi_n \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$, then $u_f(\varphi_n) \to u_f(\varphi)$, so that $u \in \mathcal{S}'(\mathbb{R}^d)$.

Next, suppose $f_n, f \in L^p(\mathbb{R}^d)$ be such that $f_n \to f$ in $L^p(\mathbb{R}^d)$. Then, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$|u_n(\varphi) - u(\varphi)| \leq \int |f_n(x) - f(x)|\,|\varphi(x)| dx \leq \|f_n - f\|_p \|\varphi\|_q \to 0.$$

Thus, $u_{f_n} \to u_f$ in $\mathcal{S}'(\mathbb{R}^d)$. ∎

Remark 4.3.16 By Theorem 4.3.15, every $f \in L^p(\mathbb{R}^d)$ can be considered as a tempered distribution and the inclusion $L^p(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$ is a (sequentially continuous) imbedding. ♢

In fact,

Theorem 4.3.17 *For every measurable $f : \mathbb{R}^d \to \mathbb{C}$ which satisfies*

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^N} dx < \infty \qquad (*)$$

for some $N \in \mathbb{N}_0$, $u_f \in \mathcal{S}'(\mathbb{R}^d)$.

Proof. Let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable such that $(*)$ is satisfied for some $N \in \mathbb{N}_0$. Then for every $\varphi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\begin{aligned} \int_{\mathbb{R}^d} |f(x)\varphi(x)| dx &= \int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^N} |(1+|x|)^N| \varphi(x)| dx \\ &\leq \sup_{x \in \mathbb{R}^d} |(1+|x|)^N| \varphi(x)| \int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^N} dx < \infty. \end{aligned}$$

Hence, $\varphi \mapsto u_f(\varphi)$ is a linear functional on $\mathcal{S}(\mathbb{R}^d)$. The above relation also shows that $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$ implies $u_f(\varphi_n) \to u_f(\varphi)$. ∎

Corollary 4.3.18 *For every $f \in C_b(\mathbb{R}^d)$, $u_f \in \mathcal{S}'(\mathbb{R}^d)$.*

Proof. Clearly, if $f \in C_b(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^N} dx < \infty$. Hence, by Theorem 4.3.17, $u_f \in \mathcal{S}'(\mathbb{R}^d)$. ∎

Exercise 4.3.19 *The space of polynomials on $\mathbb{R}^d$ is a subspace of $\mathcal{S}'(\mathbb{R}^d)$.*

Theorem 4.3.20 *If $u \in \mathcal{D}'(\mathbb{R}^d)$ with compact support, then u can be (uniquely) extended to a tempered distribution.*

Proof. Let $u \in \mathcal{D}'(\mathbb{R}^d)$ be with compact support K. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi = 1$ in an open set containing K. Then

$$\tilde{u}(\varphi) := u(\psi\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

is a tempered distribution which is an extension of u. ∎

4.4 Fourier transform of distributions

Recall that for $f \in L^1(\mathbb{R}^d)$,

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx, \quad \xi \in \mathbb{R}^d.$$

Hence, for $f \in L^1(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \hat{f}(\xi)\varphi(\xi) d\xi &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx \right) \varphi(\xi) d\xi \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(\xi) e^{-ix.\xi} d\xi \right) f(x) dx \\
&= \int_{\mathbb{R}^d} \hat{\varphi}(x) f(x) dx.
\end{aligned}$$

So, formally, we write

$$u_{\hat{f}}(\varphi) = u_f(\hat{\varphi}).$$

Formally, because, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ does not imply $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^d)$. However,

$$\varphi \in \mathcal{S}(\mathbb{R}^d) \quad \Rightarrow \quad \hat{\varphi} \in \mathcal{S}(\mathbb{R}^d).$$

Recall that for $f \in L^1(\mathbb{R}^d)$, u_f is a tempered distribution.

Theorem 4.4.1 *For each $f \in L^1(\mathbb{R}^d)$, the map $\varphi \mapsto u_f(\hat{\varphi})$ defined on $\mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution.*

Proof. Let $f \in L^1(\mathbb{R}^d)$, and let $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$. Then, it can be shown that $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}(\mathbb{R}^d)$. Since $\hat{f} \in C_0(\mathbb{R}^d)$, by Corollary 4.3.18, $u_{\hat{f}} \in \mathcal{S}'(\mathbb{R}^d)$. Hence,

$$u_f(\hat{\varphi}_n) = u_{\hat{f}}(\varphi_n) \to u_{\hat{f}}(\varphi) = u_f(\hat{\varphi}).$$

Thus, $\varphi \mapsto u_f(\hat{\varphi})$ defined on $\mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution. ∎

More generally, we have the following.

Theorem 4.4.2 *If $u \in \mathcal{S}'(\mathbb{R}^d)$, then the map $f \mapsto u(\hat{f})$ on $\mathcal{S}(\mathbb{R}^d)$ is a tempered distribution.*

Proof. Let $u \in cS'(\mathbb{R}^d)$. Let $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$. Then, it can be shown that $\hat{f}_n \to \hat{f}$ in $\mathcal{S}(\mathbb{R}^d)$. Hence,

$$u(\hat{f}_n) \to u(\hat{f}).$$

Thus, the map $f \mapsto u(\hat{f})$ on $\mathcal{S}(\mathbb{R}^d)$ is a tempered distribution. ∎

The above theorem motivates the following definition.

Definition 4.4.3 The **Fourier transform** of $u \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\hat{u}(\varphi) := u(\hat{\varphi}), \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

$\Diamond$

Remark 4.4.4 By Theorem 4.4.2, Fourier transform of every tempered distribution is a tempered distribution. $\Diamond$

Exercise 4.4.5 Prove the following.

1. For $f \in L^1(\mathbb{R}^d)$, $\widehat{u_f}(\varphi) = u_{\hat{f}}(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

2. $\hat{\delta} = 1$, a regular distribution.

4.4.1 The spaces $\mathcal{E}(\Omega)$ and $\mathcal{E}'(\Omega)$

Definition 4.4.6 The space $C^\infty(\Omega)$ with the notion of convergence defined by

> $f_n \to f \iff$ for each $\alpha \in \mathbb{N}_0$, $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on every compact $K \subseteq \Omega$

is denoted by $\mathcal{E}(\Omega)$. $\Diamond$

Clearly,

$$\mathcal{D}(\Omega) \subseteq \mathcal{E}(\Omega).$$

We also have the following.

Theorem 4.4.7 *If $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, then $\varphi_n \to \varphi$ in $\mathcal{E}(\Omega)$.*

Proof. Suppose $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Then there exists compact $K \subseteq \Omega$ such that $\mathrm{supp}(\varphi_n) \cup \mathrm{supp}(\varphi) \subseteq K$ for all $n \in \mathbb{N}$ and for each $\alpha \in \mathbb{N}_0^d$, $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on Ω. In particular, for each $\alpha \in \mathbb{N}_0^d$, $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on any compact subset $\tilde{K}$ of Ω. Hence, $\varphi_n \to \varphi$ in $\mathcal{E}(\Omega)$. ∎

Recall that for $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, fu defined by

$$(fu)(\varphi) := u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is a distribution.

Theorem 4.4.8 *Let u be a distribution. Then the map $f \mapsto fu$ from $\mathcal{E}(\Omega)$ to $\mathcal{D}'(\Omega)$ is continuous, in the sense that*

$$f_n \to f \text{ in } \mathcal{E}(\Omega) \quad \Rightarrow \quad f_n u \to fu \text{ in } \mathcal{D}'(\Omega).$$

Theorem 4.4.9 *Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Then*

$$\text{supp}(fu) \subseteq \text{supp}(f) \cap \text{supp}(u).$$

Proof. Suppose $x_0 \notin \text{supp}(f)$. Then there exists an open nbd $\Omega_0 \subseteq \Omega$ of x_0 such that $f = 0$ on Ω_0. Hence,

$$(fu)(\varphi) = u(f\varphi) = 0 \quad \forall \varphi \in \mathcal{D}(\Omega_0)$$

so that $fu = 0$ on Ω_0. Therefore, $x_0 \notin \text{supp}(fu)$. Also, $x_0 \notin \text{supp}(f)$ implies there exists an open nbd $\Omega_0 \subseteq \Omega$ of x_0 such that $u = 0$ on Ω_0 so that $fu = 0$ on Ω_0 and hence, $x_0 \notin \text{supp}(fu)$ ∎

Corollary 4.4.10 *If u is a distribution with compact support, then for any $f \in \mathcal{E}(\Omega)$, fu is also of compact support.*

Definition 4.4.11 The set of all linear functionals u on $\mathcal{E}(\Omega)$ such that

$$\varphi_n \to \varphi \text{ in } \mathcal{E}(\Omega) \Rightarrow u(\varphi_n) \to u(\varphi)$$

is denoted by $\mathcal{E}'(\Omega)$. A sequence (u_n) in $\mathcal{E}'(\Omega)$ is said to converge to $u \in \mathcal{E}'(\Omega)$, weritten $u_n \to u$ if

$$u_n(f) \to u(f) \quad \forall f \in \mathcal{E}(\Omega).$$

◊

Theorem 4.4.12 *If $u \in \mathcal{E}'(\Omega)$, then $u_0 := u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$. Further, the map $u \mapsto u_0$ is continuous from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$, in the sense that,*

$$u_n \to u \quad \text{in} \quad \mathcal{E}'(\Omega) \quad \Rightarrow \quad u_{0,n} \to u_0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Proof. Let $u \in \mathcal{E}'(\Omega)$, Let $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Then there exists a compact set $K_0 \subseteq \Omega$ such that $\text{supp}, \varphi_n, \varphi \subseteq K_0$ and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on Ω. Hence, $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on every compact subset of Ω. Thus, $\varphi_n \to \varphi$ in $\mathcal{E}(\Omega)$ so that by hypothesis, $u(\varphi_n) \to u(\varphi)$, i.e., $u_0(\varphi_n) \to u_0(\varphi)$. The last part is obvious. ∎

In view of the above theorem, we may say that

$\mathcal{E}'(\Omega)$ is embedded in $\mathcal{D}'(\Omega)$.

We shall show that the distribution u_0 in the above theorem is with compact support.

Theorem 4.4.13 *If $u \in \mathcal{D}'(\Omega)$ is with compact support, then $u \in \mathcal{E}'(\Omega)$ in the sense that there exists a unique $\tilde{u} \in \mathcal{E}'(\Omega)$ such that*

1. $\tilde{u}|_{\mathcal{D}(\Omega)} = u$ *and*
2. $f \in \mathcal{E}(\Omega)$ *with* $\mathrm{supp}(u) \cap \mathrm{supp}(f) = \emptyset$ *implies* $\tilde{u}(f) = 0$.

For proving the above theorem we shall make use of the following lemma.

Lemma 4.4.14 *If $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ are such that* $\mathrm{supp}(u) \cap \mathrm{supp}(\varphi) = \emptyset$, *then* $u(\varphi) = 0$.

Proof of Theorem 4.4.13. Suppose $u \in \mathcal{D}'(\Omega)$ is with compact support, say $K := \mathrm{supp}(u)$. Let $\psi \in \mathcal{D}(\Omega)$ be such that $\psi = 1$ on K. Then, for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$u(\varphi) = u(\psi\varphi + (1 - \psi)\varphi) = u(\psi\varphi) + u((1 - \psi)\varphi).$$

Note that $\mathrm{supp}(u) \cap \mathrm{sup}((1 - \psi)\varphi) = \emptyset$. Hence by the last lemma, $u((1 - \psi)\varphi) = 0$. Thus,

$$u(\varphi) = u(\psi\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Now, define

$$\tilde{u}(f) = u(\psi f), \quad f \in \mathcal{E}(\Omega).$$

Then we have $\tilde{u} \in \mathcal{E}'(\Omega)$ and $\tilde{u}|_{\mathcal{D}(\Omega)} = u$. [To see that $\tilde{u} \in \mathcal{E}'(\Omega)$, we may observe that $f_n \to f$ in $\mathcal{E}(\Omega)$ implies $\psi f_n \to \psi f$ in $\mathcal{D}(\Omega)$.]

To see the uniqueness, suppose $v \in \mathcal{E}'(\Omega)$ is such that

1. $v|_{\mathcal{D}(\Omega)} = u$ and
2. $f \in \mathcal{E}(\Omega)$ with $\mathrm{supp}(u) \cap \mathrm{supp}(f) = \emptyset$ implies $v(f) = 0$.

Then, for $f \in \mathcal{E}(\Omega)$, we have

$$v(f) = v(\psi f + (1 - \psi)f) = v(\psi f) + v((1 - \psi)f) = u(\psi f) + v((1 - \psi)f).$$

Since $(1 - \psi)f = 0$ on $K := \mathrm{supp}(u)$, assumption (2) on v implies $v((1 - \psi)f) = 0$. Thus, $v(f) = u(\psi f) = \tilde{u}(f)$. ∎

For the proof of Lemma 4.4.14, we make use of *partition of unity*:

Proposition 4.4.15 (Partition of unity) *Let K be a compact set and $\Omega_1, \ldots, \Omega_n$ be open subsets of $\mathbb{R}^d$ such that $K \subseteq \cup_{j=1}^n \Omega_j$. Then there exists $\psi_1, \ldots, \psi_n$ in $\mathcal{D}(\Omega_0)$ with $\Omega_0 := \cup_{j=1}^n \Omega_j$ such that* $\mathrm{supp}(\psi_j) \subseteq \Omega_j$ *and $\sum_{j=1}^n \psi_j = 1$ on K.*

Proof. Let $x \in K$. Then $x \in \Omega_i$ for some $i \in \{1, \ldots, n\}$. Let G_x be an open nbd of x such that $\overline{G}_x$ is compact and $\overline{G}_x \subseteq \Omega_i$. Since K is compact, there exist $x_1, \ldots, x_k \in K$ such that $K \subseteq \bigcup_{j=1}^k G_{x_j}$. For each $i \in \{1, \ldots, n\}$, let H_i be the union of those $\overline{G}_{x_j}$ such that $\overline{G}_{x_j} \subseteq \Omega_i$. Then each $H - i$ is compact and $H_i \subseteq \Omega_i$. Hence, there exists $g_i \in \mathcal{D}(\Omega_i)$ such that $g_i = 1$ on H_i. Note that $K \subseteq \bigcup_{i=1}^n H_i$. Now, define

$$\psi_1 = g_1, \quad \psi_2 = (1 - g_1)g_2, \quad \ldots, \quad \psi_n = (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1})g_n.$$

It can be seen by induction that

$$\psi_1 + \cdots + \psi_n = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_n).$$

Since $K \subseteq \bigcup_{i=1}^n H_i$, and since $g_i = 1$ on H_i, we obtain $\psi_1 + \cdots + \psi_n = 1$ on K. ∎

Proof of Lemma 4.4.14. Let $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ are such that $\mathrm{supp}(u) \cap \mathrm{supp}(\varphi) = \emptyset$. To prove that $u(\varphi) = 0$. For this, let $K = \mathrm{supp}(\varphi)$. For each $x \in K$, since $x \notin \mathrm{supp}(u)$, there exists open set $\Omega_x \subseteq \Omega$ such that $x \in \Omega_x$. Then $\{\Omega_x : x \in K\}$ is an open cover of K. Since K is compact, there exists $x_1, \ldots, x_n$ in K such that $K \subseteq \cup_{j=1}^n \Omega_{x_j}$. By partition of unity, there there exists $\psi_1, \ldots, \psi_n$ in $\mathcal{D}(\Omega_0)$ with $\Omega_0 := \cup_{j=1}^n \Omega_{x_j}$such that $\mathrm{supp}(\psi_j) \subseteq \Omega_{x_j}$ and $\sum_{j=1}^n \psi_j = 1$ on K. Then we have $\varphi = \sum_{j=1}^n \psi_j \varphi$ so that $u(\varphi) = \sum_{j=1}^n u(\psi_j \varphi) = 0$, since $\psi_j \varphi \in \mathcal{D}(\Omega_{x_j})$ and $\Omega_{x_j} \cap \mathrm{supp}(u) = \emptyset$. ∎

Now the theorem that we had promised:

Theorem 4.4.16 *If $u \in \mathcal{E}'(\Omega)$, then $u|_{\mathcal{D}(\Omega)}$ is a distribution with compact support.*

For its proof we use the following characterization:

Theorem 4.4.17 *Let u be a linear functional on $\mathcal{E}(\Omega)$. Then $u \in \mathcal{E}'(\Omega)$ if and only if there exists a compact $K \subseteq \Omega$, constant $C > 0$ and $m \in \mathbb{N}_0$ such that*

$$|u(f)| \le C \sum_{|\alpha|\le m} \sup_{x\in K} |(\partial^\alpha f)(x)| \quad \forall\, f \in \mathcal{E}(\Omega).$$

Proof. ($\Leftarrow$): Obvious.

($\Leftarrow$): Suppose the conclusion is not true. Then for any triple $\eta := (K, C, m)$ there exists $\varphi_\eta \in \mathcal{E}(\Omega)$ such that

$$|u(f_\eta)| > C \sum_{|\alpha|\le m} \sup_{x\in K} |(\partial^\alpha f)(x)|.$$

So, for $m \in \mathbb{N}$, let $K_m := \overline{B_m(0)}$ and $f_m \in \mathcal{E}(\Omega)$ such that

$$|u(f_m)| > m \sum_{|\alpha|\le m} \sup_{x\in K_m} |(\partial^\alpha f_m)(x)|.$$

Let $g_m = f_m/[m \sum_{|\alpha|\le m} \sup_{x\in K_m} |(\partial^\alpha f_m)(x)|]$. Then for every $\beta \in \mathbb{N}_0^d$ with $|\beta| \le m$ and $K \subseteq \Omega$ with $K \subseteq K_m$, we have

$$\sup_{x\in K} |\partial^\beta g_m|| \le \sum_{|\gamma|\le m} \sup_{x\in K_m} |(\partial^\gamma g_m)(x)| = \frac{1}{m}.$$

Thus, $f_m \to 0$ in $\mathcal{E}(\Omega)$ but $|u(f_m)| > 1$ for all $m \in \mathbb{N}$. This is a contradiction. ∎

Proof of Theorem 4.4.16. Let $u \in \mathcal{E}'(\Omega)$. We have already seen that $u|_{\mathcal{D}(\Omega)}$ is a distribution. Let K be as in Theorem 4.4.17. We claim that $\text{supp}(u) \subseteq K$. To prove this claim, suppose $x \notin K$. Then there exists an open neighbourhood $G_x \subseteq \Omega$ of x such that $G_x \cap K = \emptyset$. Hence, $\varphi \in \mathcal{D}(G_x)$ implies $\text{supp}(\varphi) \cap K = \emptyset$. Hence, from the relation

$$|u(f)| \le C \sum_{|\alpha|\le m} \sup_{x\in K} |(\partial^\alpha f)(x)| \quad \forall\, f \in \mathcal{E}(\Omega)$$

in Theorem 4.4.17, we have $u(\varphi) = 0$. Therefore, $x \notin \text{supp}(u)$. Thus we have proved that $x \notin K$ implies $x \notin \text{supp}(u)$. Equivalently, $\text{supp}(u) \subseteq K$. ∎

In view of Theorems 4.4.13 and 4.4.16, there is a one-one correspondence between $\mathcal{E}'(\Omega)$ and distributions with compact support. Therefore, distributions with compact support is also denoted by $\mathcal{E}'(\Omega)$.

4.5 Problems

Throughout, Ω denotes a nonempty open subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$.

1. Let φ be a mollifier. For $a \in \Omega$ and $\varepsilon > 0$ be such that $\overline{B_\varepsilon(a)} \subset \Omega$, let $\psi_{\varepsilon,a}(x) := \frac{1}{\varepsilon^d}\varphi(\frac{x-a}{\varepsilon}$. Show that $\psi_{\varepsilon,a} \in \mathcal{D}(\Omega)$ such that $\mathrm{supp}(\psi_{\varepsilon,a}) \subseteq B_\varepsilon(a)$ and $\int_\Omega \psi_{\varepsilon,a} dx = 1$.

2. Let $\psi_{\varepsilon,a}$ be as Problem 1, and let $\psi_\varepsilon := \psi_{\varepsilon,0}$. Prove that for $f \in C_c(\mathbb{R}^d)$, $f * \psi_\varepsilon \to f$ uniformly.

3. Show that $\mathcal{D}(\Omega)$ is sequentially complete. That is, if (φ_n) in $\mathcal{D}(\Omega)$ is such that for every $\varepsilon > 0$ and for every $\alpha \in \mathbb{N}_0^d$, there exists $N \in \mathbb{N}$ such that $\|\partial^\alpha(\varphi_n - \varphi_m)\|_\infty < \varepsilon$ for all $n \geq N$, then there exists $\varphi \in \mathcal{D}(\Omega)$ such that $\|\partial^\alpha(\varphi_n - \varphi)\|_\infty \to 0$ as $n \to \infty$ for each $\alpha \in \mathbb{N}_0^d$.

4. Corresponding to $f \in L^1_{\mathrm{loc}}(\Omega)$, let
$$u_f(\varphi) := \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega),\ x \in \Omega.$$
Show that u_f is a distribution, and it is of order 0.

5. Show that the delta-distribution is not a regular distribution.

6. Show every delta-distribution is a limit of a sequence of regular distributions.

7. Let (f_n) in $L^1_{\mathrm{loc}}(\Omega)$ and $f : \Omega \to \mathbb{C}$ be such that $f_n \to f$ a.e. on Ω and for every compact $K \subseteq \Omega$, there exists $g \in L^1(\Omega)$ such that $|f_n| \leq |g|$ a.e. on K. Prove that $f \in L^1_{\mathrm{loc}}(\Omega)$ and $f_n \to f$ in the sense of distribution.

8. Let $f_n, f \in C(\Omega)$ such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that $f_n \to f$ in the sense of distribution.

9. Let $f_n(x) := e^{inx}$, $x \in \mathbb{R}$. Show that (u_{f_n}) converges to the zero distribution.

10. Making use of necessary results, prove that for $f, g \in L^1_{\mathrm{loc}}(\Omega)$, $u_f = u_g$ implies $f = g$ a,e.

11. Let u be a linear functional on $\mathcal{D}(\Omega)$. Prove that u is a distribution if and only if for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that
$$|u(\varphi)| \le C \sum_{|\alpha| \le N} \|\partial^\alpha \varphi\|_\infty \tag{1}$$
for all $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq K$.

12. Define $u(\varphi) := \sum_{j=0}^{\infty} \varphi^{(j)}(j), \quad \varphi \in \mathcal{D}(\mathbb{R})$. Show that $u \in \mathcal{D}'(\mathbb{R})$, and it is of infinite order.

13. Prove that

    (a) $\mathrm{supp}(\delta_a) = \{a\}$.

    (b) For $f \in L^1_{\mathrm{loc}}(\Omega)$, $\mathrm{supp}(u_f) = \mathrm{supp}(f)$.

    (c) For $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, $\mathrm{supp}(u) \cap \mathrm{supp}(f) = \emptyset \Rightarrow u(\varphi) = 0$.

14. If $f \in C^\infty(\Omega)$, then prove that $f\varphi \in \mathcal{D}(\Omega)$ for every $\varphi \in \mathcal{D}(\Omega)$.

15. For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, prove that the map $\varphi \mapsto u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega)$, is a distribution.

16. If $f \in C^\infty(\Omega)$ and $a \in \Omega$, show that $f\delta_a = f(a)\delta$.

17. For $f, g \in L^1_{\mathrm{loc}}(\Omega)$, show that $fu_g = u_{fg}$.

18. Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Prove that $\mathrm{supp}(fu) \subseteq \mathrm{supp}(f) \cap \mathrm{supp}(u)$.

19. If u is a distribution with compact support, then prove that for any $f \in \mathcal{E}(\Omega)$, fu is also of compact support.

20. If $u \in \mathcal{D}'(\Omega)$ is with compact support, then prove that $u \in \mathcal{E}'(\Omega)$ in the sense that for every $u \in \mathcal{D}'(\Omega)$, there exists a unique $\tilde{u} \in \mathcal{D}'(\Omega)$ such that $u|_{\mathcal{D}(\Omega)} = \tilde{u}$.

21. If $u \in \mathcal{E}'(\Omega)$, then prove that $u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$ is with compact support.

22. Prove that $\tau_h \delta_a = \delta_{a+h}$. (Recall: For $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, the distribution $\tau_h u$ is defined by $(\tau_h u)(\varphi) := u(\tau_{-h}\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

23. For each $h \in \mathbb{R}^d$, show that the map $u \mapsto \tau_h u$ is continuous on $\mathcal{D}'(\mathbb{R}^d)$ in the sense that $u_n \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ implies $\tau_h u_n \to \tau_h u$ in $\mathcal{D}'(\mathbb{R}^d)$.

24. For $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, show that the map $\partial^\alpha u : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by $(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(O)$, is a distribution.

25. Let H be the *Heaviside function*, i.e., $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$ Show that $H' = \delta_0$.

26. For $\alpha \in \mathbb{N}_0^d$, $x_0 \in \Omega$, prove that u defined by $u(\varphi) = (\partial^\alpha \varphi)(x_0)$ defines a distribution of order α.

27. Let (x_n) be a sequence in Ω without a limit point in Ω and $(\alpha^{(n)})$ be a sequence in $\mathbb{N}_0^d$. Let $u(\varphi) := \sum_{n=1}^\infty \partial^{\alpha_n} \varphi(x_n)$. Prove that u is a distribution, and it has finite order if and only if $\sup |\alpha^{(n)}| < \infty$ and in that case the order is $\sup |\alpha^{(n)}|$.

28. If $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp}(u) \cap \operatorname{supp}(\varphi) = \emptyset$, then prove that $u(\varphi) = 0$.

29. Suppose u is a linear functional on $\mathcal{E}(\Omega)$ such that there exists compact $K \subseteq \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ satisfying

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{\infty,K} \quad \forall\, \varphi \in \mathcal{E}'(\Omega).$$

Prove that $u \in \mathcal{E}'(\Omega)$.

30. Suppose $u \in \mathcal{E}'(\Omega)$ and there exists compact $K \subseteq \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ satisfying

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{\infty,K} \quad \forall\, \varphi \in \mathcal{E}'(\Omega).$$

Prove that $u|_{\mathcal{D}(\Omega)}$ is a distribution with compact support.

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