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FOURIER ANALYSIS: ASSIGNMENT - III

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Throughout, Ω denotes a nonempty open subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$.

(1) Let φ be a mollifier. For $a \in \Omega$ and $\varepsilon > 0$ be such that $\overline{B_\varepsilon(a)} \subset \Omega$, let $\psi_{\varepsilon,a}(x) := \frac{1}{\varepsilon^d}\varphi(\frac{x-a}{\varepsilon}$. Show that $\psi_{\varepsilon,a} \in \mathcal{D}(\Omega)$ such that $\mathrm{supp}(\psi_{\varepsilon,a}) \subseteq B_\varepsilon(a)$ and $\int_\Omega \psi_{\varepsilon,a} dx = 1$.

(2) Let $\psi_{\varepsilon,a}$ be as Problem 1, and let $\psi_\varepsilon := \psi_{\varepsilon,0}$. Prove that for $f \in C_c(\mathbb{R}^d)$, $f * \psi_\varepsilon \to f$ uniformly.

(3) Show that $\mathcal{D}(\Omega)$ is sequentially complete. That is, if (φ_n) in $\mathcal{D}(\Omega)$ is such that for every $\varepsilon > 0$ and for every $\alpha \in \mathbb{N}_0^d$, there exists $N \in \mathbb{N}$ such that $\|\partial^\alpha(\varphi_n - \varphi_m)\|_\infty < \varepsilon$ for all $n \geq N$, then there exists $\varphi \in \mathcal{D}(\Omega)$ such that $\|\partial^\alpha(\varphi_n - \varphi)\|_\infty \to 0$ as $n \to \infty$ for each $\alpha \in \mathbb{N}_0^d$.

(4) Corresponding to $f \in L^1_{\mathrm{loc}}(\Omega)$, let

$$u_f(\varphi) := \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega),\, x \in \Omega.$$

Show that u_f is a distribution, and it is of order 0.

(5) Show that the delta-distribution is not a regular distribution.

(6) Show every delta-distribution is a limit of a sequence of regular distributions.

(7) Let (f_n) in $L^1_{\mathrm{loc}}(\Omega)$ and $f : \Omega \to \mathbb{C}$ be such that $f_n \to f$ a.e. on Ω and for every compact $K \subseteq \Omega$, there exists $g \in L^1(\Omega)$ such that $|f_n| \leq |g|$ a.e. on K. Prove that $f \in L^1_{\mathrm{loc}}(\Omega)$ and $f_n \to f$ in the sense of distribution.

(8) Let $f_n, f \in C(\Omega)$ such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that $f_n \to f$ in the sense of distribution.

(9) Let $f_n(x) := e^{inx}$, $x \in \mathbb{R}$. Show that (u_{f_n}) converges to the zero distribution.

(10) Making use of necessary results, prove that for $f, g \in L^1_{\mathrm{loc}}(\Omega)$, $u_f = u_g$ implies $f = g$ a,e.

(11) Let u be a linear functional on $\mathcal{D}(\Omega)$. Prove that u is a distribution if and only if for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty \tag{1}$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq K$.

(12) Define $u(\varphi) := \sum_{j=0}^\infty \varphi^{(j)}(j), \quad \varphi \in \mathcal{D}(\mathbb{R})$. Show that $u \in \mathcal{D}'(\mathbb{R})$, and it is of infinite order.

(13) Prove that
   (a) $\text{supp}(\delta_a) = \{a\}$.
   (b) For $f \in L^1_{\text{loc}}(\Omega)$, $\text{supp}(u_f) = \text{supp}(f)$.
   (c) For $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, $\text{supp}(u) \cap \text{supp}(f) = \emptyset \implies u(\varphi) = 0$.

(14) If $f \in C^\infty(\Omega)$, then prove that $f\varphi \in \mathcal{D}(\Omega)$ for every $\varphi \in \mathcal{D}(\Omega)$.

(15) For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, prove that the map $\varphi \mapsto u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega)$, is a distribution.

(16) If $f \in C^\infty(\Omega)$ and $a \in \Omega$, show that $f\delta_a = f(a)\delta$.

(17) For $f, g \in L^1_{\text{loc}}(\Omega)$, show that $fu_g = u_{fg}$.

(18) Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Prove that $\text{supp}(fu) \subseteq \text{supp}(f) \cap \text{supp}(u)$.

(19) If u is a distribution with compact support, then prove that for any $f \in \mathcal{E}(\Omega)$, fu is also of compact support.

(20) If $u \in \mathcal{D}'(\Omega)$ is with compact support, then prove that $u \in \mathcal{E}'(\Omega)$ in the sense that for every $u \in \mathcal{D}'(\Omega)$, there exists a unique $\tilde{u} \in \mathcal{D}'(\Omega)$ such that $u|_{\mathcal{D}(\Omega)} = \tilde{u}$.

(21) If $u \in \mathcal{E}'(\Omega)$, then prove that $u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$ is with compact support.

(22) Prove that $\tau_h\delta_a = \delta_{a+h}$. (Recall: For $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, the distribution $\tau_h u$ is defined by $(\tau_h u)(\varphi) := u(\tau_{-h}\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

(23) For each $h \in \mathbb{R}^d$, show that the map $u \mapsto \tau_h u$ is continuous on $\mathcal{D}'(\mathbb{R}^d)$ in the sense that $u_n \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ implies $\tau_h u_n \to \tau_h u$ in $\mathcal{D}'(\mathbb{R}^d)$.

(24) For $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, show that the map $\partial^\alpha u : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by $(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(O)$, is a distribution.

(25) Let H be the *Heaviside function*, i.e., $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$ Show that $H' = \delta_0$.

(26) For $\alpha \in \mathbb{N}_0^d$, $x_0 \in \Omega$, prove that u defined by $u(\varphi) = (\partial^\alpha \varphi)(x_0)$ defines a distribution of order α.

(27) Let (x_n) be a sequence in Ω without a limit point in Ω and $(\alpha^{(n)})$ be a sequence in $\mathbb{N}_0^d$. Let $u(\varphi) := \sum_{n=1}^\infty \partial^{\alpha_n}\varphi(x_n)$. Prove that u is a distribution, and it has finite order if and only if $\sup|\alpha^{(n)}| < \infty$ and in that case the order is $\sup|\alpha^{(n)}|$.

(28) If $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ such that $\text{supp}(u) \cap \text{supp}(\varphi) = \emptyset$, then prove that $u(\varphi) = 0$.

(29) Suppose u is a linear functional on $\mathcal{E}(\Omega)$ such that there exists compact $K \subseteq \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ satisfying $|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{\infty,K} \quad \forall\, \varphi \in \mathcal{E}'(\Omega)$. Prove that $u \in \mathcal{E}'(\Omega)$.

(30) Suppose $u \in \mathcal{E}'(\Omega)$ and there exists compact $K \subseteq \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ satisfying $|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{\infty,K} \quad \forall\, \varphi \in \mathcal{E}'(\Omega)$. Prove that $u|_{\mathcal{D}(\Omega)}$ is a distribution with compact support.

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