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FOURIER ANALYSIS: ASSIGNMENT - I

M.T. NAIR

We use the following notations: For $f \in L^1[-\pi,\pi]$, we let[1]

$$S_N(f,x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}, \quad D_N(t) := \sum_{n=-N}^{N} e^{int}, \quad K_N(t) := \frac{1}{N+1}\sum_{k=0}^{N} D_k(t).$$

- $CP[-\pi,\pi]$ denotes the vector space of all 2π-periodic complex valued functions defined on $\mathbb{R}$ whose restrictions to $[-\pi,\pi]$ are continuous.

- $CP^{(k)}[-\pi,\pi]$ for $k \in \mathbb{N}\cup\{0\}$, denotes the vector space of all 2π-periodic complex valued functions defined on $\mathbb{R}$ whose restrictions to $[-\pi,\pi]$ are k-times continuously differentiable. Note that (verify!) if f belongs to $CP^{(k)}[-\pi,\pi]$, then $f^{(j)}(-\pi) = f^{(j)}(\pi)$ for $j = 0,1,\ldots,k$.
- $AC[-\pi,\pi]$ denotes the vector space of all complex valued functions defined on $[-\pi,\pi]$ which are absolutely continuous.
- $ACP[-\pi,\pi]$ denotes the vector space of all 2π-periodic complex valued functions defined on $\mathbb{R}$ whose restrictions to $[-\pi,\pi]$ are absolutely continuous.

Problems:

(1) Prove:
  (a) Given any Riemann integrable function on $[a,b]$ and $\varepsilon > 0$, there exists a step function g on $[a,b]$ such that $\|f-g\|_1 := \int_a^b |f(x)-g(x)|dx < \varepsilon$.
  (b) If f is a step function on $[a,b]$, then
  $$\int_a^b f(t)\cos(nt)dt \to 0 \quad \text{and} \quad \int_a^b f(t)\sin(nt)dt \to 0 \quad \text{as} \quad n\to\infty.$$
  (c) If f is Riemann integrable on $[a,b]$, then
  $$\int_a^b f(t)\cos(nt)dt \to 0 \quad \text{and} \quad \int_a^b f(t)\sin(nt)dt \to 0 \quad \text{as} \quad n\to\infty.$$

(2) For $f \in L^1[-\pi,\pi]$, verify the following:
  (a) $D_N(-t) = D_N(t)$ for all $t \in [-\pi,\pi]$.
  (b) $\displaystyle\int_{-\pi}^{\pi} D_N(t)dt = 1.$
  (c) $D_N(t) = \dfrac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}, t \neq 0.$
  (d) $S_N(f,x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)D_N(x-t)dt = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt.$

[1] $D_N(\cdot)$ and $K_N(\cdot)$ are called Dirchlet kernel and Fejer kernel, respectively.

(3) Show that $\displaystyle\int_{-\pi}^{\pi} |D_N(t)|dt \geq \frac{8}{\pi}\sum_{k=1}^{N}\frac{1}{k}$.

(4) Let $f \in L^1[-\pi,\pi]$. If $\sum_{n=1}^{\infty} |\hat{f}(n)| < \infty$, then the Fourier series of f converges uniformly on $[-\pi,\pi]$ to a continuous function. Why?

(5) Justify the following statements:
   (a) If f is Lipschits continuous at a point[2] $x \in [-\pi,\pi]$, then the Fourier series of f converges at x.
   (b) If f is Holder continuous at a point[3] $x \in [-\pi,\pi]$, then the Fourier of f converges at x.
   (c) If f is Lipschits continuous[4] on $[-\pi,\pi]$, then the Fourier of f converges uniformly.
   (d) If f is Holder continuous[5] on $[-\pi,\pi]$, then the Fourier of f converges uniformly.

(6) Prove the following:
   (a) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1$.
   (b) $K_N(t)$ is an even function and $K_N(t) \geq 0$ for all $t \in [-\pi,\pi]$.
   (c) For $0 < \delta \leq \pi$, $\displaystyle\int_{\delta}^{\pi} K_N(t)dt \to 0$ as $N \to \infty$ on $[\delta,\pi]$.

(7) Let $u_n(x) = e^{inx}$ for $x \in [-\pi,\pi,]$, $n \in \mathbb{Z}$ and $\mathcal{P}_T[-\pi,\pi] :=$ span$\{u_n : n \in \mathbb{Z}\}$, i.e., the space of all trigonometric polynomials. Justify the following:
   (a) $\mathcal{P}_T[-\pi,\pi]$ is dense in $CP[-\pi,\pi]$ with respect to $\|\cdot\|_\infty$.
   (b) $\mathcal{P}_T[-\pi,\pi]$ is dense in $L^2[-\pi,\pi]$ with respect to $\|\cdot\|_2$.

(8) Justify the following:
   (a) If $f \in L^1[-\pi,\pi]$ and the Fourier series of f converges uniformly, say to g, then $\hat{g}(m) = \hat{f}(m)$ for all $m \in \mathbb{Z}$.
   (b) If $f \in L^2[-\pi,\pi]$ and if the Fourier series of f converges uniformly, say to g, then $f = g$ a.e.
   (c) If $f \in CP[-\pi,\pi]$ and if the Fourier series of f converges uniformly, say to g, then $f = g$.

(9) Let $f \in L^1[-\pi,\pi]$ and $0 < \delta \leq \pi$. Show that (you may use Riemann Lebesgue lemma) for every $x \in [-\pi,\pi]$,
   (a) $\displaystyle\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} [f(x) - f(x-t)]D_N(t)dt \to 0$ as $N \to \infty$.
   (b) $\displaystyle\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} [f(x+t) - f(x)]D_N(t)dt \to 0$ as $N \to \infty$.

(10) Let $f \in L^1[-\pi,\pi]$. Prove the following (you may use Problem 9 and ideas from Dini criterion):
   (a) If $f(x+) := \lim_{t\to 0+} f(x+t)$, $f'(x+) := \lim_{t\to 0+}\dfrac{f(x+t) - f(x)}{t}$ exist at a point $x \in [-\pi,\pi)$, then $S_N(f,x) \to f(x+)$ as $N \to \infty$.
   (b) If $f(x-) := \lim_{t\to 0+} f(x-t)$, $f'(x-) := \lim_{t\to 0+}\dfrac{f(x) - f(x-t)}{t}$ exist at a point $x \in (-\pi,\pi]$, then $S_N(f,x) \to f(x-)$ as $N \to \infty$.

[2] There exists $\kappa > 0$ such that $|f(x) - f(y)| \leq \kappa|x-y|$ for all $y \in [-\pi,\pi]$, where κ may depend on x

[3] There exists $\kappa > 0$ and $\alpha > 0$ such that $|f(x) - f(y)| \leq \kappa|x-y|^\alpha$ for all $y \in [-\pi,\pi]$, where κ and α may depend on x.

[4] There exists $\kappa > 0$ such that $|f(x) - f(y)| \leq \kappa|x-y|$ for all $x, y \in [-\pi,\pi]$

[5] There exists $\kappa > 0$ and $\alpha > 0$ such that $|f(x) - f(y)| \leq \kappa|x-y|^\alpha$ for all $x, y \in [-\pi,\pi]$

(c) If the limits in (a) and (b) above exist exist at a point $x \in (-\pi, \pi)$, then $S_N(f,x) \to \frac{1}{2}[f(x+) + f(x-)]$ as $N \to \infty$.

(11) Prove the following[6]:

(a) If $f \in ACP[-\pi,\pi]$ is absolutely continuous on $[-\pi,\pi]$, then $\widehat{f'}(n) = (in)\hat{f}(n)$ for all $n \in \mathbb{Z}$.

(b) If $f \in CP^{k-1}[-\pi,\pi]$ and $f^{(k-1)} \in ACP[-\pi,\pi]$, then $\widehat{f^{(k)}}(n) = (in)^k \hat{f}(n)$ for all $n \in \mathbb{Z}$.

(c) If $f \in CP^1[-\pi,\pi]$, $f' \in AC[-\pi,\pi]$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$.

(d) If $f \in L^1[-\pi,\pi]$ and if $\hat{f}(n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ a.e.

(12) *Hardy's Tauberian Theorem*:

Suppose $f \in L^1[-\pi,\pi]$ and $\hat{f}(n) = O(1/n)$. If $\sigma_N(f,x) \to f(x)$ at some point $x \in [-\pi,\pi]$ (resp. uniformly on $[-\pi,\pi]$), then $S_N(f,x) \to f(x)$ (resp. uniformly on $[-\pi,\pi]$).

Prove that if f is absolutely continuous on $[-\pi,\pi]$, then Fourier series of f converges uniformly to f.

(13) Show that, for each $x \in [-\pi,\pi]$ and $N \in \mathbb{N}$, the map $f \mapsto S_N(f,x)$ is a continuous linear transformation on $CP[-\pi,\pi]$ with respect to the norm $\|\cdot\|_\infty$.

(14) Prove that

$$\sup_{f \in CP[-\pi,\pi],\, \|f\|_\infty \le 1} |S_N(f,0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt.$$

(15) Prove that[7] there exists $f \in CP[-\pi,\pi]$ such that $S_N(f,0) \not\to f(0)$ as $N \to \infty$.

(16) Prove that $x^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$ and deduce

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(17) Prove that $x = 2\sum_{n=1} \frac{(-1)^{n+1}}{n} \sin nx$ and deduce the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Department of Mathematics, I.I.T. Madras, Chennai-600 036, INDIA

E-mail address: mtnair@iitm.ac.in

[6]Recall *Fundamental theorem of Lebesgue Integration*: A function $f : [a,b] \to \mathbb{C}$ is absolutely continuous if and only if there exists $g \in L^1[a,b]$ such that $f(x) = f(a) + \int_a^x g(t)dt$ for all $x \in [a,b]$, and in that case f is differentiable a.e., and $f' = g$ a.e.

[7]You may use a special case of *Uniform boundedness principle*: Let $\mathcal{X}$ be a a Banach space and (φ_n) be a sequence of continuous linear functionals on $\mathcal{X}$ which is pointwise bounded on $\{u \in \mathcal{X} : \|u\| \le 1\}$. Then (φ_n) is uniformly bounded on $\{u \in \mathcal{X} : \|u\| \le 1\}$.