Topics in Fourier Analysis-I[1]

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1 Fourier Series

1.1 Motivation through heat equation

The consideration of *Fourier Series* can be traced back to the situation which Fourier encountered in the beginning of last century while solving heat equation:

Consider a thin metallic wire of length ℓ. Suppose an initial temperature is supplied to it, and suppose the temperature at both the end points kept at 0. Then one would like to know the temperature at each point of the string at a particular time.

Let us represent the string as an interval $[0, \ell]$. Let $u(x, t)$ be the temperature at the point $x \in [0, \ell]$ at time t. It is known that $u(\cdot, \cdot)$ satisfies the partial equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < \ell, \tag{1.1}$$

where $c > 0$ is the heat conductivity of the material. Since the temperature at both the end points kept at 0, we have

$$u(0, t) = 0 = u(\ell, t), \quad t > 0. \tag{1.2}$$

Let the initial temperature at the point x be $f()x)$, i.e.,

$$u(x, 0) = f(x), \quad 0 \leq x \leq \ell. \tag{1.3}$$

Exercise 1.1. Equation (1.1) satisfying (1.2) and (1.3) cannot have more than one solution. $\diamondsuit$

In order to find $u(x, t)$, we use a procedure called *method of separation of variables.* In this method, we assume first that $u(x, t)$ is of the form:

$$u(x, t) = \phi(x)\psi(t).$$

Then we have

$$\frac{\partial u}{\partial t} = \phi(x)\psi'(t), \quad \frac{\partial^2 u}{\partial t^2} = \phi''(x)\psi(t).$$

Hence, from (1.1),

$$\phi(x)\psi'(t) = c^2\phi''(x)\psi(t).$$

Hence,

$$\frac{\psi'(t)}{c^2\psi(t)} = \frac{\phi''(x)}{\phi(x)} = K, \text{const.}$$

Hence,

$$\psi'(t) = Kc^2\psi(t), \quad \phi''(x) = K\phi(x). \tag{1.4}$$

Let us consider different cases:

<u>Case(i): $K = 0$</u>: In this case, $\phi''(x) = 0$ so that ϕ is of the form

$$\phi(x) = ax + b.$$

By (1.2), $\phi(0)\psi(t) = 0 = \phi(\ell)\psi(t)$.

$$b\psi(t) = 0 = (a\ell + b)\psi(t).$$

Thus, we arrive at either $\psi = 0$ or $\phi = 0$.

Case(ii): $K > 0$: In this case, $K = \alpha^2$ for some $\alpha \neq 0$. Then we have

$$\phi''(x) - \alpha^2\phi(x) = 0$$

so that ϕ is of the form

$$\phi(x) = ae^{\alpha x} + be^{-\alpha x}.$$

Again, by (1.2), $\phi(0)\psi(t) = 0 = \phi(\ell)\psi(t)$ so that

$$(a + b)\psi(t) = 0 = (ae^{\alpha\ell} + be^{-\alpha\ell})\psi(t).$$

This again lead to $u(x, t) = 0$.

Case(iii): $K < 0$: In this case, $K = -\alpha^2$ for some $\alpha \neq 0$. Then we have

$$\phi''(x) + \alpha^2\phi(x) = 0$$

so that ϕ is of the form

$$\phi(x) = a\cos\alpha x + b\sin\alpha x.$$

By (1.2), $\phi(0)\psi(t) = 0 = \phi(\ell)\psi(t)$ so that if $\psi \neq 0$, we obtain $a = 0$ and $b\sin\alpha\ell = 0$. Assuming $b \neq 0$ (otherwise $\phi = 0$), we have $\alpha\ell = n\pi$, $n \in \mathbb{Z}$. Thus, $\alpha \in \{n\pi/\ell : n \in \mathbb{Z}\}$.

Now, from (1.4),

$$\psi'(t) = -\alpha^2c^2\psi(t)$$

with $\alpha \in \{n\pi/\ell : n \in \mathbb{Z}\}$. Hence,

$$\psi(t) = ae^{-\alpha^2c^2t},$$

and hence, u is of the form

$$u(x, t) = ae^{-\alpha^2c^2t}\sin\alpha x, \quad \alpha \in \{n\pi/\ell : n \in \mathbb{Z}\}.$$

Thus, for each $n \in \mathbb{Z}$,

$$u_n(x, t) = a_ne^{-\lambda_n^2c^2t}\sin\lambda_nx, \quad \lambda_n := \frac{n\pi}{\ell},$$

with $a_n \in \mathbb{R}$ satisfies (1.1) and (1.2). But, this u_n need not satisfy (1.3), unless $f(x) = a_n\sin(n\pi x/\ell)$ for some $n \in \mathbb{Z}$. If f is of the form

$$f(x) = \sum_{n=1}^{k} a_n\sin(n\pi x/\ell) \tag{1.5}$$

for some $k \in \mathbb{N}$, then we see that

$$u(x, t) := \sum_{n=1}^{k} a_ne^{-\lambda_n^2c^2t}\sin(n\pi x/\ell)$$

satisfies (1.1), (1.2) and (1.3).

What can we say if f is, in some sense, arbitrary? The consideration of the functions of the form in (1.5) suggests the following query:

> If f is of the form $f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/\ell)$, can we say that
> $$u(x,t) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 c^2 t} \sin(n\pi x/\ell)$$
> is a solution of (1.1) satisfying (1.2) and (1.3) with appropriate notion of convergence?

As a first step, let us assume that f is of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{\ell}). \tag{1.6}$$

Assume further that, term by term integration of the above series is possible. Then, we have

$$\int_0^\ell f(x) \sin(\frac{m\pi x}{\ell})dx = \sum_{n=1}^{\infty} a_n \int_0^\ell \sin(\frac{n\pi x}{\ell}) \sin(\frac{m\pi x}{\ell})dx.$$

Since

$$\int_0^\ell \sin(\frac{n\pi x}{\ell}) \sin(\frac{m\pi x}{\ell})dx = 0 \quad \text{for} \quad m \neq n,$$

and

$$\int_0^\ell \sin^2(\frac{n\pi x}{\ell})dx = \int_0^\ell \frac{1-\cos 2(\frac{n\pi x}{\ell})}{2}dx = \frac{\ell}{2},$$

we obtain

$$a_n = \frac{2}{\ell} \int_0^\ell f(x) \sin(\frac{n\pi x}{\ell})dx. \tag{1.7}$$

Note that, if f has the form as in (1.6), then f is 2ℓ-periodic, i.e.,

$$f(x+2\ell) = f(x) \quad \text{for all} \quad x \in \mathbb{R}$$

and f is an odd function, i.e.,

$$f(-x) = -f(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

If f is as in (1.6), we we may define

$$u(x,t) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 c^2 t} \sin(n\pi x/\ell), \quad \lambda_n := n\pi/\ell.$$

Assuming that the above series is convergent and can be differentiated term by term, we see that $u(\cdot,\cdot)$ is a solution of (1.1) satisfying (1.2) and (1.3).

Exercise 1.2. Show that each $\lambda_n := n\pi/\ell$ is an eigenvalue of the operator $\frac{d^2}{dx^2}$ wit corresponding eigenvector $\sin \lambda_n x$. ♢

1.2 Fourier Series of 2π-Periodic functions

In the last section, we assumed that the function f can be represented as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{m\pi x}{\ell}).$$

If $\ell = \pi$, then the above series takes the form

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx, \quad A_n := \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

The above series is a special case of the *Fourier series* that we are going to introduce. Let us consider a few definitions.

Definition 1.3. A function of the form

$$c_0 + \sum_{n=1}^{k} (a_n \cos nx + b_n \sin nx).$$

where c_0, $a_n, b_n \in \mathbb{R}$, is called a **trigonometric polynomial**, and a series of the form

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with c_0, $a_n, b_n \in \mathbb{R}$ is called a **trigonometric series**. $\diamondsuit$

Note that a trigonometric polynomial is a special case of a trigonometric series.

We observe that trigonometric polynomials are 2π*-periodic* on $\mathbb{R}$, i.e., if $f(x)$ is a trigonometric polynomial, then

$$f(x + 2\pi) = f(x) \quad \forall x \in \mathbb{R}.$$

From this, we can infer that, if the trigonometric series

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges at a point $x \in \mathbb{R}$, then it has to converge at $x + 2\pi$ as well; and hence at $x + 2n\pi$ for all integers n. This shows that we can restrict the discussion of convergence of a trigonometric series to an interval of length 2π. Hence, we cannot expect to have a trigonometric series expansion for a function $f : \mathbb{R} \to \mathbb{R}$ if it is not a 2π-periodic function.

We know that a convergent trigonometric series is 2π-periodic. What about the converse?

> Suppose that f is a 2π-periodic function. Is it possible to represent f as a trigonometric series?

Suppose, for a moment, that we can write

$$f(x) = c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

for all $x \in \mathbb{R}$. Then what should be the coefficients c_0, a_n, b_n? To answer this question, let us further assume that

> f is integrable on $[-\pi, \pi]$ and the series can be integrated term by term.

For instance if the above series is uniformly convergent to f in $[-\pi, \pi]$, then term by term integration is possible. By Weierstrass test,we have the following result:

> If $\sum_{n=0}^{\infty}(|a_n| + |b_n|)$ converges, then $c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is a dominated series on $\mathbb{R}$ and hence it is uniformly convergent.

For $n, m \in \mathbb{N} \cap \{0\}$, we observe the following *orthogonality relations*:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos nx \cos mx dx &= \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m \neq 0, \\ 2\pi, & \text{if } n = m = 0, \end{cases} \\
\int_{-\pi}^{\pi} \sin nx \sin mx dx &= \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m, \end{cases} \\
\int_{-\pi}^{\pi} \cos nx \sin mx dx &= 0.
\end{aligned}$$

Thus, under the assumption that f is integrable on $[-\pi, \pi]$ and the series can be integrated term by term, we obtain

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Definition 1.4. The **Fourier series** of a 2π-periodic function f is the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$ and this fact is written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The numbers a_n and b_n are called the **Fourier coefficients** of f. ♢

If f is a trigonometric polynomial, then its Fourier series is itself.

Writing

$$\cos nx = \frac{1}{2}[e^{inx} + e^{-inx}], \quad \sin nx = \frac{1}{2i}[e^{inx} - e^{-inx}],$$

we have

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \frac{a_n}{2}[e^{inx} + e^{-inx}] + \frac{b_n}{2i}[e^{inx} - e^{-inx}] \\ &= \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-inx}. \end{aligned}$$

Thus, writing

$$c_n := \frac{a_n}{2} + \frac{b_n}{2i}, \quad c_{-n} := \frac{a_n}{2} - \frac{b_n}{2i},$$

we have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Now, suppose $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$ with $c_n \in \mathbb{C}$, and this series can be integrated term by term. Then, we have

$$\int_{-\pi}^{\pi} f(x) e^{-imx}\, dx = \sum_{n \in \mathbb{Z}} c_n \int_{-\pi}^{\pi} e^{i(n-m)x} dx.$$

But,

$$\int_{-\pi}^{\pi} e^{i(n-m)} dx = \begin{cases} 0 & if\, m = m, \\ 2\pi & if\, m \neq m. \end{cases}$$

Hence, $\int_{-\pi}^{\pi} e^{i(n-m)} dx = 2\pi c_m$, i.e.,

$$c_n = \frac{1}{2\pi} \int_{-2\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}.$$

The following theorem show that there is a large class of functions which can be represented by their Fourier series (see Bhatia [1]). We shall come back to this theorem at a later stage.

THEOREM 1.5. (Dirichlet's theorem) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function which is piecewise differentiable on $(-\pi, \pi)$. Then the Fourier series of f converges, and the limit function $\tilde{f}(x)$ is given by*

$$\tilde{f}(x) = \begin{cases} f(x) & \textit{if } f \textit{ is continuous at } x, \\ \frac{1}{2}[f(x-) + f(x+)] & \textit{if } f \textit{ is not continuous at } x. \end{cases}$$

In Theorem 1.5 we used the terminology *piecewise differentiable* as per the following definition.

Definition 1.6. A function $f : [a, b] \to \mathbb{R}$ is said to be **piecewise differentiable** if f' exists and is piecewise continuous on $[a, b]$ except possibly at a finite number of points. $\diamondsuit$

Remark 1.7. It is known that there are continuous functions f defined on $[-\pi, \pi]$ whose Fourier series does not converge pointwise to f. Its proof relies on UBP (see [2]). We shall consider this at a later occasion. $\Diamond$

Although each term and the partial sums of a Fourier series are infinitely differentiable, the sum function need not be even continuous at certain points. This fact is illustrated by the following example.

Example 1.8. Let $f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ 1, & 0 < x \le \pi. \end{cases}$ By Dirichlet's theorem (Theorem 1.5), the Fourier series of f converges to $f(x)$ for every $x \neq 0$, and at the point 0, the series converges to 1/2. Note that

$$a_n = \frac{1}{\pi} \int_0^\pi \cos nx dx = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

and for $n \in \mathbb{N}$,

$$b_n = \frac{1}{\pi} \int_0^\pi \sin nx dx = \frac{1}{\pi} \left[\frac{1 - \cos n\pi}{n} \right] = \frac{1}{\pi} \left[\frac{1 - (-1)^n}{n} \right] = \begin{cases} \dfrac{2}{\pi n}, & n \text{ odd}, \\ \\ 0, & n \text{ even}. \end{cases}$$

Thus, Fourier series of f is

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^\infty \frac{\sin(2n+1)x}{(2n+1)}.$$

In particular, for $x = \pi/2$,

$$1 = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^\infty \frac{\sin[(2n+1)\pi/2]}{(2n+1)} = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)}$$

which leads to the *Madhava–Nilakaṅtha series*

$$\frac{\pi}{4} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)}.$$

$\Diamond$

1.3 Fourier Series for Even and Odd Functions

The following can be verified easily:

- Suppose f is an even function, i.e.,

$$f(-x) = f(x) \quad \forall\, x \in X.$$

Then $f(x) \cos nx$ is an even function and $f(x) \sin nx$ is an odd function, so that

$$a_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos nx dx = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx dx.$$

- Suppose f is an odd function, i.e.,

$$f(-x) = -f(x) \quad \forall\, x \in X.$$

Then $f(x)\cos nx$ is an odd function and $f(x)\sin nx$ is an even function, so that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx.$$

Thus, we have the following:

(1) Suppose f is an even function. Then the Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{with} \quad a_n := \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx.$$

In particular,

$$f(0) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n, \quad f(\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (-1)^n a_n,$$

respectively.

(2) Suppose f is an odd function. Then the Fourier series of f is

$$\sum_{n=1}^{\infty} b_n \sin nx \quad \text{with} \quad b_n := \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx,$$

In particular,

$$f(\pi/2) = \sum_{n=0}^{\infty} (-1)^n b_{2n+1}.$$

Example 1.9. Consider the function f defined by

$$f(x) = |x|, \quad x \in [-\pi, \pi].$$

In this case, f is an even function. Hence, the Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [-\pi, \pi]$$

with

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

and for $n = 1, 2, \ldots$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \frac{2}{\pi}\left\{\left[x\,\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx\right\} \\ &= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] \end{aligned}$$

Thus,

$$a_{2n} = 0, \quad a_{2n+1} = \frac{-4}{\pi(2n+1)^2}, \quad n = 1, 2, \ldots$$

so that

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}, \quad x \in [-\pi, \pi].$$

Taking $x = 0$ (using Dirichlet's theorem), we obtain

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \qquad \Diamond$$

Example 1.10. Let $f(x) = x, \quad x \in [-\pi, \pi]$. In this case, f is an odd function. Hence, the Fourier series is

$$\sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [-\pi, \pi]$$

with

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi} \left\{ \left[-x \frac{\cos nx}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx \right\} \\ &= \frac{2}{\pi} \left\{ -\pi \frac{\cos n\pi}{n} \right\} = \frac{(-1)^{n+1} 2}{n}. \end{aligned}$$

Thus the Fourier series is

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

In particular (using Dirichlet's theorem), with $x = \pi/2$ we obtain the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \qquad \Diamond$$

Example 1.11. Let $f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 1, & 0 \le x \le \pi. \end{cases}$ In this case, f is an odd function. Hence, the Fourier series is

$$\sum_{n=1}^{\infty} b_n \sin nx,$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2}{\pi}(1 - \cos n\pi) = \frac{2}{\pi}[1 - (-1)^n].$$

Thus

$$f(x) \sim \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}.$$

Taking $x = \pi/2$, again we obtain the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \qquad \Diamond$$

Example 1.12. Let $f(x) = x^2, \quad x \in [-\pi, \pi]$. Since f is an even function, its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [-\pi, \pi], \quad a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx.$$

It can be see that $a_0 = 2\pi^2/3$, and $a_n = (-1)^n 4/n^2$. Thus

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad x \in [-\pi, \pi].$$

Taking $x = 0$ and $x = \pi$ (using Dirichlet's theorem), we have

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \qquad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

respectively. ♢

1.4 Sine and Cosine Series Expansions

Suppose a function f is defined on $[0, \pi]$. By extending it to $[-\pi, \pi]$ so that the extended function is an odd function, we obtain *Forier sine series* of f, and by extending it to $[-\pi, \pi]$ so that the extended function is an even function, we obtain *Fourier cosine series* of f.

The *odd extension* and *even extension* of f, denoted by f_{odd} and f_{even} are defined by

$$f_{odd}(x) = \begin{cases} f(x) & \text{if } 0 \le x < \pi, \\ -f(-x) & \text{if } -\pi \le x < 0, \end{cases},$$

$$f_{even}(x) = \begin{cases} f(x) & \text{if } 0 \le x < \pi, \\ f(-x) & \text{if } -\pi \le x < 0, \end{cases}$$

respectively. Therefore,

$$f(x) = f_{odd}(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [0, \pi]$$

and

$$f(x) = f_{even}(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [0, \pi]$$

with

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Example 1.13. Let $f(x) = x^2$, $x \in [0, \pi]$. The even extension of f is itself. Its odd extension is:

$$f_{odd}(x) = \begin{cases} x^2, & \text{if } 0 \le x < \pi, \\ -x^2, & \text{if } -\pi \le x < 0. \end{cases},$$

Hence,

$$f(x) = f_{odd}(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [0, \pi],$$

with

$$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,dx = \frac{2}{\pi}\left\{\left[-x^2\frac{\cos nx}{n}\right]_0^\pi + \int_0^\pi 2x\frac{\cos nx}{n}\,dx\right\}.$$

Note that

$$\left[-x^2\frac{\cos nx}{n}\right]_0^\pi = -\pi^2\frac{\cos n\pi}{n} = \pi^2\frac{(-1)^{n+1}}{n},$$

$$\int_0^\pi 2x\frac{\cos nx}{n}\,dx = \left[2x\frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi 2\frac{\sin nx}{n}dx = 2\left[\frac{\cos nx}{n^2}\right]_0^\pi = 2\left[\frac{(-1)^n - 1}{n^2}\right].$$

Thus,

$$b_n = \frac{2}{\pi}\left\{\pi^2\frac{(-1)^{n+1}}{n} + 2\left[\frac{(-1)^n - 1}{n^2}\right]\right\} = 2\pi\frac{(-1)^{n+1}}{n} + \frac{4}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right].$$

◊

Example 1.14. Let $f(x) = x$, $x \in [0, \pi]$. Its odd extension is itself, and

$$f_{even}(x) = |x|, \quad x \in [-\pi, \pi].$$

From Examples 1.10 and 1.9, we obtain

$$x \sim 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin nx, \quad x \in [0, \pi]$$

and

$$x \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^\infty \frac{\cos(2n+1)x}{(2n+1)^2}, \quad x \in [0, \pi].$$

◊

Example 1.15. Let us find the sine series expansion of the function

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2, \\ 1, & \text{if } \pi/2 \le x < \pi. \end{cases}$$

The sine series of f is given by

$$f(x) \sim \sum_{n=1}^\infty b_n \sin nx, \quad x \in [0, \pi],$$

where

$$b_n = \frac{2}{\pi}\int_{\pi/2}^\pi \sin nx\,dx = -\frac{2}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^\pi = \frac{2}{\pi}\left[\frac{\cos n\pi/2 - \cos n\pi}{n}\right].$$

Note that $b_{2n-1} = \dfrac{2}{(2n-1)\pi}$ and

$$b_{2n} = \frac{2}{2n\pi}[(-1)^n - 1] = \begin{cases} -\frac{2}{n\pi} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}$$

Thus, for $x \in [0, \pi]$, we have

$$\begin{aligned} \frac{\pi}{2}f(x) \quad \sim \quad & \frac{\sin x}{1} - \frac{\sin 2x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(4n-3)x}{4n-3} \\ & - \frac{\sin(4n-2)x}{4n-2} + \frac{\sin(4n-1)x}{4n-1} + \frac{\sin(4n+1)x}{4n+1} + \cdots. \end{aligned}$$

◊

1.5 Fourier Series of 2ℓ-Periodic Functions

Suppose f is a T-periodic function. We may write $T = 2\ell$. Then we may consider the change of variable $t = \pi x/\ell$ so that the function

$$f(x) := f(\ell t/\pi),$$

as a function of t is 2π-periodic. Hence, its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right)\cos ntdt = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right)\sin ntdt = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx. \end{aligned}$$

In particular,

$$f \quad \text{even} \quad \Longrightarrow \quad b_n = 0 \quad \text{and} \quad a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx,$$

$$f \quad \text{odd} \quad \Longrightarrow \quad a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx.$$

Example 1.16. Let $f(x) = 1 - |x|, \quad -1 \le x \le 1$. Taking $\ell = 1$, we have

$$a_n = \int_{-1}^{1} (1-|x|)\cos n\pi x\, dx = 2\int_0^1 (1-|x|)\cos n\pi x\, dx$$

and

$$b_n = \int_{-1}^{1} (1-|x|)\sin n\pi x\, dx = 0.$$

Now,

$$\int_0^1 \cos n\pi x\, dx = \left[\frac{\sin n\pi x}{n}\right]_0^1 = 0,$$

$$\int_0^1 x\cos n\pi x\, dx = \left[x\frac{\sin n\pi x}{n}\right]_0^1 - \int_0^1 \frac{\sin n\pi x}{n}\, dx = \left[\frac{\cos n\pi x}{n}\right]_0^1 = \frac{(-1)^n - 1}{n}.$$

Hence,

$$a_n = 2\int_0^1 (1-|x|)\cos n\pi x\, dx = \frac{2}{n}[1-(-1)^n] = \begin{cases} 0, & n\text{even}, \\ 4/n, & n \text{ odd}. \end{cases}$$

Thus,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{4}{2n+1}\cos n\pi x. \qquad \Diamond$$

1.6 Fourier Series on Arbitrary Intervals

Suppose a function f is defined in an interval $[a, b]$. We can obtain Fourier expansion of it on $[a, b]$ as follows:

Method 1: Let us consider a change of variable as $y = x - \frac{a+b}{2}$. Let

$$\varphi(y) := f(x) = f(y + \frac{a+b}{2}), \quad \text{where} \quad -\ell \leq y \leq \ell$$

with $\ell = (b - a)/2$. We can extend φ as a 2ℓ-periodic function and obtain its Fourier series as

$$\varphi(y) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi}{\ell} y + b_n \sin \frac{n\pi}{\ell} y \Big)$$

where

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(y) \cos \frac{n\pi y}{\ell} dy, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(y) \sin \frac{n\pi y}{\ell} dy.$$

Thus,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi}{\ell} y + b_n \sin \frac{n\pi}{\ell} y \Big)$$

where

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi y}{\ell} dx, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi y}{\ell} dx$$

with $\ell = (b - a)/2$ and $y = x - \frac{a+b}{2}$.

Method 2: Considering the change of variable as $y = x - a$ and $\ell := b - a$, we define $\varphi(y) := f(x) = f(y + a)$ where $0 \leq y < \ell$. We can extend φ as a 2ℓ-periodic function in any manner and obtain its Fourier series. Here are two specific cases:

(a) For $y \in [-\ell, 0]$, define $\tilde{f}_e(y) = \varphi f(-y)$. Thus $\tilde{f}_e$ on $[-\ell, \ell]$ is an even function. In this case,

$$\varphi(y) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{\ell} y$$

where $\ell = (b - a)/2$ and

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(y) \cos \frac{n\pi y}{\ell} dy.$$

(b) For $y \in [-\ell, 0]$, define $\tilde{f}_o(y) = -\varphi(-y)$. Thus $\tilde{f}_o$ on $[-\ell, \ell]$ is an odd function. In this case,

$$\varphi(y) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\ell} y$$

where

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} \varphi(y) \sin \frac{n\pi y}{\ell} dy.$$

From the series of φ we can recover the corresponding series of f on $[a, b]$ by writing $y = x - a$.

1.7 Exercises

The following are taken from [3].

1. Find the Fourier series of the 2π- period function f such that:

   (a) $f(x) = \begin{cases} 1, & \frac{-\pi}{2} \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \frac{3\pi}{2}. \end{cases}$

   (b) $f(x) = \begin{cases} x, & \frac{-\pi}{2} \leq x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \frac{3\pi}{2}. \end{cases}$

   (c) $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi. \end{cases}$

   (d) $f(x) = \frac{x^2}{4}$, $-\pi \leq x \leq \pi$.

2. Using the Fourier series in Exercise 1, find the sum of the following series:

   (a) $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots,$ (b) $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \ldots.$

   (c) $1 - \dfrac{1}{4} + \dfrac{1}{9} - \dfrac{1}{16} + \ldots,$ (d) $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots.$

3. If $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} \leq x < \frac{\pi}{2} \end{cases}$, then show that

$$f(x) \sim \frac{8}{\pi} \cos \frac{\pi}{4} \left[\frac{\sin x}{1.3} + \frac{\sin 3x}{5.7} + \frac{\sin 10x}{9.11} + \ldots \right].$$

4. Show that for $0 < x < 1$,

$$x - x^2 = \frac{8}{\pi^2} \left[\frac{\sin x\pi}{1^3} + \frac{\sin 3\pi x}{3^3} + \frac{\sin 5\pi x}{5^3} + \ldots \right].$$

5. Show that for $0 < x < \pi$,

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots = \frac{\pi}{4}.$$

6. Show that for $-\pi < x < \pi$,

$$x \sin x = 1 - \frac{1}{2} \cos x - \frac{2}{1.3} \cos 2x + \frac{2}{2.4} \cos 3x - \frac{2}{3.5} \cos 4x + \ldots,$$

   and find the sum of the series

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots.$$

7. Show that for $0 \leq x \leq \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots \right],$$

$$x(\pi - x) = \frac{8}{\pi} \left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots \right].$$

8. Assuming that the Fourier series of f converges uniformly on $[-\pi, \pi)$, show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

9. Using Exercises 7 and 8 show that

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$

10. Write down the Fourier series of $f(x) = x$ for $x \in [1, 2)$ so that it converges to $1/2$ at $x = 1$.

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