TOPICS IN FOURIER ANALYSIS-II

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Contents

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1. Trigonometric series and Fourier series

Definition 1.1. *A series of the form*

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

is called a **trigonometric series***, where* c_0, a_n, b_n *are real numbers.*

- If (1.1) converges on $[-\pi, \pi]$ to a an integrable function f and if it can be integrated term by term, then
$$f(-\pi) = f(\pi),$$
and
$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$
- If the (1.1) converges (pointwise) on $[-\pi, \pi]$ to a function f, then f can be extended as a 2π-periodic function by defining
$$f(x + 2n\pi) = f(x), \quad n \in \mathbb{Z}.$$
- If the series $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$ converges, then (1.1) converges uniformly on $[-\pi, \pi]$ and it can be integrated term by term. We know that if $f \in L^1[-\pi, \pi]$, then the function $\tilde{f} : [-\pi, \pi] \to \mathbb{C}$ defined by
$$\tilde{f}(x) = \begin{cases} f(x), & x \in [-\pi, \pi), \\ f(-\pi), & x = \pi \end{cases}$$
satisfies
$$\tilde{f}(-\pi) = \tilde{f}(\pi) \quad \text{and} \quad \tilde{f} = f \; a.e.$$
- The series (1.1) can be written as
$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Definition 1.2. *Let* $f \in L^1[-\pi, \pi]$*. The* **Fourier series** *of* f *is the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.2}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \tag{1.3}$$

The series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \textit{with} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{1.4}$$

is also called the **Fourier series** *of f. The coefficients c_n are called the* **Fourier coefficient** *and are usually denoted by $\hat{f}(n)$, i.e.,*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}.$$

The sum

$$S_N(f,x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

is called the N-th partial sum of the Fourier series (1.4)).

Notation: In the above and in the following, the integral are w.r.t. the Lebesgue measure.

The fact that (1.2) is the Fourier series of f is usually written as

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Equivalently,

$$f(x) \approx \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

Since $\cos nx, \sin x, e^{inx}$ are 2π-periodic functions, we can talk about Fourier series of 2π-periodic functions. If (1.2) (resp. (1.4)) converges at a point $x \in [-\pi,\pi]$, then it converges at $x + 2k\pi$ for every $k \in \mathbb{Z}$.

- The Fourier series (1.4)) converges at $x \in [-\pi,\pi]$ if and only if $S_N(f,x) \to f(x)$ as $N \to \infty$.
- If $f \in L^1[-\pi,\pi]$, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.
- If $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges, then $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ converges uniformly.

Suppose Fourier series of $f \in L^1[-\pi,\pi]$ converges uniformly, say to g. Then g is continuous and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-imx} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}dx = \hat{f}(m),$$

i.e., $\hat{g}(m) = \hat{f}(m)$ for all $m \in \mathbb{Z}$. A natural question would be whether $f = g$ a.e. We shall answer this affirmatively.

We know that if the Fourier series of $f \in L^1[-\pi, \pi]$ converges, then

$$\hat{f}(n) \to 0 \quad \text{as} \quad n \to \infty.$$

Can we assert this for every $f \in L^1[-\pi, \pi]$? The answer is in the affirmative as proved in the next section.

2. Riemann Lebesgue Lemma

Theorem 2.1. (Riemann Lebesgue lemma) *Let $f \in L^1[a, b]$. Then*

$$\int_a^b f(t)\cos(\lambda t)dt \to 0 \quad \text{and} \quad \int_a^b f(t)\sin(\lambda t)dt \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Corollary 2.2. (Riemann Lebesgue lemma) *Let $f \in L^1[a, b]$. Then*

$$\int_a^b f(t)\cos(nt)dt \to 0 \quad \text{and} \quad \int_a^b f(t)\sin(nt)dt \to 0 \quad \text{as} \quad n \to \infty.$$

For the proof of the Theorem 2.1, we shall make use of

LEMMA 2.3. *The span of all step functions*[1] *on $[a, b]$ is dense in $L^1[a, b]$.*

Proof of Theorem 2.1. First we observe that if for every $\varepsilon > 0$, there exists a function $g \in L^1[a, b]$ such that $\|f - g\|_1 < \varepsilon$ and the the result is true for g, then the result is true for f also.

Indeed,

$$\begin{aligned}\left|\int_a^b f(t)\cos(\lambda t)dt\right| &\leq \left|\int_a^b [f(t) - g(t)]\cos(\lambda t)dt\right| + \left|\int_a^b g(t)\cos(\lambda t)dt\right| \\ &\leq \varepsilon + \left|\int_a^b g(t)\cos(\lambda t)dt\right|.\end{aligned}$$

Let $\lambda_0 > 0$ be such that $\left|\int_a^b g(t)\cos(\lambda t)dt\right| < \varepsilon$ for all $\lambda \geq \lambda_0$. Then we have

$$\left|\int_a^b f(t)\cos(\lambda t)dt\right| < 2\varepsilon \quad \forall\, \lambda \geq \lambda_0$$

[1]Step functions are finite linear combinations of characteristic functions. Also, recall that $L^1[a, b]$ is the vector space of all Lebesgue measurable complex valued functions f such that $\|f\|_1 := \int_a^b |f(x)|dx < \infty$. Here, dx stands for the Lebesgue measure.

so that $\int_a^b f(t)\cos(\lambda t)dt \to 0$ as $\lambda \to \infty$. Similarly, $\int_a^b f(t)\sin(\lambda t)dt \to 0$ as $\lambda \to \infty$.

Hence, it is enough to prove the result for step functions. Since every step function is a finite linear combination of characteristic functions on intervals, it is enough to prove for f of the form $f = \chi_{[c,d]}$, $[c,d] \subseteq [a,b]$. Note that

$$\begin{aligned} \left|\int_a^b \chi_{[c,d]}\cos(\lambda t)dt\right| &= \left|\int_c^d \cos(\lambda t)dt\right| \\ &= \left|\frac{\sin(\lambda d) - \sin(\lambda c)}{\lambda}\right| \\ &\leq \frac{2}{|\lambda|} \to 0 \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

Similarly, $\left|\int_a^b \chi_{[c,d]}\sin(\lambda t)dt\right| \to 0$ as $\lambda \to \infty$. □

Remark 2.4. *If f is Riemann integrable on $[a,b]$, then there exists a sequence of (f_n) of step functions such that $\|f - f_n\|_1 \to 0$. Thus, conclusion in Theorem 2.1 holds if f is Riemann integrable.*

Proof of Lemma 2.3. If $f \in L^1[a,b]$ with $f \geq 0$, then there exists an increasing sequence of non-negative simple measurable functions $\varphi_n, n \in \mathbb{N}$ such that $\varphi_n \to f$ pointwise. Hence, by DCT, $\int_a^b |f - \varphi_n| \to 0$. From this, for any complex valued $f \in L^1[a,b]$, there exists a sequence (φ_n) of simple complex measurable functions

$$\int_a^b |f - \varphi_n| \to 0.$$

We observe (see [3]):

(1) Every simple real valued measurable function is a finite linear combination of characteristic function of measurable sets.
(2) For every measurable set $E \subseteq (a,b)$ and $\varepsilon > 0$, there exists an open set $G \supseteq E$ such that $m(G \setminus E) < \varepsilon$. Hence,

$$\int_a^b |\chi_G - \chi_E| = \int_a^b |\chi_{(G\setminus E)}| \leq m(G \setminus E) < \varepsilon.$$

(3) If $G \subseteq (a,b)$ is an open set, then $G = \bigcup_{k=1}^\infty I_n$, where $\{I_n\}$ is a countable disjoint family of open intervals in (a,b);

$$\chi_G = \lim_{n\to\infty} \psi_n, \quad \psi_n = \sum_{k=1}^n \chi_{I_k},$$

Since $0 \leq \psi_n \leq \chi_G$, by DCT,

$$\int |\chi_G - \psi_n| \to 0.$$

(4) By (1)-(3), if φ is a simple measurable function and $\varepsilon > 0$, there exists a step function ψ such that

$$\int_a^b |\varphi - \psi| < \varepsilon.$$

Thus, the lemma is proved. □

3. Dirichlet kernel

Note that

$$S_N(f,x) := \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} \int_{-\pi}^{\pi} f(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) dt,$$

where

$$D_N(t) := \sum_{n=-N}^{N} e^{int}.$$

Redefining f at the end-points if necessary, and extending it as a 2π-periodic function, we can also write (verify!),

$$S_N(f,x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt.$$

Notation: We denote by T the unit circle $T := \{e^{it} : -\pi \leq t < \pi\}$. Note that if $f : T \to \mathbb{C}$ and if we define $\tilde{f} : \mathbb{R} \to \mathbb{C}$ by $\tilde{f}(t) = f(e^{it})$, then

$$\tilde{f}(-\pi) = \tilde{f}(\pi) \quad \text{and} \quad \tilde{f}(t + 2n\pi) = f(t) \quad \text{for all} \quad n \in \mathbb{Z}.$$

That is, $\tilde{f}$ is a 2π-periodic function. In the due course, we shall identify 2π-periodic functions with functions on T. We shall denote $L^1(T)$ for the space of all 2π-periodic (complex valued) functions on $\mathbb{R}$ (with equality replaced equal a.e.) which are integrable on $[-\pi, \pi]$ with norm

$$f \mapsto \|f\|_1 := \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx.$$

Analogously, for $1 \leq p < \infty$, $L^p(T)$ denotes the space of all 2π-periodic (complex valued) functions f on $\mathbb{R}$ such that $|f|^p$ is integrable on $[-\pi, \pi]$ with norm

$$f \mapsto \|f\|_p := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right)^{1/p}$$

The space $L^2(T)$ is also a Hilbert space with inner product

$$(f,g) \mapsto \langle f,g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

Definition 3.1. *The function $D_N(\cdot)$ is called the* **Dirichlet kernel**.

We observe that,

- $D_N(-t) = D_N(t)$ for all $t \in [-\pi,\pi]$ and
- $\displaystyle\int_{-\pi}^{\pi} D_N(t)dt = 1.$
- $\displaystyle D_N(t) = \sum_{n=-N}^{N} e^{int} = 1 + \sum_{n=1}^{N}[e^{int} + e^{-int}] = 1 + 2\sum_{n=1}^{N}\cos nt.$

Remark 3.2. *We shall see that $\displaystyle\int_{-\pi}^{\pi}|D_N(t)|dt \to \infty$ as $N \to \infty$.*

Theorem 3.3.

$$D_N(t) = \begin{cases} 2N+1, & t = 0, \\ \dfrac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}, & t \neq 0. \end{cases}$$

Proof. Clearly, $D_N(0) = 2N+1$. So, let $t \neq 0$. Note that

$$(e^{it}-1)D_N(t) = \sum_{n=-N}^{N}[e^{i(n+1)t} - e^{int}] = e^{i(N+1)t} - e^{-iNt}.$$

But,

$$(e^{it}-1)D_N(t) = e^{it/2}(e^{it/2} - e^{-it/2})D_N(t) = 2ie^{it/2}\sin(t/2)D_N(t).$$

Thus,

$$2i\sin(t/2)D_N(t) = e^{-it/2}[e^{i(N+1/2)t} - e^{-i(N+1/2)t}] = 2i\sin(N+1/2)t.$$

i.e.,

$$D_N(t) = \frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}, \quad t \neq 2k\pi.$$

□

4. Dirichlet-Dini criterion for convergence

We investigate the convergence:

$$S_N(f,x) \to f(x).$$

Since $\int_{-\pi}^{\pi} D_N(t)dt = 1$ and $S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt$, we have

$$f(x) - S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) - f(x-t)]D_N(t)dt.$$

Theorem 4.1. (Dirichlet-Dini criterion) *Let $f \in L^1(T)$. If f satisfies*

$$\int_{-\pi}^{\pi} \left| \frac{f(x) - f(x-t)}{t} \right| dt < \infty \qquad (*)$$

at a point $x \in [-\pi, \pi]$, then

$$S_N(f,x) \to f(x).$$

If $(*)$ hods uniformly for $x \in [-\pi,\pi]$, then the convergence $\{S_N(f,x)\}$ to $f(x)$ is uniform.

Remark 4.2. *In the above theorem, by* $\dfrac{f(x) - f(x-t)}{t}$, *we mean the function*

$$\varphi(t) = \begin{cases} \frac{f(x)-f(x-t)}{t}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Proof of Theorem 4.1. We observe that

$$\begin{aligned} f(x) - S_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) - f(x-t)]D_N(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) - f(x-t)]\frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \left\{\frac{f(x) - f(x-t)}{t}\right\}\left\{\frac{t/2}{\sin(\frac{t}{2})}\right\}\sin(N+\frac{1}{2})tdt \end{aligned}$$

Since $(t/2)/[\sin(t/2)]$ is bounded, in view of Riemann Lebesgue lemma, we have the following. □

The following corollaries are immediate from Theorem 4.1.

Corollary 4.3. *Suppose f is Lipschitz at a point*[2] *$x \in [-\pi,\pi]$. Then*

$$S_N(f,x) \to f(x) \quad \text{as} \quad N \to \infty.$$

[2]A function $\varphi : I \to \mathbb{C}$ is said to be *Lipschitz at a point* $x_0 \in I$ if there exists $K_0 > 0$ such that $|\varphi(x) - \varphi(x_0)| \leq K_0|x - x_0|$ for all $x \in I$.

Corollary 4.4. *Suppose f is Lipschitz*[3] *on $[-\pi,\pi]$. Then*

$$S_N(f,x)\to f(x) \quad as \quad N\to\infty$$

uniformly on $[-\pi,\pi]$.

Notation: We denote by $C(T)$ the space of all 2π-periodic continuous functions on $\mathbb{R}$, and by $C^k(T)$ for $k\in\mathbb{N}\cup\{0\}$, the space of all 2π-periodic functions on $\mathbb{R}$ which are k-times continuously differentiable on $\mathbb{R}$.

Corollary 4.5. *If $f\in C^1(T)$, then*

$$S_N(f,x)\to f(x) \quad as \quad N\to\infty$$

uniformly on $\mathbb{R}$.

Now obtain a more general result.

Theorem 4.6. *Suppose f is a 2π-periodic function such that the following limits exist at a point $x\in\mathbb{R}$:*

$$f(x+):=\lim_{t\to 0+}f(x+t),\quad f(x-):=\lim_{t\to 0+}f(x-t),$$

$$f'(x+):=\lim_{t\to 0+}\frac{f(x+t)-f(x+)}{t},\quad f'(x-):=\lim_{t\to 0+}\frac{f(x-)-f(x-t)}{t}.$$

Then

$$S_N(f,x)\to\frac{f(x+)+f(x-)}{2}\quad as \quad N\to\infty.$$

Proof. Since $D_N(t)=D_N(-t)$, we have

$$\begin{aligned}S_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-t)D_N(t)dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{0}f(x-t)D_N(t)dt+\frac{1}{2\pi}\int_{0}^{\pi}f(x-t)D_N(t)dt\\ &= \frac{1}{2\pi}\int_{0}^{\pi}f(x+t)D_N(t)dt+\frac{1}{2\pi}\int_{0}^{\pi}f(x-t)D_N(t)dt\\ &= \frac{1}{2\pi}\int_{0}^{\pi}[f(x+t)+f(x-t)]D_N(t)dt\end{aligned}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(t)dt=\frac{2}{2\pi}\int_{0}^{\pi}D_N(t)dt.$$

[3]A function $\varphi: I\to\mathbb{C}$ is said to be *Lipschitz on I* if there exists $K>0$ such that $|\varphi(x)-\varphi(x_0)|\le K_0|x-x_0|$ for all $x\in I$.

Hence, for any $\beta \in \mathbb{R}$,

$$S_N(f,x) - \beta = \frac{1}{2\pi}\int_0^\pi [f(x+t) + f(x-t) - 2\beta] D_N(t)dt.$$

Taking $\beta = \frac{f(x+)+f(x-)}{2}$, we have

$$f(x+t) + f(x-t) - 2\beta = [f(x+t) - f(x+)] - [f(x-) - f(x-t)].$$

Thus,

$$S_N(f,x) - \beta = A_N + B_N,$$

where

$$A_N = \frac{1}{2\pi}\int_0^\pi [f(x+t) - f(x+)] D_N(t)dt, \quad B_N = \frac{1}{2\pi}\int_0^\pi [f(x-) - f(x-t)] D_N(t)dt.$$

Note that

$$\begin{aligned} A &= \frac{1}{2\pi}\int_0^\pi [f(x+t) - f(x+)] D_N(t)dt \\ &= \frac{1}{2\pi}\int_0^\pi [f(x+t) - f(x+)] \frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})} dt \\ &= \frac{1}{\pi}\int_0^\pi \left\{\frac{f(x+t) - f(x+)}{t}\right\}\left\{\frac{t/2}{\sin(\frac{t}{2})}\right\} \sin(N+\frac{1}{2})t dt \end{aligned}$$

Since $\frac{f(x+t)-f(x+)}{t} \to f'(x+)$ as $t \to 0+$, there exists $\delta > 0$ such that

$$\begin{aligned} 0 < t < \delta \quad &\Longrightarrow \quad \left|\frac{f(x+t) - f(x+)}{t} - f'(x+)\right| \le 1 \\ &\Longrightarrow \quad \left|\frac{f(x+t) - f(x+)}{t}\right| \le 1 + |f'(x+)|. \end{aligned}$$

Hence, the function

$$t \mapsto \left\{\frac{f(x+t) - f(x+)}{t}\right\}\left\{\frac{t/2}{\sin(\frac{t}{2})}\right\}, \quad t \ne 0,$$

is bounded on $(0,\delta)$, and hence, belongs to $L^1(T)$. Therefore,by Riemann Lebesgue lemma, $A_N \to 0$ as $N \to \infty$. Similarly, we see that, $B_N \to 0$ as $N \to \infty$. □

An immediate corollary:

Corollary 4.7. *If $f \in C(T)$ and has left and right derivative at a point x, then $S_N(f,x) \to f(x)$ as $N \to \infty$.*

The following result is known as *localization lemma*.

LEMMA 4.8. *For $0 < r < \pi$ and $x \in [-\pi, \pi]$,*

$$\int_{r\le|t|\le\pi} f(x-t)D_N(t)dt \to 0 \quad as \quad N\to\infty.$$

Proof. Observe that

$$\int_{r\le|t|\le\pi} f(x-t)D_N(t)dt = \int_{r\le|t|\le\pi} g(x,t)\sin(N+1/2)tdt,$$

where

$$g(x,t) = \begin{cases} f(x-t)/\sin(t/2), & r\le|t|\le\pi,\\ 0, & |t|\le r.\end{cases}$$

Since $g(x,\cdot)$ is integrable, by Riemann Lebesgue lemma,

$$\int_{r\le|t|\le\pi} g(x,t)\sin(N+1/2)tdt \to 0 \quad \text{as} \quad N\to\infty.$$

□

Proof of Corollary 4.4 using localization lemma. Suppose f is Lipschitz at a point $x\in[-\pi,\pi]$ with Lipschitz constant K_x, i.e., there exists $\delta>0$ such that

$$|f(x)-f(x-t)| \le K_x|t| \quad \text{whenever} \quad |t|<\delta.$$

Now,

$$\begin{aligned} f(x) - S_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-f(x-t)]D_N(t)dt \\ &= \frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt \\ &\quad +\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt \end{aligned}$$

By Lemma 4.8,

$$\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt \to 0 \quad \text{as} \quad N\to\infty.$$

Hence, for a given $\varepsilon>0$, there exists $N_0\in\mathbb{N}$ such that for all $N\ge N_0$,

$$\left|\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt\right| < \varepsilon/2.$$

Also, But,

$$\left|\frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt\right| \le \frac{1}{2\pi}\int_{0\le|t|<\delta}|f(x)-f(x-t)|\,|D_N(t)|dt,$$

$$\frac{1}{2\pi}\int_{0\le|t|<\delta}|f(x)-f(x-t)|\,|D_N(t)|dt \le K_x\frac{1}{2\pi}\int_{0\le|t|<\delta}|t|\,|D_N(t)|dt,$$

$$|t|\,|D_N(t)| = |t|\left|\frac{\sin(N+\frac{1}{2})t}{\sin(\frac{t}{2})}\right| = 2\left|\frac{t/2}{\sin(\frac{t}{2})}\right|\,|\sin(N+\frac{1}{2})t| \le 2M,$$

where M is a bound for $\left|\frac{t/2}{\sin(\frac{t}{2})}\right|$ | on $0<|t|\le\delta$. Hence,

$$\left|\frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt\right| \le \frac{4MK_x\delta}{2\pi} = \frac{2MK_x\delta}{\pi}.$$

We may take δ such that $\frac{2MK_x\delta}{\pi}<\varepsilon/2$. Hence,

$$\begin{aligned}|f(x)-S_N(f,x)| &\le \left|\frac{1}{2\pi}\int_{0\le|t|<\delta}[f(x)-f(x-t)]D_N(t)dt\right| \\ &\quad +\left|\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}[f(x)-f(x-t)]D_N(t)dt\right| \\ &< \varepsilon \quad \text{for all} \quad N\ge N_0.\end{aligned}$$

□

Exercise 4.9. *Suppose f is 2π-periodic and Hölder continuous at x, i.e., there exist $M>0$ and $\alpha>0$ such that $|f(x)-f(y)|\le M|x-y|^\alpha$ for all $y\in[-\pi,\pi]$. Then show that $S_N(f,x)\to f(x)$ as $N\to\infty$.*

Exercise 4.10. *Suppose f is 2π-periodic and Hölder continuous on $[-\pi,\pi]$, i.e., there exist $M>0$ and $\alpha>0$ such that $|f(x)-f(y)|\le M|x-y|^\alpha$ for all $x,y\in[-\pi,\pi]$. Then show that $S_N(f,x)\to f(x)$ uniformly.*

5. Cès̀aro summablity of Fourier series

Theorem 5.1. (Fejér's theorem) *If $f\in C(T)$, then the Fourier series of f is uniformly Cesàro summable on $[-\pi,\pi]$, that is,*

$$\sigma_N(f,x) := \frac{1}{N+1}\sum_{k=0}^{N} S_k(f,x) \to f(x) \quad \text{as} \quad N\to\infty$$

uniformly on $[-\pi,\pi]$.

Recall that

$$S_k(f,x) := \sum_{n=-k}^{k}\hat{f}(n)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_k(t)dt.$$

Hence,

$$\sigma_N(f,x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(f,x) = \int_{-\pi}^{\pi} f(x-t)\left\{\frac{1}{N+1}\sum_{k=0}^{N} D_k(t)\right\}.$$

Thus,

$$\sigma_N(f,x) = \int_{-\pi}^{\pi} f(x-t)K_N(t)dt,$$

where

$$K_N(t) := \frac{1}{N+1}\sum_{k=0}^{N} D_k(t).$$

Definition 5.2. *The function $K_N(t)$ defined above is called the* **Fejér** *kernel.*

We observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1.$$

Hence,

$$f(x) - \sigma_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x) - f(x-t)]K_N(t)dt.$$

For the proof of Theorem 5.1, we shall make use of the following lemma.

LEMMA 5.3. *The following results hold.*

(1) *For $t \neq 0$,*

$$K_N(t) = \frac{1}{N+1}\frac{1-\cos(N+1)t}{1-\cos t} = \frac{1}{N+1}\frac{\sin^2[(N+1)t/2]}{\sin^2(t/2)}.$$

(2) *$K_N(t)$ is an even function and $K_N(t) \geq 0$ for all $t \in [-\pi,\pi]$.*

(3) *For $0 < \delta \leq \pi$,*

$$K_N(t) \leq \frac{1}{N+1}\left(\frac{1}{\sin^2(\delta/2)}\right).$$

In particular, K_N is positive and $K_N(t) \to 0$ as $N \to \infty$ uniformly on $0 < \delta \leq |t| \leq \pi$.

Proof of Theorem 5.1. Sine $K_N(t)$ is a non-negative function (see Lemma 5.3), we have

$$|f(x) - \sigma_N(f,x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x) - f(x-t)|K_N(t)dt.$$

Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta \in (0,\pi]$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad |x-y| < \delta.$$

Hence,

$$\frac{1}{2\pi}\int_{|t|<\delta}|f(x) - f(x-t)|K_N(t)dt < \frac{\varepsilon}{2\pi}\int_{|t|<\delta} K_N(t)dt = \varepsilon.$$

Also, since f is uniformly bounded there exists $M > 0$ such that $|f(y)| \leq M$ for all $x \in [-\pi, \pi]$.

$$\frac{1}{2\pi}\int_{|t|\geq\delta} |f(x) - f(x-t)| K_N(t)dt \leq \frac{2M}{2\pi}\int_{|t|\geq\delta} K_N(t)dt.$$

We have observed in Lemma 5.3 that $K_N(t)$ is an even function and $K_N(t) \to 0$ as $N \to \infty$ uniformly on $[\delta, \pi]$. Hence, there exists N_0 such that

$$\frac{1}{2\pi}\int_{|t|\geq\delta} |f(x) - f(x-t)| K_N(t)dt \leq \frac{4M}{2\pi}\int_{\delta}^{\pi} K_N(t)dt < \varepsilon \quad \text{for all} \quad N \geq N_0.$$

Hence,

$$|f(x) - \sigma_N(f,x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f(x-t)| K_N(t)dt < 2\varepsilon$$

for all $N \geq N_0$. Note that N_0 is independent of the point x. Thus, we have proved that $S_N(f,x) \to f(x)$ as $N \to \infty$ uniformly for $x \in [-\pi, \pi]$. □

Remark 5.4. The proof of Theorem 5.1 reveals more:

> If f is peace-wise continuous and 2π-periodic, and continuous at x, then $\sigma_N(f,x) \to f(x)$ as $N \to \infty$.

Notation:

- $u_n(x) := e^{inx}$, $n \in \mathbb{Z}$.
- $AC(T)$ denotes the vector space of all 2π-periodic complex valued functions defined on $\mathbb{R}$ which are absolutely continuous.
- $\text{span}\{u_n : n \in \mathbb{Z}\}$ is the space (over $\mathbb{C}$) of all trigonometric polynomials.

Corollary 5.5. *The space of all trigonometric polynomials is dense in $C(T)$ with respect to the uniform norm, and hence dense in $L^p(T)$ w.r.t. $\|\cdot\|_p$ for $1 \leq p < \infty$.*

Proof. By Theorem 5.1, space of all trigonometric polynomials is dense in $C(T)$ with respect to the uniform norm $\|\cdot\|_\infty$. Hence, for any $f \in C(T)$, there exists a sequence (f_n) of trigonometric polynomials such that

$$\|f - f_n\|_p^p = \int_{-\pi}^{\pi} |f(x) - f_n(x)|^p dx \leq 2\pi\|f - f_n\|_\infty^p \to 0$$

as $n \to \infty$. □

Corollary 5.6. *If $f \in L^2(T)$ for some $1 \leq p < \infty$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.*

Proof. Suppose $f \in L^2(T)$ for some $1 \leq p < \infty$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, i.e., $\langle f, u_n \rangle = 0$ for all $n \in \mathbb{Z}$. By Corollary 5.5, it follows that $\|f\|_{L^2} = 0$. Hence, $f = 0$ a.e. $\square$

Corollary 5.7. *If $f \in C(T)$ such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$. In particular, if $f, g \in C(T)$ such that $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f = g$.*

Proof. Suppose $f \in C(T)$ such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Thus, $\langle f, u_n \rangle_{L^2} = 0$ for all $n \in \mathbb{Z}$. Since $C(T) \subseteq L^2[-\pi, \pi]$, $f \in L^2[-\pi, \pi]$. Hence by Corollary, $f = 0$ a.e. Since f is continuous, $f = 0$. $\square$

The above corollary shows:

The Fourier coefficients of $f \in C(T)$ determines f uniquely.

Corollary 5.8. *If $f \in C^2(T)$, then*

$$\widehat{f''}(n) = (in)^2 \hat{f}(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

In particular, $\hat{f}(n) = o(\frac{1}{n^2})$, and the Fourier series of f converges uniformly to f.

Proof. Let $f \in C^2(T)$. Then, using integration by parts, we obtain,

$$\begin{aligned} 2\pi \hat{f}(n) &= \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \Big[f(x) \frac{e^{-inx}}{-in} \Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x) \Big[\frac{e^{-inx}}{-in} \Big] dx \\ &= \frac{1}{in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\ &= \frac{1}{in} \Big[f'(x) \frac{e^{-inx}}{-in} \Big]_{-\pi}^{\pi} - \frac{1}{in} \int_{-\pi}^{\pi} f''(x) \Big[\frac{e^{-inx}}{-in} \Big] dx \\ &= \frac{1}{(in)^2} \int_{-\pi}^{\pi} f''(x) e^{-inx} dx. \end{aligned}$$

Hence, $\widehat{f''}(n) = (in)^2 \hat{f}(n)$ for all $n \in \mathbb{Z}$. In particular, $\hat{f}(n) = o(1/n^2)$. Therefore, $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|$ converges, and hence the Fourier series converges uniformly. Suppose $S_N(f, x) \to g(x)$ uniformly. Then it follows that $g \in C(T)$ and $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. Therefore, by Corollary 5.7, $g = f$. $\square$

Following the same arguments as in the proof of Corollary 5.8, we obtain:

Corollary 5.9. *If $f \in C^1(T)$ and f' is absolutely continuous, then f'' exists almost everywhere, $f'' \in L^1[-\pi, \pi]$ and*

$$\widehat{f''}(n) = (in)^2 \hat{f}(n) \quad \textit{for all} \quad n \in \mathbb{Z},$$

and the Fourier series of f converges uniformly to f.

More generally,

Theorem 5.10. *If $f \in C^{k-1}(T)$ and $f^{(k-1)}$ is absolutely continuous for some $k \in \mathbb{N}$, then $f^{(k)}$ exists almost everywhere $f^{(k)} \in L^1(T)$ and*

$$\widehat{f^{(k)}}(n) = (in)^k \hat{f}(n) \quad \textit{for all} \quad n \in \mathbb{Z}.$$

Proof of Lemma 5.3. We have

$$K_N(t) := \frac{1}{N+1} \sum_{k=0}^{N} D_k(t) \quad \text{where} \quad D_k(t) = \frac{\sin(k+1/2)t}{\sin t/2}$$

Hence,

$$(N+1)K_N(t) = \sum_{k=0}^{N} \frac{\sin(k+1/2)t}{\sin t/2} = \sum_{k=0}^{N} \frac{e^{i(k+1/2)t} - e^{-i(k+1/2)t}}{e^{it/2} - e^{-it/2}}$$

But,

$$\frac{e^{i(k+1/2)t} - e^{-i(k+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{e^{i(k+1)t} - e^{-ikt}}{e^{it} - 1},$$

$$\frac{e^{i(k+1/2)t} - e^{-i(k+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{e^{ikt} - e^{-i(k+1)t}}{1 - e^{-it}},$$

Therefore,

$$[e^{it} - 1](N+1)K_N(t) = \sum_{k=0}^{N} [e^{i(k+1)t} - e^{-ikt}], \tag{1}$$

$$[1 - e^{-it}](N+1)K_N(t) = \sum_{k=0}^{N} [e^{ikt} - e^{-i(k+1)t}] \tag{2}$$

Subtracting the (2) from (1),

$$[2\cos t - 2](N+1)K_N(t) = 2\sum_{k=0}^{N} [\cos(k+1)t - \cos kt] = 2[\cos(N+1)t - 1]$$

Thus,

$$K_N(t) = \frac{1}{N+1} \frac{\cos(N+1)t - 1}{\cos t - 1} = \frac{1}{N+1} \frac{\sin^2[(N+1)t/2]}{\sin^2(t/2)}.$$

Thus, we have proved (1). It is clear that $K_N(t)$ is even and non-negative. Now, for $0 < \delta \leq \pi$, $\sin^2(t/2) \geq \sin\delta/2$, so that

$$\int_\delta^\pi K_N(t)dt = \frac{1}{N+1}\int_\delta^\pi \frac{\sin^2[(N+1)t/2]}{\sin^2(t/2)}dt \leq \frac{1}{N+1}\int_\delta^\pi \frac{1}{\sin^2(\delta/2)}dt.$$

Thus,

$$\int_\delta^\pi K_N(t)dt \leq \frac{\pi-\delta}{(N+1)\sin^2(\delta/2)} \to 0 \quad \text{as} \quad N \to \infty.$$

□

Exercise 5.11. *Suppose f is piecewise continuous and 2π-periodic. If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(x) = 0$ for all x at which f is continuous.*

Exercise 5.12. *If $f \in C^1(T)$, then $\hat{f}(n) = O(1/n)$. More generally, $f \in C^k(T)$ implies $\hat{f}(n) = O(1/n^k)$.*

Example 5.13. Let $f(x) = x^2$, $|x| \leq \pi$. Note that

$$2\pi\hat{f}(0) = \int_{-\pi}^{\pi} x^2 dx = 2\frac{\pi^3}{3}$$

so that $\hat{f}(0) = \pi^2/3$, and for $n \neq 0$,

$$\begin{aligned}
2\pi\hat{f}(n) &= \int_{-\pi}^{\pi} x^2 e^{-inx}dx \\
&= \left[x^2\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{e^{-inx}}{-in}dx \\
&= \left[x^2\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \left[2x\frac{e^{-inx}}{(-in)^2}\right]_{-\pi}^{\pi} \\
&= -\left[2x\frac{e^{-inx}}{(-in)^2}\right]_{-\pi}^{\pi} = \left[2x\frac{e^{-inx}}{n^2}\right]_{-\pi}^{\pi} = 4\pi\frac{e^{inx}}{n^2} \\
&= 4\pi\frac{(-1)^n}{n^2}
\end{aligned}$$

Hence, for $n \neq 0$,

$$\hat{f}(n) = 2\frac{(-1)^n}{n^2}.$$

Thus,

$$x^2 \approx \frac{\pi^2}{3} + 2\sum_{n\neq 0}^{\infty}\frac{(-1)^n}{n^2}e^{inx} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

Since the series of coefficients converges absolutely, we have

$$f(x) = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

Taking $x = 0$,

$$0 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Taking $x = \pi$,

$$\pi^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 5.14. Let $f(x) = x$, $x \in [-\pi, \pi]$. Note that $\hat{f}(0) = 0$ and for $n \neq 0$,

$$2\pi\hat{f}(n) = \int_{-\pi}^{\pi} xe^{-inx}dx = \Big[x\frac{e^{-inx}}{-in}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in}dx = \Big[x\frac{e^{-inx}}{-in}\Big]_{-\pi}^{\pi}.$$

Thus,

$$2\pi\hat{f}(n) = \Big[x\frac{e^{-inx}}{-in}\Big]_{-\pi}^{\pi} = \frac{1}{-in}[\pi e^{-in\pi} + \pi e^{in\pi}] = 2\pi\frac{e^{in\pi}}{-in}$$

so that

$$\hat{f}(n) = \frac{(-1)^n}{-in\pi} = \frac{(-1)^{n+1}}{in\pi}.$$

Hence,

$$x = \sum_{n\neq 0}^{\infty} \frac{(-1)^{n+1}}{in}e^{inx} = \sum_{n=1} \frac{(-1)^{n+1}}{in}[e^{inx} - e^{-inx}] = 2\sum_{n=1} \frac{(-1)^{n+1}}{n}\sin nx$$

Taking $x = \pi/2$ we obtain the *Madhava-Nīlakaṅtha* series

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$ ◊

6. Divergence of Fourier series

Theorem 6.1. *There exists $f \in C(T)$ such that $\{S_N(f, 0)\}$ is unbounded; in particular, the Fourier series of f does not converge to f at 0.*

For this we shall make use of the *Uniform Boundedness Principle* from Functional Analysis:

Theorem 6.2. (Uniform Boundedness Principle) *Let (T_n) be a sequence of continuous linear transformations from a Banach space X to a normed linear space Y. If for each $u \in X$, the set $\{\|T_n u\| : n \in \mathbb{N}\}$ is bounded, then there exists $M > 0$ such that*

$$\sup_{\|u\|\leq 1} \|T_n u\| \leq M \quad \forall\, n \in \mathbb{N}.$$

Let

$$\varphi_N(f) := S_N(f, 0), \quad f \in C(T).$$

We see that $\varphi_N : C(T) \to \mathbb{C}$ is a linear functional for each $N \in \mathbb{N}$ and

$$|\varphi_N(f)| = |S_N(f,0)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)D_N(t)dt\right| \leq \|f\|_\infty \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt\right).$$

Hence, each φ_N is a continuous linear functional on $C(T)$ and

$$\sup_{\|u\|_\infty \leq 1} |\varphi_N(f)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt.$$

In fact,

Theorem 6.3.

$$\sup_{\|u\|_\infty \leq 1} |\varphi_N(f)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt$$

and

$$\int_{-\pi}^{\pi} |D_N(t)|dt \geq \frac{8}{\pi}\sum_{k=1}^{N}\frac{1}{k}.$$

Proof of Theorem 6.1. By Theorem 6.3, there does not exist $M > 0$ such that $\sup_{\|u\|_\infty \leq 1} |\varphi_N(f)| \leq M$ for all $n \in \mathbb{N}$. Hence, by Theorem 6.2, there exists $f \in C(T)$ such that $\{|\varphi_n(f)| : n \in \mathbb{N}\}$ is unbounded. Hence, there exists $f \in C(T)$ such that Fourier series of f diverges at 0. □

Remark 6.4. Let $\mathcal{D} := \{f \in C(T) : \{S_N(f,0)\}$ does not converge$\}$. Then $C(T) \setminus \mathcal{D}$ is a subspace of $C(T)$, and by Theorem 6.1, $C(T) \setminus \mathcal{D}$ is a proper subspace. Hence, $C(T) \setminus \mathcal{D}$ is nowhere dense, and hence $\mathcal{D}$ is dense in $C(T)$. Thus, we have proved the following:

> There exists a dense subset $\mathcal{D}$ of $C(T)$ such that for each $f \in \mathcal{D}$, the Fourier series of f diverges at 0.

In place of 0, we can take any point in $[-\pi, \pi]$ and obtain similar divergence result at that point.

7. Uniqueness

Theorem 7.1. (Uniqueness of Fourier series) *Let $f \in L^1(T)$. If $\hat{f}(n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ a.e.*

Proof. Let

$$g(t) = \int_{-\pi}^{t} f(x)dx, \quad t \in [-\pi, \pi].$$

Then, by Fundamental Theorem of Lebesgue Integration (FTLI), g is absolutely continuous, g' exists a.e. and $g' = f$ a.e. Note that

$$g(t+2\pi) - g(t) = \int_{t}^{t+2\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx = 2\pi \hat{f}(0) = 0.$$

Hence g is 2π-periodic. Let

$$h(t) = \int_{-\pi}^{t} g(x)dx, \quad t \in [-\pi, \pi].$$

Then we see that

$$h(t+2\pi) - h(t) = \int_{t}^{t+2\pi} g(x)dx = \int_{-\pi}^{\pi} g(x)dx = 2\pi \hat{g}(0).$$

Taking

$$G(t) = \int_{-\pi}^{t} [g(x) - \hat{g}(0)]dx, \quad t \in [-\pi, \pi],$$

we have

$$G(t+2\pi) - G(t) = \int_{-\pi}^{\pi} [g(x) - \hat{g}(0)]dx = 2\pi[\hat{g}(0) - \hat{g}(0)] = 0.$$

Thus, G is 2π-periodic, and $G'' = f$ a.e.Hence,

$$\hat{f}(n) = \widehat{G''}(n) = (in)^2 \hat{G}(n) \quad \text{for all} \quad n \neq 0.$$

Therefore, $\hat{G}(n) = 0$ for all $n \neq 0$. Hence, by Corollary 5.9, $G(x) = \hat{G}(0)$, and hence $G'' = 0$, so that $f = 0$ a.e. □

Recall that for each $f \in L^1(T)$,

$$\hat{f}(n) \to 0 \quad \text{as} \quad |n| \to \infty.$$

Thus, $(\hat{f}(n)) \in c_0(\mathbb{Z})$ for every $f \in L^1[-\pi, \pi]$.

Notation: $c_0(\mathbb{Z})$ is the set of all sequences $\varphi : \mathbb{Z} \to \mathbb{C}$ such that $\varphi(n) \to 0$ as $|n| \to \infty$.

Theorem 7.2. *The map* $\mathcal{F} : L^1(T) \to c_0(\mathbb{Z})$ *be defined by*

$$\mathcal{F}(f) = (\hat{f}(n)), \quad f \in L^1(T)$$

is an injective continuous linear operator which is not onto.

Proof. For $f, g \in L^1(T)$ and $\alpha \in \mathbb{C}$, we have

$$\widehat{(f+g)}(n)) = \hat{f}(n) + \hat{g}(n) \quad \text{for all} \quad n \in \mathbb{Z},$$

$$\widehat{\alpha f}(n)) = \alpha \hat{f}(n). \quad \text{for all} \quad n \in \mathbb{Z},$$

Thus, $\mathcal{F}$ is a linear operator. Note that

$$|\hat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx.$$

Thus, if we endow $L^1[-\pi, \pi]$ with the norm

$$\|f\|_{L^1} := \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx, \quad f \in L^1(T),$$

then we see that $\mathcal{F}$ is a continuous linear operator. By Theorem 7.1, $\mathcal{F}$ is injective. So, it remains to show that $\mathcal{F}$ is not onto. If it is onto, then my Bounded Inverse Theorem, its inverse is also continuous. Note that

$$\mathcal{F}(D_N) = \{\widehat{D_N}(n)\}$$

and

$$\widehat{D}_N(n) = 1 \quad \text{for} \quad |n| \geq N$$

so that

$$\|(\mathcal{F}(D_N))\|_\infty = 1 \quad \text{for all} \quad N \in \mathbb{N}.$$

If $\mathcal{F}$ is onto, then, by *Bounded Inverse Theorem*[4] its inverse $\mathcal{F}^{-1}$ is continuous so that $(\|D_N\|) = \{\|\mathcal{F}^{-1}(\mathcal{F}(D_N)\|\}$ is bounded, which is not true. □

By the above theorem there exists $(c_n) \in c_0(\mathbb{Z})$ such that there is no $f \in L^(T)$ satisfying $c_n = \hat{f}(n)$ for all $n \in \mathbb{N}$. It is a natural urge to have an example of such a sequence $c_n)$. We shall show that $c_n)$ with

$$c_n = \begin{cases} 1/\log(n), & n \geq 2, \\ 0, & n \leq 1, \end{cases}$$

is such a sequence. This is a consequence of the first part of the following theorem.

[4]If X and Y are Banach spaces and $T : X \to Y$ is a continuous bijective linear operator, then T^{-1} is also continuous.

Theorem 7.3. *Let $f \in L^1(T)$. Then* $\sum_{n\neq 0} \frac{\hat{f}(n)}{n} e^{inx}$ *converges at every $x \in \mathbb{R}$ and*

$$\int_a^b f(x)dx = \sum_{n\in\mathbb{Z}} \int_a^b \hat{f}(n)e^{inx}\, dx.$$

For proving the above theorem we shall make use of the following theorem:

Theorem 7.4. (Jordan) *If $f \in L^1(T)$ is of bounded variation*[5]*, then for every $x \in \mathbb{R}$,*

$$S_N(f,x) \to \frac{1}{2}(f(x+) + f(x-) \quad as \quad N \to \infty.$$

In particular, if $f \in AC(T)$, then

$$S_N(f,x) \to f(x) \quad as \quad N \to \infty$$

for every $x \in \mathbb{R}$.

It can be easily shown that:

Every absolutely continuous function is of bounded variation.

Proof of Theorem 7.3. Let

$$g(t) = \int_{-\pi}^{t} [f(x) - \hat{f}(0)]dx.$$

Then g is absolutely continuous and g is 2π-periodic, i.e., $g \in AC(T)$, $g' \in L^1(T)$ and $g' = f - \hat{f}(0)$ a.e. Therefore, $\hat{g'}(n) = in\hat{g}(n)$ for all $n \neq 0$ so that

$$\hat{g}(n) = \frac{\hat{f}(n)}{in}, \quad n \neq 0.$$

By Jordan's theorem,

$$g(x) = \hat{g}(0) + \sum_{n\neq 0} \hat{g}(n)e^{inx} = \hat{g}(0) + \sum_{n\neq 0} \frac{\hat{f}(n)}{in}\, e^{inx}.$$

In particular, $\sum_{n\neq 0} \frac{\hat{f}(n)}{n} e^{inx}$ converges. Also,

$$g(x) - g(y) = \sum_{n\neq 0} \frac{\hat{f}(n)}{in}\, [e^{inx} - e^{inx}] = \sum_{n\neq 0} \hat{f}(n) \int_y^x e^{int}dt.$$

[5]A function $f : [a,b] \to \mathbb{C}$ is of *bounded variation* if there exits $\kappa > 0$ such that for every partition $x_0 < x_1 < \cdots < x_n = b$, $\sum_{k=1}^n |f(x_{k+1}) - f(x_k| \leq \kappa$.

But,

$$g(x) - g(y) = \int_y^x g'(t)dt = \int_y^x [f(t) - \hat{f}(0)]dt = \int_y^x f(t)dt - \hat{f}(0)(x-y)dt.$$

Hence,

$$\int_y^x f(t)dt = \sum_{n\in\mathbb{Z}} \int_a^b \hat{f}(n)e^{int}\, dt.$$

This competes the theorem. □

Corollary 7.5. *Let* (c_n) *be with*

$$c_n = \begin{cases} 1/\log(n), & n \geq 2, \\ 0, & n \leq 1, \end{cases}$$

Then there is no $f \in L^1(T)$ *satisfying* $c_n = \hat{f}(n)$ *for all* $n \in \mathbb{N}$.

Proof. Suppose $f \in L^1(T)$ satisfying $c_n = \hat{f}(n)$ for all $n \in \mathbb{N}$. Then by the first part of Theorem 7.3, the series $\sum_{n=2}^\infty \frac{e^{inx}}{n \log n}$ converges. In particular, taking $x = 0$, $\sum_{n=2}^\infty \frac{1}{n \log n}$ converges, which is not true (e.g., by integral test).

□

8. Convolution

Given $f, g \in L^1(T)$, it can be shown that

$$(x, y) \mapsto f(x-y)g(y)$$

is measurable on $\mathbb{R} \times \mathbb{R}$, and hence, for each $x \in [-\pi, \pi]$, the integral

$$\int_{-\pi}^{\pi} f(x-y)g(y)dy$$

converges.

Definition 8.1. *The convolution of* $f, g \in L^1(T)$ *is defined by*

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y)dy, \quad x \in [-\pi, \pi].$$

We observe the following:

(1) $f * g \in L^1(T)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$:

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x-y)|\,|g(y)|dydx &= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} |f(x-y)|\,dx\right]|g(y)|dy \\ &= \int_{-\pi}^{\pi} 2\pi\|f\|_1\,|g(y)|dy \\ &= (2\pi)^2\|f\|_1\|g\|_1.\end{aligned}$$

(2) $f * g = g * f$:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x-y)g(y)dy &= \int_{-\pi}^{\pi} f(\tau)g(x-\tau)dy \\ &= \int_{x-\pi}^{x+\pi} f(\tau)g(x-\tau)dy \\ &= \int_{-\pi}^{\pi} f(\tau)g(x-\tau)dy.\end{aligned}$$

(3) $\widehat{f * g}(n) = \hat{g}(n\hat{f}(n)$ for all $n \in \mathbb{Z}$:

$$\widehat{f * g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f * g)(x)e^{-inx}dx,$$

$$\begin{aligned}(f * g)(x)e^{-inx} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)e^{-inx}dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)e^{-in(x-y)}e^{-iny}dy,\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi}(f * g)(x)e^{-inx}dx &= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} f(x-y)g(y)e^{-in(x-y)}e^{-iny}dy\right]dx \\ &= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} f(x-y)e^{-in(x-y)}dx\right]g(y)e^{-iny}dy \\ &= 2\pi\int_{-\pi}^{\pi}\hat{f}(n)g(y)e^{-iny}dy \\ &= (2\pi)^2\hat{f}(n)\hat{g}(n).\end{aligned}$$

(4) $(f * g) * h = f * (g * h)$:

$$\begin{aligned}
\int_{-\pi}^{\pi} (f * g)(x-y)h(y)dy &= \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} f(x-y-t)g(t)dt\right] h(y)dy \\
&= \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} f(x-\tau)g(\tau-y)d\tau\right] h(y)dy \\
&= \int_{-\pi}^{\pi} f(x-\tau) \left[\int_{-\pi}^{\pi} g(\tau-y)h(y)dy\right] d\tau \\
&= 2\pi \int_{-\pi}^{\pi} f(x-\tau)(g * h)(\tau)d\tau \\
&= (2\pi)^2 [f * (g * h)](x).
\end{aligned}$$

Theorem 8.2. *With respect to convolution as multiplication, $L^1(T)$ is a Banach algebra.*

- The Banach algebra $L^1(T)$ does not have a multiplicative identity:

  Suppose there exists $\varphi \in L^(T)$ such that $f * \varphi = f$ for all $f \in L^1(T)$. Then $\hat{f}(n)\hat{\varphi}(n) = \hat{f}(n)$ for all $f \in L^1(T)$. Hence, $\hat{\varphi}(n) = 1$ whenever $\hat{\varphi}(n) \neq 0$. But, $\hat{\varphi}(n) \to 0$ as $|n| \to \infty$. Hence, there exists $N \in \mathbb{N}$ such that $\hat{\varphi}(n) = 0$ for all $n \geq N$. Let $f \in L^1(T)$ be such that $\hat{f}(n) \neq 0$ for some $n \geq N$. Then for such n, we obtain

  $$0 = \hat{f}(n)\hat{\varphi}(n) = \hat{f}(n) \neq 0,$$

  which is a contradiction.

However,

- There exists (φ_n) in $L^1(T)$ such that $\|f * \varphi_n - f\|_1 \to 0$.

In fact, we have the following.

Theorem 8.3. *Let K_n be the Fejér kernel. Then, for every $f \in L^1(T)$,*

$$\|f * K_n - f\|_1 \to 0 \quad as \quad N \to \infty.$$

Proof. Recall that if $g \in C(T)$, then $\|g * \varphi_n - g\|_1 \to 0$. Let $f \in L^1(T)$ and $\varepsilon > 0$ be given. Let $g \in C(T)$ be such that $\|f - g\|_1 < \varepsilon$, and let $N \in \mathbb{N}$ be such that

$\|g * \varphi_n - g\|_1 < \varepsilon$ for all $n \geq N$. Then, for $n \geq N$, we have

$$\begin{aligned} \|f * K_n - f\|_1 &\leq \|f * K_n - g * K_n\|_1 + \|g * K_n - g\|_1 + \|g - f\|_1 \\ &\leq \|(f-g) * K_n\|_1 + \varepsilon + \varepsilon \\ &\leq \|(f-g) * K_n\|_1 + 2\varepsilon \\ &\leq \|f - g\|_1 \|K_n\|_1 + 2\varepsilon \\ &\leq 3\varepsilon. \end{aligned}$$

The last inequality is due the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)|dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$. □

9. L^2-Theory

The norm on $L^2(T)$ is given by

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{1/2}.$$

Observe:

(1) If $u_n(x) := e^{inx}$, $n \in Z$, then the set $\{u_n : n \in \mathbb{Z}\}$ is an orthonormal set and span$\{u_n : n \in \mathbb{Z}\}$, the space of all trigonometric polynomials, is dense in $L^2(T)$.

(2) Let $f \in L^2(T)$ and $S_N(f) := S_N(f, \cdot)$. Then

(a) $S_N(f) = \displaystyle\sum_{n=-N}^{N} \langle f, u_n\rangle u_n$.

(b) $\|S_N(f)\|_2^2 = \displaystyle\sum_{n=-N}^{N} |\hat{f}(n)|^2$.

(c) $\|f - S_N(f)\|_2^2 = \|f\|_2^2 - \|S_N(f)\|_2^2 = \|f\|_2^2 - \displaystyle\sum_{n=-N}^{N} |\hat{f}(n)|^2$.

(d) **(Bessel's inequality)**: $\displaystyle\sum_{n=-N}^{N} |\hat{f}(n)|^2 \leq \|f\|_2^2 \quad \forall N \in \mathbb{N}$. In particular, $\hat{f}(n) \to 0$ as $|n| \to \infty$.

(e) $\langle f - S_N(f), u_n\rangle = 0 \quad \forall |n| \leq N\}$.

(f) $\|f - S_N(f)\|_2 \leq \|f - g\| \quad \forall g \in \text{span}\{u_n : n \in \mathbb{Z}, |n| \leq N\}$.

Only (2)(f) requires some explanation.

Note that for every $g \in \text{span}\{u_n : n \in \mathbb{Z}, |n| \leq N\}$,

$$\|f - g\|_2^2 = \|f - S_N(f)\|_2^2 + \|S_N(f) - g\|_2^2,$$

because, in view of (2)(e), $\langle f - S_N(f), S_N(f) - g\rangle = 0$.

- The result in (2)(d) gives another proof for the Riemann Lebesgue lemma, because $L^2(T)$ is dense in $L^1(T)$.
- In view of (2)(f),

$$\|f - S_N(f)\|_2 = \inf\{\|f-g\|_2 : g \in \mathcal{X}_N\},$$

where $\mathcal{X}_N := \text{span}\{u_n : n \in \mathbb{Z},\, |n| \leq N\}$. In other words, $S_N(f)$ is the (unique!) *best approximation* of f from $\mathcal{X}_N$. Uniqueness is due to the following: Suppose φ be in $\mathcal{X}_N$ such that

$$\|f-\varphi\|_2 = \inf\{\|f-g\|_2 : g \in \mathcal{X}_N\}.$$

Then,

$$\|f-\varphi\|_2^2 = \|f - S_N(f)\|_2^2 + \|S_N(f)-\varphi\|_2^2$$

since $\langle f - S_N(f),\, S_N(f)-\varphi\rangle = 0$ so that we obtain $\|S_N(f)-\varphi\|_2 = 0$.

Theorem 9.1. *Let $f \in L^2(T)$. Then we have the following:*

(1) span$\{u_n : n \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(T)$*, i.e., a maximal orthonormal set in* $L^2(T)$.
(2) **(Fourier expansion)** $f = \sum_{n\in\mathbb{Z}} \hat{f}(n) u_n$ *in* $L^2(T)$.
(3) **(Parseval's formula)** $\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.

Proof. (1) It can be seen that $\langle f, u_n\rangle = 0$ for all $n \in \mathbb{Z}$ implies $f = 0$ in $L^2(T)$. Hence, span$\{u_n : n \in \mathbb{Z}\}$ is a maximal orthonormal set in $L^2(T)$.

(3) We observe that, for $n > m$,

$$\|S_n(f) - S_m(f)\|^2 \leq \sum_{n\leq|k|\leq m} |\hat{f}(n)|^2.$$

Hence, $\{S_n(f)\}$ is a cauchy sequence in $L^2(T)$. Therefore, it converges to some $g \in L^2(T)$. It can be seen that $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. Therefore, $g = f$ in $L^2(T)$.

(3) Follows from (2). □

Now, we give another proof for the following theorem:

Theorem 9.2. *If $f \in C^1(T)$, then the Fourier series of f converges absolutely, and uniformly to f. Further,*

$$\|f - S_N(f,\cdot)\|_\infty = O\left(\frac{1}{\sqrt{N}}\right).$$

Proof.

$$f \in C^1(T) \quad \Longrightarrow \quad \hat{f'}(n) = in\hat{f}(n).$$

Hence,

$$\sum_{n\neq 0}|\hat{f}(n)| = \sum_{n\neq 0}\frac{1}{n}|\,|in\hat{f}(n)| = \sum_{n\neq 0}\frac{1}{n}|\,|\hat{f'}(n)| \leq \left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\|\hat{f'}\|_2 = \frac{\pi}{\sqrt{3}}\|\hat{f'}\|_2.$$

Hence the Fourier series of f converges absolutely, and uniformly to a continuous function, say $g \in C(T)$. Since $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$, we obtain $g = f$. We also observe that, for all $x \in \mathbb{R}$,

$$|f(x) - S_N(f,x)| \leq \sum_{|n|>N}|\hat{f}(n)| = \sum_{n\neq 0}\frac{1}{n}|\,|\hat{f'}(n)| \leq \left(\sum_{|n|>N}\frac{1}{n^2}\right)^{1/2}\|\hat{f'}\|_2 \leq \frac{\|\hat{f'}\|_2}{\sqrt{N}}.$$

This completes the proof. □

References

[1] R. Bhatia, *Fourier Series*, TRIM, Hindustan Book Agency, 1993 (Second Edition: 2003) .

[2] S. Kesavan, *Lectures on Fourier Series*, Notes: Third Annual Foundation School, December, 2006.

[3] M.T. Nair, *Measure and Integration*, Notes for the MSc. Course, Jan-may, 2014.

[4] R. Radha & S. Thangavelu, *Fourier Series*, Web-Course, NPTEL, IIT Madras, 2013.

[5] B. O. Turesson, *Fourier Analysis, Distribution Thoery, and Wavelets*, Lecture Notes, March, 2012.

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