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FOURIER ANALYSIS: ASSIGNMENT - II

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(1) Let $f \in L^1(\mathbb{R}^d)$. Prove the following:
   (a) $\hat{f}(\xi)$ is well-defined for every $\xi \in \mathbb{R}^d$.
   (b) $\sup_{\xi \in \mathbb{R}^d)} |\hat{f}(\xi)| \leq \|f\|_1$.
   (c) $\xi \to \hat{f}(\xi)$ is uniformly continuous on $\mathbb{R}^d$.
   (d) The map $f \mapsto \hat{f}$ is a linear operator from $L^1(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$ with norm at most 1.
   (e) $|\hat{f}(\xi)| \to 0$ as $|\xi| \to 0$.

(2) Let $1 \leq p < \infty$. For $f \in L^p(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, let

$$(\tau_y f)(x) := f(x - y), \quad x \in \mathbb{R}^d.$$

For each $f \in L^p(\mathbb{R}^d)$, prove the following:
   (a) $\tau_y f \in L^p(\mathbb{R}^d)$ for every $y \in \mathbb{R}^d$.
   (b) The map $y \mapsto \tau_y f$ from $\mathbb{R}^d$ to $L^p(\mathbb{R}^d)$ is continuous.

(3) Prove that, if $f \in L^1(\mathbb{R})$ is differentiable with $f' \in L^1(\mathbb{R})$, then

$$\widehat{f'}(\xi) = (i\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

(4) Suppose $f \in L^1(\mathbb{R})$ such that $x \mapsto g(x) := xf(x)$ belongs to $L^1(\mathbb{R})$. Prove that $\hat{f}$ is differentiable and

$$(\hat{f})'(\xi) = i\hat{g}(\xi), \quad \xi \in \mathbb{R}.$$

(5) Let

$$(e_h f)(x) = e^{ih\cdot x} f(x), \qquad (\tau_h f)(x) = f(x - h),$$
$$(\mathcal{R}f)(x) = f(-x), \qquad (D_t f)(x) = f(tx).$$

For $f \in L^1(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, $0 \neq t \in \mathbb{R}$, prove the following.
   (a) $\widehat{e_h f} = \tau_h \hat{f}$,
   (b) $\widehat{\tau_h f} = e_{-h}\hat{f}$,
   (c) $\widehat{\mathcal{R}f} = \mathcal{R}\hat{f}$,
   (d) $\widehat{D_t f} = |t|^{-d} D_{1/t}\hat{f}$, $0 \neq t \in \mathbb{R}$.

(6) Prove that the operator $f \mapsto \hat{f}$ from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$ is not onto.

(7) Suppose $f \in L^1(\mathbb{R})$. If $\hat{f} \in L^1(\mathbb{R})$ and if

$$g(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt, \quad x \in \mathbb{R},$$

then prove that $f = g$ a.e. Also deduce the following.

(a) If $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt \quad \forall\, x \in \mathbb{R}.$$

(b) $f \mapsto \hat{f}$ is an injective operator.

(8) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $0 < \varphi(x) \leq 1$ for all $x \in \mathbb{R}$ and $\varphi(\lambda x) := \varphi(\lambda x) \to 1$ as $\lambda \to 0+$. Prove that, for every $g \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(\lambda t) g(t) e^{itx} dt \to \int_{\mathbb{R}} g(t) e^{itx} dt.$$

(9) Let φ be as in Problem(8) and let ψ be defined by $\psi(x) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} \varphi(t) e^{itx} dt$ for $x \in \mathbb{R}$. If $f \in L^1(\mathbb{R})$ is such that $\hat{f} \in L^1(\mathbb{R})$ and if $\int_{\mathbb{R}} \psi(x) dx = 1$, then prove that

$$\frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) \hat{f}(t) e^{itx} dt = \int_{\mathbb{R}} f(x - \lambda s) \psi(s) ds.$$

(10) Let $\psi \in L^1(\mathbb{R})$ be a non-negative function such that $\int_{\mathbb{R}} \psi(x) dx = 1$ and for $f \in L^1(\mathbb{R})$ and $\lambda > 0$, let

$$h_\lambda(x) := \int_{\mathbb{R}} f(x - \lambda s) \psi(s) ds.$$

Prove that $\|h_\lambda - f\|_1 \to 0$ as $\lambda \to 0$. Deduce the following.

(a) There exists a sequence (λ_n) of positive real numbers such that $h_{\lambda_n} \to f$ a.e.

(b) If φ is as in Problem(8), $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, and

$$f_\lambda(x) := \int_{\mathbb{R}} \varphi(\lambda t) \hat{f}(t) e^{itx} dt,$$

then $\|f_\lambda - f\|_1 \to 0$ as $\lambda \to 0$.

(11) Let $\varphi(x) = e^{-|x|}$. Verify the following.

(a) φ satisfies the conditions in Problem(8).

(b) $\psi(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t) e^{itx} dt$, $x \in \mathbb{R}$ satisfies $\int_{\mathbb{R}} \psi(x) dx = 1$.

(12) Let ψ be as in Problem(9). If $f \in L^p(\mathbb{R})$ for $1 \leq p < \infty$, then prove that $f * \psi_\lambda \in L^p(\mathbb{R})$ and $\lim\limits_{\lambda \to 0} \|f * \psi_\lambda - f\|_p = 0$.

(13) Let ψ be as in Problem(9) and $f \in L^\infty(\mathbb{R})$. Prove that if f is continuous at a point $x \in \mathbb{R}$, then $\lim\limits_{\lambda \to 0} (f * \psi_\lambda)(x) = f(x)$, where ψ_λ is as in Problem(9).

(14) Prove the following.

(a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = 2\pi \|f\|_2$.

(b) The set $Y := \{\hat{f} : f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$.

(15) Prove that there exists a unique surjective continuous linear operator Φ from $L^2(\mathbb{R})$ onto itself such that $\Phi(f) = \hat{f}$ for every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Show also that $\|\Phi(f)\|_2 = 2\pi \|f\|_2$ for every $f \in L^2(\mathbb{R})$.

(16) Let $f \in L^1(\mathbb{R}^d)$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \le n\}$. Prove that

$$\int_{E_n} f(x) e^{-ix.\xi} dx \to \int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx = \hat{f}(\xi) \quad \text{as} \quad n \to \infty$$

for every $\xi \in \mathbb{R}^d$.

(17) Let $f \in L^2(\mathbb{R})$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \le n\}$. Prove the following.

(a) $f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $n \in \mathbb{N}$.

(b) $\|f - f_n\|_2 \to 0$ as $n \to \infty$.

(c) $(\hat{f}_n)$ is a Cauchy sequence in $L^2(\mathbb{R})$.

(d) $\|\hat{f}_n - \Phi(f)\|_2 \to 0$ as $n \to \infty$, where Φ is the Fourier-Plancheral transform.

(e) There exists a subsequence $(\hat{f}_{k_n})$ for $(\hat{f}_n)$ such that $\hat{f}_{k_n} \to \Phi(f)$ a.e. on $\mathbb{R}$.

(18) For $f \in L^2(\mathbb{R})$ and $r > 0$, let

$$g_r(\xi) := \int_{-r}^{r} f(x) e^{-ix\xi} dx, \quad h_r(x) := \int_{-r}^{r} \Phi(f)(\xi) e^{ix\xi} dt$$

for all $x, \xi \in \mathbb{R}$. Then $g_r \in L^2(\mathbb{R})$, $h_r \in L^2(\mathbb{R})$, and

$$\|g_r - \Phi(f)\|_2 \to 0, \quad \|h_r - f\|_2 \to 0 \quad \text{as} \quad r \to \infty.$$

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