TOPICS IN FOURIER ANALYSIS-IV

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Contents

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1. Test functions and distributions

Let Ω be an open subset of $\mathbb{R}^d$. We shall denote the vector space $C_c^\infty(\Omega)$ by $\mathcal{D}(\Omega)$, and call this space as **space of test functions**.

Definition 1.1. A sequence (φ_n) in $\mathcal{D}(\Omega)$ is said to converge to $\varphi \in \mathcal{D}(\Omega)$ if

(1) there exists a compact set $K \subset \Omega$ such that $\mathrm{supp}(\varphi_n) \subseteq K$ for all $n \in \mathbb{N}$ and
(2) $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on Ω for every $\alpha \in \mathbb{N}_0^d$.

Notation 1.2. For $x_0 \in \mathbb{R}^d$ and $r > 0$, we denote:

$$B_r(x_0) := \{x \in \mathbb{R}^d : |x| < r\}$$

and its closure by $\overline{B_r(x_0)}$, i.e., $\overline{B_r(x_0)} := \{x \in \mathbb{R}^d : |x| \le r\}$.

Let us give an example of a function in $\mathcal{D}(\mathbb{R}^d)$:

Example 1.3. Let

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

Then $\psi \in \mathcal{D}(\mathbb{R}^d)$ with $\mathrm{supp}(\psi) \subset \overline{B_1(0)}$. For $\varepsilon > 0$, let

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right).$$

Then $\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ with $\mathrm{supp}(\psi_\varepsilon) \subset \overline{B_e(0)}$. □

Definition 1.4. A **distribution** on Ω is a linear functional u on $\mathcal{D}(\Omega)$ such that for every (φ_n) in $\mathcal{D}(\Omega)$, $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ implies $u(\varphi_n) \to u(\varphi)$.

The set of all distributions on Ω is denoted by $\mathcal{D}'(\Omega)$.

Definition 1.5. A sequence (u_n) of distributions on Ω is said to **converge** to a distribution u on Ω if

$$u_n(\varphi) \to u(\varphi) \quad \text{for every} \quad \varphi \in \mathcal{D}(\Omega).$$

Notation 1.6. For $1 \le p < \infty$, $L^1_{\mathrm{loc}}(\Omega)$ denotes the the space of all complex valued measurable functions f on Ω such that

$$\int_K |f(x)|dx < \infty \quad \text{for all compact} \quad K \subseteq \Omega.$$

Recall that K is compact if and only if K contains all its boundary points, i.e., points x such that $B_r(x) \cap K$ and $B_r(x) \cap K^c$ are nonempty for every $r > 0$.

Observe that $L^p(\Omega) \subseteq L^1_{\mathrm{loc}}(\Omega)$ for every p with $1 \le p < \infty$.

Example 1.7. Corresponding to $f \in L^1_{\text{loc}}(\Omega)$, let

$$u_f(\varphi) := \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega),\, x \in \Omega.$$

Then u_f is a distribution: Clearly, u_f is a linear functional on $\mathcal{D}(\Omega)$. Now, let (φ_n) in $\mathcal{D}(\Omega)$ be such that $\varphi_n \to \varphi$ for some $\varphi \in \mathcal{D}(\Omega)$. Then we have

$$\begin{aligned} |u_f(\varphi_n) - u_f(\varphi)| &= |u_f(\varphi_n \varphi)| \\ &\leq \int_{\mathbb{R}^d} |f(x)|\,|(\varphi_n \varphi)(x)|dx \\ &\leq \|\varphi_n - \varphi\|_\infty \int_\Omega |f(x)|dx. \end{aligned}$$

Hence, $u(\varphi_n) \to u(\varphi)$.

Definition 1.8. A distribution u on Ω is called a **regular distributions** if $u = u_f$ for some $f \in L^1_{\text{loc}}(\Omega)$, and in that case u_f is said to be the distribution[1] generated by f.

There are distributions that are not regular.

Example 1.9. Let φ be as in Example 1.21. For $a \in \Omega$, let

$$\delta_a(\varphi) := \varphi(a), \quad \varphi \in \mathcal{D}(\Omega).$$

It is easily seen that δ_a is a distribution on Ω. But it is not a regular distribution: To see this, suppose there exists $f \in L^1_{\text{loc}}(\Omega)$ such that $\delta_a = u_f$, i.e.,

$$\varphi(a) = \int_\Omega f(x)\varphi(x)dx \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega).$$

Let φ be as in Example 1.21 and let $\varepsilon > 0$ be small enough such $B_\varepsilon(a) \subseteq \Omega$. Let

$$\tilde{\varphi}_\varepsilon(x) := \varphi(\frac{x-a}{\varepsilon}).$$

Then $\tilde{\varphi}_\varepsilon \in \mathcal{D}(\Omega)$ and $\text{supp}(\tilde{\varphi}_\varepsilon) \subset \{x \in \mathbb{R}^d : |x-a| < \varepsilon\}$ and we have

$$\tilde{\varphi}_\varepsilon(a) = \int_\Omega f(x)\tilde{\varphi}_\varepsilon(x)dx = \int_{|x-a|<\varepsilon} f(x)\tilde{\varphi}_\varepsilon(x)dx$$

Note that

$$\left| \int_{|x-a|<\varepsilon} f(x)\tilde{\varphi}_\varepsilon(x)dx \right| \leq \int_{|x-a|<\varepsilon} |f(x)|dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus,

$$|\tilde{\varphi}_\varepsilon(a)| \leq \int_{|x-a|<\varepsilon} |f(x)|dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This is a contradiction, since $\tilde{\varphi}_\varepsilon(0) \neq 0$. □

[1] We shall prove that a regular distribution can be generated by only one function in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Definition 1.10. The distribution δ_a in Example 1.9 is called a **delta distribution**.

In view of Example 1.9, a delta-distribution is not a regular distribution. However, we have the following:

Theorem 1.11. *There exists a sequence u_n of regular distributions which converge to a delta distribution. In fact, taking $f_n := \frac{n}{2}\chi_{E_n}$, where $E_n := \{x \in \Omega : |x-a| < 1/n\}$,*

$$u_{f_n} \to \delta_a \quad as \quad n \to \infty.$$

Proof. Let $f_n := \frac{n}{2}\chi_{E_n}$, where $E_n := \{x \in \Omega : |x-a| < 1/n\}$, and let $u_n := u_{f_n}$. Let $\varphi \in \mathcal{D}(\Omega)$. Then

$$u_n(\varphi) = \frac{n}{2}\int_{|x-a|<1/n} \varphi(x)dx.$$

Note that

$$u_n(\varphi) = \frac{n}{2}\int_{|x-a|<1/n} \varphi(x)dx = \frac{n}{2}\int_{|x-a|<1/n} [\varphi(x) - \varphi(a)]dx + \varphi(a)$$

and

$$\frac{n}{2}\int_{|x-a|<1/n} |\varphi(x) - \varphi(a)|dx \le \max_{|x-a|<1/n} |\varphi(x) - \varphi(a)| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $u_n(\varphi) \to \varphi(a)$ as $n \to \infty$. □

Example 1.12. For $n \in \mathbb{N}$, let

$$u_n(\varphi) := \int_{\mathbb{R}} \varphi(x)e^{inx}dx, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Note that, defining $f_n(x) := e^{inx}$, $x \in \mathbb{R}$, we see that $u_n = u_{f_n}$. Thus u_n is a regular distribution for every $n \in \mathbb{N}$. Further, by Riemann-Lebesgue lemma,

$$u_n(\varphi) = \int_{\mathbb{R}} \varphi(x)e^{inx}dx \to 0 \quad \text{as} \quad n \to \infty$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$. Thus, (u_n) converges t the zero distribution.

Remark 1.13. In the books on *signals and systems* one comes across a function called **implulse function**.

It is defined as a function $\delta : \mathbb{R} \to [0, \infty]$ such that

(1) $\displaystyle\int_{-\infty}^{\infty} \delta(x)dx = 1$,
(2) $\delta(x) = 0 \quad \text{for} \quad x \neq 0$, and
(3) $\delta(0) = \infty$.

Unfortunately, there is no function having the above two properties!

Even though we can define a function $\delta : \mathbb{R} \to [0, \infty]$ satisfying

(1) $\delta(x) = 0 \quad \text{for} \quad x \neq 0$, and
(2) $\delta(0) = \infty$,

such a function cannot satisfy the requirement $\displaystyle\int_{-\infty}^{\infty} \delta(x)dx = 1$.

Then what does one have?

We can only have an **ε-impulse function** which can be defined as follows:

Definition 1.14. For $\varepsilon > 0$, an **ε-impulse function** is a non-negative function $\delta_\varepsilon(x)$ defined for $-\infty < x < \infty$ such that

(1) $\displaystyle\int_{-\infty}^{\infty} \delta_\varepsilon(x)dx = 1$,
(2) $\delta_\varepsilon(x) = 0 \quad \text{for} \quad |x| > \varepsilon$.
(3) $\delta_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$.

Example 1.15. (i) Define $\delta_\varepsilon(x)$ to be a function whose graph is an isosceles triangle with base $[-\varepsilon, \varepsilon]$ and hight $1/\varepsilon$. Then δ_ε is an ε-impulse function.

(ii) Define $\delta_\varepsilon(x)$ to be $1/2\varepsilon$ in the interval $[-\varepsilon, \varepsilon]$ and 0 elsewhere. Then δ_ε is an ε-impulse function.

Theorem 1.16. *For $\varepsilon > 0$, if δ_ε is an ε-impulse function, then $u_{\delta_\varepsilon} \to \delta_0$ as $\varepsilon \to 0$, where δ_0 is the delta-distribution at* 0.

Proof. The proof is along the same line as that of Theorem 1.11:

Let φ be a continuous function defined on $\mathbb{R}$ and $\delta_\varepsilon(x)$ is an ε-impulse function. Then we have

$$\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx = \int_{-\varepsilon}^{\varepsilon} \varphi(x)\delta_\varepsilon(x)dx.$$

Hence,

$$\left| \int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx - \varphi(0) \right| = \left| \int_{-\varepsilon}^{\varepsilon} \varphi(x)\delta_\varepsilon(x)dx - \int_{\varepsilon}^{\varepsilon} \varphi(0)\delta_\varepsilon(x)dx \right|.$$

Thus we have

$$\left| \int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx - \varphi(0) \right| \leq \int_{-\varepsilon}^{\varepsilon} |\varphi(x) - \varphi(0)|\delta_\varepsilon(x)dx.$$

Since φ is continuous, for any given $\alpha > 0$, there is an $\varepsilon > 0$ such that

$$|\varphi(x) - \varphi(0)| < \alpha \quad \text{whenever} \quad |x| < \varepsilon.$$

Hence, for such an $\varepsilon > 0$, we have

$$\left|\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx - \varphi(0)\right| \leq \int_{-\varepsilon}^{\varepsilon} |\varphi(x) - \varphi(0)|\delta_\varepsilon(x)dx \leq \alpha \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x)dx = \alpha.$$

That is, for every $\alpha > 0$, there is an $\varepsilon > 0$ such that

$$\left|\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx - \varphi(0)\right| < \alpha.$$

Thus,

$$\int_{-\infty}^{\infty} \varphi(x)\delta_\varepsilon(x)dx \to \varphi(0) \quad \text{as} \quad \varepsilon \to 0.$$

and hence, $u_{\delta_\varepsilon}(\varphi) \to \delta_0(\varphi)$ as $\varepsilon \to 0$. where δ_0 is the *delta distribution* at 0. □

In view of the following theorem, regular distributions can be identified with the functions that correspond to them. That is, regular distributions are uniquely defined by functions in $L^1_{\rm loc}(\Omega)$.

Theorem 1.17. (Uniqueness theorem) *For $f, g \in L^1_{\rm loc}(\Omega)$,*

$$u_f = u_g \quad \Longrightarrow \quad f = g \quad a.e.$$

Before proving the above we shall introduce some definitions and consider some results.

Throughout, we shall make use of a special type of function in $C_c^\infty(\Omega)$, called a *mollifier*. In the due course it wil be showed why such functions are called mollifiers.

Definition 1.18. A non-negative function φ defined on $\mathbb{R}^d$ is called a **mollifier** if

$$\varphi \in C_c^\infty(\mathbb{R}^d), \quad \text{supp}(\varphi) \subseteq \overline{B_1(0)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi(x)dx = 1.$$

Here is an example of a mollifier.

Example 1.19. Let ψ be as in Example 1.21, and let

$$\varphi(x) = C_0\psi(x) \quad \text{where} \quad C_0 := 1\Big/\int_{\mathbb{R}} \psi(x)dx.$$

Then φ is a mollifier.

In fact functions $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, $\varepsilon > 0$, with

$$\text{supp}(\varphi) \subseteq \overline{B_\varepsilon(0)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi_\varepsilon(x)dx = 1$$

are also called **mollifiers**. Such mollifiers can be constructed from a given mollifer by defining

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x}{\varepsilon}\Big).$$

Clearly,

$$\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\varphi_\varepsilon) \subset \overline{B_\varepsilon(0)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi_\varepsilon(x)dx = 1.$$

Also, for any $a \in \mathbb{R}^d$ and $\varepsilon > 0$, the function $\varphi_{\varepsilon,a}$ defined by

$$\varphi_{\varepsilon,a}(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x-a}{\varepsilon}\Big)$$

satisfies

$$\varphi_{\varepsilon,a} \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\varphi_{\varepsilon,a}) \subset \overline{B_\varepsilon(a)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi_{\varepsilon,a}(x)dx = 1.$$

Observe that

$$\varphi_{\varepsilon,a}(a) := \frac{\varphi(0)}{\varepsilon^d}.$$

In particular,

$$\varphi_{\varepsilon,a}(a) \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

Definition 1.20. A non-negative function φ defined on $\mathbb{R}^d$ is called a **mollifier** if

$$\varphi \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\varphi) \subseteq \overline{B_1(0)} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi(x)dx = 1.$$

Example 1.21. Let

$$\varphi_0(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

and

$$\varphi(x) = C_0\varphi_0(x) \quad \text{with} \quad C_0 := 1\Big/\int_{\mathbb{R}} \varphi_0(x)dx.$$

Then φ is a mollifier on $\mathbb{R}^d$.

Suppose φ is a mollifier and $\varepsilon > 0$. Let

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x}{\varepsilon}\Big).$$

Then

$$\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d) \quad \text{and} \quad \mathrm{supp}(\varphi_\varepsilon) \subset \overline{B_\varepsilon(0)}.$$

Also, for any $a \in \mathbb{R}^d$ and $\varepsilon > 0$, the function $\varphi_{\varepsilon,a}$ defined by

$$\varphi_{\varepsilon,a}(x) := \frac{1}{\varepsilon^d}\varphi\Big(\frac{x-a}{\varepsilon}\Big)$$

satisfies

$$\varphi_{\varepsilon,a} \in C_c^\infty(\mathbb{R}^d) \quad \text{and} \quad \mathrm{supp}(\varphi_\varepsilon) \subset \overline{B_\varepsilon(a)}.$$

2. Convolution revisited

Proof of the following theorem is easy and hence we mot the proof.

Theorem 2.1. *If $f, g \in L^1(\mathbb{R}^d)$, then* $\sup(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.
In particular, if $1 \leq p < \infty$ and $f, g \in L^p(\mathbb{R}^d)$ are with compact support, then $\sup(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.

Proposition 2.2. *Suppose $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$ and $g \in C^k(\mathbb{R}^d)$ with $\partial^\alpha g \in L^q(\Omega)$ for $|\alpha| \leq k$. Then $f * g \in C^k(\mathbb{R}^d)$ and $\partial^\alpha(f * g) = f * \partial^\alpha g$ for $|\alpha| \leq k$.*

In particular, for $1 \leq p < \infty$, if $f \in L^p(\mathbb{R}^d)$ is with compact support and $g \in C_c^\infty(\mathbb{R}^d)$, then $f * g \in C_c^\infty(\mathbb{R}^d)$ and and $\partial^\alpha(f * g) = f * \partial^\alpha g$ for all $\alpha \in \mathbb{N}_0^d$.

Proof. We prove the case for $p = 1$ and $k = 1$, i.e., $|\alpha| = 1$. Proof of the case of $k > 1$ will follow similarly. The case of $p > 1$ involves more calculations.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ and let j be such that $\alpha_j = 1$ and $\alpha_i = 0$ for $i \neq j$. We have to show that

$$\lim_{h \to 0} \frac{(f * g)(x + he_j) - (f * g)(x)}{h} \quad \text{exists}$$

and it is equal to $(f * \partial_j g)(x)$. Note that

$$\frac{(f * g)(x + he_j) - (f * g)(x)}{h} = \int_{\mathbb{R}^d} f(x - y) \frac{g(y + he_j) - g(y)}{h} dy.$$

Since

$$\frac{g(y + he_j) - g(y)}{h} \to \partial_j g(y) \quad \text{as} \quad h \to 0 \quad \text{and} \quad \partial_j g \in L^\infty(\Omega),$$

there exists $\alpha > 0$ such that for all h with $|h| \leq \alpha$,

$$|f(x - y)| \left| \frac{g(y + he_j) - g(y)}{h} \right| \leq |f(x - y)|(|\partial_j g(y)| + 1).$$

Since $y \mapsto |f(x - y)|(|\partial_j g(y)| + 1)$ belongs to $L^1(\Omega)$, by DCT, we have

$$\int_{\mathbb{R}^d} f(x - y) \frac{g(y + he_j) - g(y)}{h} dy \to \int_{\mathbb{R}^d} f(x - y) \partial_j g(y) dy.$$

Thus, $\partial_j(f * g)$ exists and $\partial_j(f * g) = f * \partial_j g$. □

Proposition 2.3. *If K is a compact subset of Ω, then there exists $\psi \in \mathcal{D}(\Omega)$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on K.*

Proof. Let K be a compact subset of Ω and let $\delta := \operatorname{dist}(K, \Omega^c)$. Let $\alpha := \delta/3$ and G_α be the α-neighbourhood of K, i.e.,

$$G_\alpha := \{x \in \Omega : \operatorname{dist}(x, \Omega) < \alpha\}.$$

Let φ be a mollifier and for $\varepsilon > 0$, let $\psi_\varepsilon := \varphi_\varepsilon * \chi_\alpha$, where $\chi_\alpha := \chi_{G_\alpha}$ and $\varphi_\varepsilon := (1/\varepsilon^d)\varphi(x/\varepsilon)$. Since $\chi_\alpha \in L^1(\mathbb{R}^d)$, by Proposition 2.2, $\psi_\varepsilon \in C^\infty(\mathbb{R}^d)$. Note that

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)\chi_\alpha(y)dy \le \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)dy = 1.$$

Further, if $x \in K$ and $\varepsilon \le \alpha$, then

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\chi_\alpha(x-y)dy = \int_{B_\varepsilon(0)} \varphi_\varepsilon(y)\chi_\alpha(x-y)dy = 1,$$

since

$$x \in K, \quad y \in B_\varepsilon(0) \quad \text{implies} \quad x - y \in G_\alpha.$$

Thus, $0 \le \psi_\alpha \le 1$ and $\psi_\alpha = 1$ on K.

Also,

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)\chi_\alpha(y)dy = 0$$

whenever x is not in the ε-neighbourhood of G_α. Since α-neighbourhood of G_α is contained in the 2α-neighbourhood of K, taking $\varepsilon < \alpha$, we have $\mathrm{supp}(\psi_\varepsilon) \subseteq G_{2\alpha}$. □

Theorem 2.4. *Let $1 \le p < \infty$. If $f \in L^p(\mathbb{R}^d)$ for and $g \in L^1(\mathbb{R}^d)$, then*

$$f * g \in L^p(\mathbb{R}^d) \quad \textit{and} \quad \|f * g\|_p \le \|f\|_p \|g\|_1.$$

In particular,

$$f \in L^p(\mathbb{R}^d) \implies f * \varphi_\varepsilon \in L^p(\mathbb{R}^d) \quad \textit{and} \quad \|f * \varphi_\varepsilon\|_p \le \|f\|_p.$$

Proof. Let $f \in L^p(\mathbb{R}^d)$ for and $g \in L^1(\mathbb{R}^d)$. First let $= 1$. Then,

$$\begin{aligned} \int |f * g)(x)| &\le \int \left(\int |f(x-y)g(y)|dy \right) dx \\ &\le \int \left(\int |f(x-y)|dx \right) |g(y)|dy \\ = \|f\|_1 \|g\|_1. \end{aligned}$$

Next, let $1 < p < \infty$ and let q such that $(1/p) + (1/q) = 1$. Then

$$\begin{aligned} |f * g)(x)| &\le \int |f(x-y)g(y)|dy \\ &\le \int |f(x-y)|\, |g(y)|^{1/p} |g(y)|^{1/q} dy \\ &\le \left(\int |f(x-y)|)|^p |g(y)|)dy \right)^{1/p} \left(\int |g(y)|dy \right)^{1/q} \\ &= \left(\int |f(x-y)|)|^p |g(y)|dy \right)^{1/p} \|g\|_1^{1/q}. \end{aligned}$$

Hence,

$$\begin{aligned}\int |(f*g)(x)|^p dx &= \|g\|_1^{p/q} \int \left(\int |f(x-y)|)|^p |g(y)|dy\right) dx \\ &= \|g\|_1^{p/q} \int \left(\int |f(x-y)|)|^p dx\right) |g(y)|dy \\ &= \|g\|_1^{1+\frac{p}{q}} \|f\|_p^p\end{aligned}$$

so that

$$\left(\int |(f*g)(x)|^p dx\right)^{1/p} = \|g\|_1^{\frac{1}{p}\frac{1}{q}} \|f\|_p = \|g\|_1 \|f\|_p.$$

Thus, $f*g \in L^p(\mathbb{R}^d)$ and $\|f*g\|_p \le \|f\|_p \|g\|_1$. □

Corollary 2.5. *If $f \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$, then*

$$f*\varphi_\varepsilon \in L^p(\mathbb{R}^d) \quad \textit{and} \quad \|f*\varphi_\varepsilon\|_p \le \|f\|_p.$$

Theorem 2.6. *Let $L^p(\Omega)$ for $1 \le p < \infty$. Then $f*\varphi_\varepsilon \in C^\infty(\Omega) \cap L^p(\mathbb{R}^d)$ and*

$$\|f*\varphi_\varepsilon - f\|_p \to 0 \quad \textit{as} \quad \varepsilon \to 0.$$

Proof. By Proposition 2.2, $f*\varphi_\varepsilon \in C^\infty(\Omega)$. If $\Omega \ne \mathbb{R}^d$, then we extend f to all of $\mathbb{R}^d$ by defining it to be zero on Ω^c. First let $p=1$. Then we have

$$\begin{aligned}\int |(f*\varphi_\varepsilon)(x)|dx &\le \int \left|\int [f(x)-f(x-y)]\varphi_\varepsilon(y)dy\right| dx \\ &= \int\int |f(x)-f(x-y)|\varphi_\varepsilon(y)dydx \\ &\le \int \left(\int |f(x)-f(x-y)|dx\right) \varphi_\varepsilon(y)dy \\ &= \int \|f-\tau_y f\|_1 \varphi_\varepsilon(y)dy.\end{aligned}$$

Next let $1 < p < \infty$. Then we have

$$\begin{aligned}|f(x)-(f*\varphi_\varepsilon)(x)| &\le \int |f(x)-f(x-y)|\varphi_\varepsilon(y)dy \\ &\le \int |f(x)-f(x-y)|[\varphi_\varepsilon(y)]^{1/p}[\varphi_\varepsilon(y)]^q dy \\ &\le \left(\int |f(x)-f(x-y)|)|^p \varphi_\varepsilon(y)dy\right)^{1/p} \left(\int \varphi_\varepsilon(y)dy\right)^{1/q} \\ &= \left(\int |f(x)-f(x-y)|)|^p \varphi_\varepsilon(y)dy\right)^{1/p}.\end{aligned}$$

Hence,

$$\begin{aligned}\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx &\leq \int \left(\int |f(x)-f(x-y)|)|^p \varphi_\varepsilon(y)dy\right) dx \\ &= \int \left(\int |f(x)-f(x-y)|)|^p dx\right) \varphi_\varepsilon(y)dy \\ &= \int \|f-\tau_y f\|_p^p \varphi_\varepsilon(y)dy.\end{aligned}$$

Thus, for $1 \leq p < \infty$, we have

$$\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx \leq \int \|f-\tau_y f\|_p^p \varphi_\varepsilon(y)dy.$$

Now, recall that $\|f-\tau_y f\|_p^p \to 0$ as $y \to x$. Therefore, for any given $\eta > 0$, there exists $\delta > 0$ such that

$$\|f-\tau_y f\|_p^p < \eta \quad \text{whenever} \quad |y| < \delta.$$

Also, we know that $\|\tau_y f\|_p = \|f\|_p$ and for any $r > 0$,

$$\int_{|y|\geq r} \varphi_\varepsilon(y)dy \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence, there exists $\varepsilon_0 > 0$ such that

$$\int_{|y|\geq \delta} \varphi_\varepsilon(y)dy < \eta \quad \text{whenever} \quad 0 < \varepsilon < \varepsilon_0.$$

Thus, we obtain

$$\begin{aligned}\int |f(x)-(f*\varphi_\varepsilon)(x)|^p dx &\leq \int \|f-\tau_y f\|_p^p \varphi_\varepsilon(y)dy \\ &= \int_{|y|<\delta} \|f-\tau_y f\|_p^p \varphi_\varepsilon(y)dy + \int_{|y|\geq\delta} \|f-\tau_y f\|_p^p \varphi_\varepsilon(y)dy \\ &\leq \eta \int_{|y|<\delta} \varphi_\varepsilon(y)dy + (2\|f\|_p)^p \int_{|y|\geq\delta} \varphi_\varepsilon(y)dy \\ &\leq (1+(2\|f\|_p)^p)\eta\end{aligned}$$

whenever $\varepsilon < \varepsilon_0$. Thus, we have proved that $f * \varphi_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f * \varphi_\varepsilon - f\|_p \to 0$ as $\varepsilon \to 0$. □

Theorem 2.7. *$C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. The proof involves the following two steps:

(1) For every $f \in L^p(\Omega)$ and $\varepsilon > 0$, there exists $g \in L^p(\Omega)$ with compact support such that $\|f-g\| < \varepsilon$.

(2) For every $g \in L^p(\Omega)$ with compact support, $g*\varphi_\varepsilon \in C_c^\infty(\Omega)$ and $\|g - g*\varphi_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Proof of Step (1): Let $f \in L^p(\Omega)$. For $n \in \mathbb{N}$, let

$$K_n = \{x \in \Omega : |x| \le n, \, \text{dist}(x, \Omega^c) \ge 1/n\}.$$

Then each K_n is a compact subset of Ω. Taking $f_n := f\chi_{K_n}$, we see that $f_n \in L^p(\Omega)$ with $\text{supp}(f_n) \subseteq K_n$ and

$$\|f - f_n\|_p \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, given $\varepsilon > 0$, there exists $g := f_N$ such that $\|f - g\|_p < \varepsilon$.

Proof of Step (2): Let $g \in L^p(\Omega)$ with compact support. Let φ be a mollifier and $\varepsilon > 0$ be given. By Proposition 2.2, $g * \varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, where $\varphi_\varepsilon(x) := (1/\varepsilon^d)\varphi(x/\varepsilon)$. We may take ε small enough such that $\text{supp}(g * \varphi_\varepsilon) \subseteq \Omega$. Also, by Theorem 2.6,

$$\|g - (g * \varphi_\varepsilon)\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Now, let $f \in L^p(\Omega)$ and $\varepsilon > 0$. Then by Step (1), there exists $g \in L^p(\Omega)$ with compact support such that $\|f - g\|_p < \varepsilon$ and by Step (2), $g * \varphi_\varepsilon \in C_c^\infty(\Omega)$ and $\|g - g * \varphi_\varepsilon\| \to 0$ as $\varepsilon \to 0$. Thus,

$$\|f - g * \varphi_\varepsilon\|_p \le \|f - g\|_p + \|g - g * \varphi_\varepsilon\|_p \to 0$$

as $\varepsilon \to 0$. This completes the proof. □

We have proved in Theorem 2.6 that $\|f - f * \varphi_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$ for evert $f \in L^p(\Omega)$ with $1 \le p < \infty$. Thne next theorem show that the convergence can be strongerif $f \in C_c(\Omega)$.

Theorem 2.8. *Suppose $f \in C_c(\Omega)$. Then $f * \varphi_\varepsilon \to f$ uniformly on Ω.*

Proof. For $x \in \Omega$, we have

$$|f(x) - (f * \varphi_\varepsilon)(x)| \le \int |f(x) - f(x-y)|\varphi_\varepsilon(y)dy.$$

Since f is uniformly on $\text{supp}(f)$,

$$\begin{aligned} \int |f(x) - f(x-y)|\varphi_\varepsilon(y)dy &\le \int_{|y|<\varepsilon} |f(x) - f(x-y)|\varphi_\varepsilon(y)dy \\ &\le \sup\{|f(x) - f(x-y)| : x \in \text{supp}(f), \, |y| < \varepsilon\} \\ &\quad \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

□

3. Proof of uniqueness theorem

Proof of Theorem 1.17. It is enough to proof that

$$f \in L^1_{\text{loc}}(\Omega), \quad u_f = 0 \quad \implies f = 0 \quad \text{a.e.}$$

So, let $f \in L^1_{\text{loc}}(\Omega)$ such that $u_f = 0$, i.e., $\int_\Omega f(x)\varphi(x)dx = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Let K be a compact subset of Ω and ψ be as in Proposition 2.3. Then $f\psi \in L^1(\mathbb{R}^d)$. This is seen as follows: Let $K_\psi := \text{supp}(\psi)$. Then

$$\int_{\mathbb{R}^d} |f\psi| = \int_{K_\psi} |f\psi| \leq \|\psi\|_\infty \int_{K_\psi} |f| < \infty.$$

Let φ be a mollifier on $\mathbb{R}^d$ and $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$. Then we have

$$(\varphi_\varepsilon * f\psi)(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)f(y)\psi(y)dy = 0$$

for every $x \in \mathbb{R}^d$ since $y \mapsto \varphi_\varepsilon(x-y)\psi(y)$ belongs to $\mathcal{D}(\Omega)$. Also, by Theorem 2.6, we have

$$\|\varphi_\varepsilon * f\psi - f\psi\|_1 \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence, $f\psi = 0$ in $L^1(\mathbb{R}^d)$ so that $f = 0$ a.e. on K. Since Ω can be written as a countable union of compact subsets it follows that $f = 0$ a.e. on Ω. ☐

Example 3.1. For each $k \in \mathbb{N}$, let

$$f_k(x) := \sum_{n=-k}^{k} e^{inx}, \quad x \in \mathbb{R}.$$

Then, we have

$$u_{f_k}(\varphi) = \int_{\mathbb{R}} f_k(x)\varphi(x) = \sum_{n=-k}^{k} \int_{\mathbb{R}} \varphi(x)e^{inx}dx = 2\pi \sum_{n=-k}^{k} \hat{\varphi}(-n).$$

Hence, for every $\varphi \in \mathcal{D}(\mathbb{R})$,

$$u_{f_k}(\varphi) \to 2\pi \sum_{n\in\mathbb{N}} \hat{\varphi}(n) = 2\pi\varphi(0) = 2\pi\delta_0(\varphi).$$

Thus, $u_{f_k} \to 2\pi\delta_0$ as $k \to \infty$. Identifying u_{f_k} with f_k, we may write the above fact as

$$\sum_{n\in\mathbb{Z}} e_n = 2\pi\delta_0,$$

where $e_n(x) := e^{inx}$.

4. A characterization of distributions

First a characterization theorem.

Theorem 4.1. *Let u be a linear functional on $\mathcal{D}(\Omega)$. Then u is a distribution if and only if for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that*

$$|u(\varphi)| \le C \sum_{|\alpha|\le N} \|\partial^\alpha \varphi\|_\infty \tag{1}$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq K$.

Proof. Suppose u is a distribution. Assume for a moment that there exists a compact $K \subseteq \Omega$ such that (1) is not satisfied for any $C > 0$ and $N \in \mathbb{N}$. Then for every $N \in \mathbb{N}$ and $C > 0$, there exists φ, depending on (N, C), such that $\operatorname{supp}(\varphi) \subseteq K$ and

$$|u(\varphi)| > C \sum_{|\alpha|\le N} \|\partial^\alpha \varphi\|_\infty.$$

In particular, for every $N \in \mathbb{N}$, there exists φ_N such that $\operatorname{supp}(\varphi_N) \subseteq K$ and

$$|u(\varphi_N)| > N \sum_{|\alpha|\le N} \|\partial^\alpha \varphi_N\|_\infty.$$

Let $\tilde\varphi_N := \varphi_N/|u(\varphi_N)|$, $N \in \mathbb{N}$. Then we have

$$1 = |u(\tilde\varphi_N)| > N \sum_{|\alpha|\le N} \|\partial^\alpha \tilde\varphi_N\|_\infty \ge N\|\partial^\alpha \tilde\varphi_N\|_\infty$$

for all $N \in \mathbb{N}$. Hence, $\tilde\varphi_N \to 0$ in $\mathcal{D}(\Omega)$ as $N \to \infty$. But, $u(\tilde\varphi_N) = 1$ for all $N \in \mathbb{N}$. Thus, we arrived at a contradiction to the fact that u is a distribution.

Conversely, let (φ_n) in $\mathcal{D}(\Omega)$ such that $\varphi_n \to 0$ in $\mathcal{D}(\Omega)$. Let $K \subseteq \Omega$ be a compact set with $\operatorname{supp}(\varphi_n) \subseteq K$ for all $n \in \mathbb{N}$. Suppose that there exists a constant $C > 0$ and $N \in \mathbb{N}_0$ such that

$$|u(\varphi)| \le C \sum_{|\alpha|\le N} \|\partial^\alpha \varphi\|_\infty$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq K$. Then we have

$$|u(\varphi_n)| \le C \sum_{|\alpha|\le N} \|\partial^\alpha \varphi_n\|_\infty.$$

By the assumption on (φ_n), $u(\varphi_n) \to 0$. □

5. Distributions of finite and infinite orders

Definition 5.1. Let u be a distribution on Ω. Then u is said to be of **finite order**, if there exists $N \in \mathbb{N}_0$ satisfying the condition in Theorem 4.1 which is valid for all compact set $K \subseteq \Omega$, and in that case, the infimum of all such N is called the order of u. If u is not of finite order, then it is said to be of **infinite order**.

Example 5.2. Every regular distribution is of finite order: To see this, let $f \in L^1_{\rm loc}(\Omega)$. Then for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$|u_f(\varphi)| \leq \int_\Omega |f(x)|\,|\varphi)(x)|dx \leq \|\varphi\|_\infty \int_\Omega |f(x)|dx.$$

Thus, (1) in Theorem 4.1 is satisfied with $N = 0$ and $C = \int_\Omega |f(x)|dx$.

Example 5.3. Define

$$u(\varphi) := \sum_{j=0}^{\infty} \varphi^{(j)}(j), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Note that, since φ is with compact support, the above is a finite sum for each φ. More precisely, if $\operatorname{supp}(\varphi) \subseteq [-k, k]$ for some $k \in \mathbb{N}$, then

$$u(\varphi) = \sum_{j=0}^{k-1} \varphi^{(j)}(j).$$

Further, if K is a compact set and if $K \subseteq [-k, k]$ for some $k \in \mathbb{N}$, then we have

$$|u(\varphi)| \leq \sum_{j=0}^{k-1} \|\varphi^{(j)}\|_\infty$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subseteq K$. Hence, by Theorem 4.1, $u \in \mathcal{D}'(\mathbb{R})$. This distribution is of infinite order (Why?).

Exercise 5.4. *Show that the delta-distribution is of* 0 *order.*

Exercise 5.5. *Show that the distribution in Example 5.3 is of infinite order.*

6. Restrictions and support of distributions

Definition 6.1. Let u be a distribution on Ω and Ω_0 be an open subset of Ω. Then **restriction** of u to Ω_0, denoted by u_{Ω_0} is a distribution on Ω_0 defined by

$$u_{\Omega_0}(\varphi) := u(\varphi) \quad \text{for every} \quad \varphi \in \mathcal{D}(\Omega_0).$$

Definition 6.2. Let u be a distribution on Ω. Then the support of u is the set

$$\operatorname{supp}(u) := \{x \in \Omega : u_G \neq 0 \text{ for every open set } G \subset \Omega \text{ with } x \in G\}.$$

Note that, for $u \in \mathcal{D}'(\Omega)$ and $x \in \Omega$,

$$x \notin \text{supp}(u) \iff \exists \text{ open set } G \subset \Omega \text{ with } x \in G \text{ such that } u_G = 0.$$

Hence,

$$\text{supp}(u) = \Omega \setminus \bigcup \{G : u_G = 0\}.$$

Thus, $\text{supp}(u)$ is a closed subset of Ω.

Exercise 6.3. $\text{supp}(\delta_a) = \{a\}$.

Exercise 6.4. *For* $f \in L^1_{\text{loc}}(\Omega)$, $\text{supp}(u_f) = \text{supp}(f)$.

Exercise 6.5. *For* $u \in \mathcal{D}'(\Omega)$ *and* $\varphi \in \mathcal{D}(\Omega)$, $\text{supp}(u) \cap \text{supp}(f) = \emptyset \implies u(\varphi) = 0$.

7. Multiplication by C^∞ functions

Theorem 7.1. *If* $f \in C^\infty(\Omega)$, *then* $f\varphi \in \mathcal{D}(\Omega)$ *for every* $\varphi \in \mathcal{D}(\Omega)$.

Proof. Exercise. □

Theorem 7.2. *For* $f \in C^\infty(\Omega)$ *and* $u \in \mathcal{D}'(\Omega)$, *then the map*

$$\varphi \mapsto u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is a distribution.

Proof. Suppose $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Then it can be seen that $f\varphi_n \to f\varphi$ in $\mathcal{D}(\Omega)$. Hence, $u(f\varphi_n) \to u(f\varphi)$. □

Notation 7.3. For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, the distribution $f \mapsto f\varphi$ as in Theorem 7.2 is denoted by fu.

Example 7.4. $f \in C^\infty(\Omega)$ and $a \in \Omega$, we have

$$(f\delta_a)(\varphi) = \delta_a(f\varphi) = f(a)\varphi(a) = f(a)\delta_a(\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence, $f\delta_a = f(a)\delta$.

Example 7.5. $f, g \in L^1_{\text{loc}}(\Omega)$, we have

$$(fu_g)(\varphi) = u_g(f\varphi) = \int g(x)f(x)\varphi(x)dx = u_{fg}(\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence, $fu_g = u_{fg}$.

Theorem 7.6. *Let* $f \in C^\infty(\Omega)$. *Then the map* $u \mapsto fu$ *is continuous in the sense that*

$$u_n \to u \text{ in } \mathcal{D}'(\Omega) \implies fu_n \to fu \text{ in } \mathcal{D}'(\Omega).$$

8. Translation of distributions

We observe that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$,

$$u_{\tau_h f}(\varphi) = \int (\tau_h f)(x)\varphi(x)dx = \int f(x-h)\varphi(x)dx = \int f(x)\varphi(x+h)dx = u_f(\tau_{-h}\varphi).$$

Identifying L^1_{loc}-functions with the corresponding distributions, we may write the above as

$$(\tau_h f)(\varphi) = f(\tau_{-h}\varphi) \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Motivated by this, for $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, we may define

$$\boxed{(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).}$$

Theorem 8.1. *If $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, then $\tau_h u$ defined by*

$$(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

is s distribution.

Definition 8.2. For $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, the distribution $\tau_h u$ defined by

$$(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

is called the **translation** of u by h.

Example 8.3. Observe that

$$(\tau_h \delta_a)(\varphi) = \delta_a(\tau_{-h}\varphi) = (\tau_{-h}\varphi)(a) = \varphi(a+h) = \delta_{a+h}(\varphi).$$

Hence, $\tau_h \delta_a = \delta_{a+h}$.

Theorem 8.4. *For each $h \in \mathbb{R}^d$, the map $u \mapsto \tau_h u$ is continuous on $\mathcal{D}'(\mathbb{R}^d)$ in the sense that $u_n \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ implies $\tau_h u_n \to \tau_h u$ in $\mathcal{D}'(\mathbb{R}^d)$.*

9. The spaces $\mathcal{E}(\Omega)$ and $\mathcal{E}'(\Omega)$

Definition 9.1. The space $C^\infty(\Omega)$ with the notion of convergence defined by

$$f_n \to f \iff \partial^\alpha f_n \to \partial^\alpha f \quad \text{uniformly on every compact } K \subseteq \Omega \quad \forall\, \alpha \in \mathbb{N}_0^d$$

is denoted by $\mathcal{E}(\Omega)$.

Clearly,

$$\mathcal{D}(\Omega) \subseteq \mathcal{E}(\Omega).$$

Theorem 9.2. *Let u be a distribution. Then the map $f \mapsto fu$ from $\mathcal{E}(\Omega)$ to $\mathcal{D}'(\Omega)$ is continuous in the sense that*

$$f_n \to f \text{ in } \mathcal{E}(\Omega) \quad \Longrightarrow \quad f_n u \to fu \text{ in } \mathcal{D}'(\Omega).$$

Recall that $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, fu defined by

$$(fu)(\varphi) := u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega),$$

is a distribution.

Theorem 9.3. *Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Then*

$$\operatorname{supp}(fu) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(u).$$

Proof. Suppose $x_0 \notin \operatorname{supp}(f)$. Then there exists an open nbd $\Omega_0 \subseteq \Omega$ of x_0 such that $f = 0$ on Ω_0. Hence,

$$(fu)(\varphi) = 0 \quad \forall \varphi \in \mathcal{D}(\Omega_0)$$

so that $fu = 0$ on Ω_0. Therefore, $x_0 \notin \operatorname{supp}(fu)$. Also, $x_0 \notin \operatorname{supp}(f)$ implies there exists an open nbd $\Omega_0 \subseteq \Omega$ of x_0 such that $u = 0$ on Ω_0 so that $fu = 0$ on Ω_0 and hence, $x_0 \notin \operatorname{supp}(fu)$ □

Corollary 9.4. *If u is a distribution with compact support, then for any $f \in \mathcal{E}(\Omega)$, fu is also of compact support.*

Definition 9.5. The set of all linear functionals u on $\mathcal{E}(\Omega)$ such that

$$\varphi_n \to \varphi \text{ in } \mathcal{E}(\Omega) \implies u(\varphi_n) \to u(\varphi)$$

is denoted by $\mathcal{E}'(\Omega)$. A sequence (u_n) in $\mathcal{E}'(\Omega)$ is said to converge to $u \in \mathcal{E}'(\Omega)$, weritten $u_n \to u$ if

$$u_n(f) \to u(f) \quad \forall\, f \in \mathcal{E}(\Omega).$$

Theorem 9.6. *If $u \in \mathcal{E}'(\Omega)$, then $u_0 := u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$. Further, the map $u \mapsto u_0$ is continuous from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$, in the sense that,*

$$u_n \to u \quad in \quad \mathcal{E}'(\Omega) \quad \implies \quad u_{0,n} \to u_0 \quad in \quad \mathcal{D}'(\Omega).$$

Proof. Let $u \in \mathcal{E}'(\Omega)$, Let $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Then there exists a compact set $K_0 \subseteq \Omega$ such that supp, $\varphi_n, \varphi \subseteq K_0$ and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on Ω. Hence, $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on every compact subset of Ω. Thus, $\varphi_n \to \varphi$ in $\mathcal{E}(\Omega)$ so that by hypothesis, $u(\varphi_n) \to u(\varphi)$, i.e., $u_0(\varphi_n) \to u_0(\varphi)$. The last part is obvious. □

In view of the above theorem, we may say that

$\mathcal{E}'(\Omega)$ is embedded in $\mathcal{D}'(\Omega)$.

We shall show that the distribution u_0 in the above theorem is with compact support.

Theorem 9.7. *If $u \in \mathcal{D}'(\Omega)$ is with compact support, then $u \in \mathcal{E}'(\Omega)$ in the sense that there exists a unique $\tilde{u} \in \mathcal{E}'(\Omega)$ such that*

(1) $\tilde{u}|_{\mathcal{D}(\Omega)} = u$ *and*
(2) $f \in \mathcal{E}(\Omega)$ *with* $\operatorname{supp}(u) \cap \operatorname{supp}(f) = \emptyset$ *implies* $\tilde{u}(f) = 0$.

For proving the above theorem we shall make use of the following lemma.

Lemma 9.8. *If* $u \in \mathcal{D}'(\Omega)$ *and* $\varphi \in \mathcal{D}(\Omega)$ *are such that* $\operatorname{supp}(u) \cap \operatorname{supp}(\varphi) = \emptyset$*, then* $u(\varphi) = 0$.

Proof of Theorem 9.7. Suppose $u \in \mathcal{D}'(\Omega)$ is with compact support, say $K := \operatorname{supp}(u)$. Let $\psi \in \mathcal{D}(\Omega)$ be such that $\psi = 1$ on K. Then, for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$u(\varphi) = u(\psi\varphi + (1-\psi)\varphi) = u(\psi\varphi) + u((1-\psi)\varphi).$$

Note that $\operatorname{supp}(u) \cap \sup((1-\psi)\varphi) = \emptyset$. Hence by the last lemma, $u((1-\psi)\varphi) = 0$. Thus,

$$u(\varphi) = u(\psi\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Now, define

$$\tilde{u}(f) = u(\psi f), \quad f \in \mathcal{E}(\Omega).$$

Then we have $\tilde{u} \in \mathcal{E}'(\Omega)$ and $\tilde{u}|_{\mathcal{D}(\Omega)} = u$. [To see that $\tilde{u} \in \mathcal{E}'(\Omega)$, we may observe that $f_n \to f$ in $\mathcal{E}(\Omega)$ implies $\psi f_n \to \psi f$ in $\mathcal{D}(\Omega)$.]

To see the uniqueness, suppose $v \in \mathcal{E}'(\Omega)$ is such that

(1) $v|_{\mathcal{D}(\Omega)} = u$ and
(2) $f \in \mathcal{E}(\Omega)$ with $\operatorname{supp}(u) \cap \operatorname{supp}(f) = \emptyset$ implies $v(f) = 0$.

Then, for $f \in \mathcal{E}(\Omega)$, we have

$$v(f) = v(\psi f + (1-\psi)f) = v(\psi f) + v((1-\psi)f) = u(\psi f) + v((1-\psi)f).$$

Since $(1-\psi)f = 0$ on $K := \operatorname{supp}(u)$, assumption (2) on v implies $v((1-\psi)f) = 0$. Thus, $v(f) = u(\psi f) = \tilde{u}(f)$. □

For the proof of Lemma 9.8, we make use of *partition of unity*:

Proposition 9.9. (Partition of unity) *Let K be a compact set and $\Omega_1, \ldots, \Omega_n$ be open subsets of $\mathbb{R}^d$ such that $K \subseteq \cup_{j=1}^n \Omega_j$. Then there exists $\psi_1, \ldots, \psi_n$ in $\mathcal{D}(\Omega_0)$ with $\Omega_0 := \cup_{j=1}^n \Omega_j$ such that* $\operatorname{supp}(\psi_j) \subseteq \Omega_j$ *and* $\sum_{j=1}^n \psi_j = 1$ *on* K.

Proof. Let $x \in K$. Then $x \in \Omega_i$ for some $i \in \{1, \ldots, n\}$. Let G_x be an open nbd of x such that $\overline{G}_x$ is compact and $\overline{G}_x \subseteq \Omega_i$. Since K is compact, there exist $x_1, \ldots, x_k \in K$ such that $K \subseteq \bigcup_{j=1}^k G_{x_j}$. For each $i \in \{1, \ldots, n\}$, let H_i be the union of those $\overline{G}_{x_j}$

such that $\overline{G}_{x_j} \subseteq \Omega_i$. Then each $H - i$ is compact and $H_i \subseteq \Omega_i$. Hence, there exists $g_i \in \mathcal{D}(\Omega_i)$ such that $g_i = 1$ on H_i. Note that $K \subseteq \bigcup_{i=1}^n H_i$. Now, define

$$\psi_1 = g_1, \quad \psi_2 = (1-g_1)g_2, \quad \ldots, \quad \psi_n = (1-g_1)(1-g_2)\cdots(1-g_{n-1})g_n.$$

It can be seen by induction that

$$\psi_1 + \cdots + \psi_n = 1 - (1-g_1)(1-g_2)\cdots(1-g_n).$$

Since $K \subseteq \bigcup_{i=1}^n H_i$, and since $g_i = 1$ on H_i, we obtain $\psi_1 + \cdots + \psi_n = 1$ on K. □

Proof of Lemma 9.8. Let $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ are such that $\mathrm{supp}(u) \cap \mathrm{supp}(\varphi) = \emptyset$. To prove that $u(\varphi) = 0$. For this, let $K = \mathrm{supp}(\varphi)$. For each $x \in K$, since $x \notin \mathrm{supp}(u)$, there exists open set $\Omega_x \subseteq \Omega$ such that $x \in \Omega_x$. Then $\{\Omega_x : x \in K\}$ is an open cover of K. Since K is compact, there exists $x_1, \ldots, x_n$ in K such that $K \subseteq \cup_{j=1}^n \Omega_{x_j}$. By partition of unity, there there exists $\psi_1, \ldots, \psi_n$ in $\mathcal{D}(\Omega_0)$ with $\Omega_0 := \cup_{j=1}^n \Omega_{x_j}$such that $\mathrm{supp}(\psi_j) \subseteq \Omega_{x_j}$ and $\sum_{j=1}^n \psi_j = 1$ on K. Then we have $\varphi = \sum_{j=1}^n \psi_j \varphi$ so that $u(\varphi) = \sum_{j=1}^n u(\psi_j \varphi) = 0$, since $\psi_j \varphi \in \mathcal{D}(\Omega_{x_j})$ and $\Omega_{x_j} \cap \mathrm{supp}(u) = \emptyset$. □

Now the theorem that we had promised:

Theorem 9.10. *If $u \in \mathcal{E}'(\Omega)$, then $u|_{\mathcal{D}(\Omega)}$ is a distribution with compact support.*

For its proof we use the following characterization:

Theorem 9.11. *Let u be a linear functional on $\mathcal{E}(\Omega)$. Then $u \in \mathcal{E}'(\Omega)$ if and only if there exists a compact $K \subseteq \Omega$, constant $C > 0$ and $m \in \mathbb{N}_0$ such that*

$$|u(f)| \leq C \sum_{|\alpha| \leq m} \sup_{x \in K} |(\partial^\alpha f)(x)| \quad \forall\, f \in \mathcal{E}(\Omega).$$

Proof. (⇐): Obvious.

(⇐): Suppose the conclusion is not true. Then for any triple $\eta := (K, C, m)$ there exists $\varphi_\eta \in \mathcal{E}(\Omega)$ such that

$$|u(f_\eta)| > C \sum_{|\alpha| \leq m} \sup_{x \in K} |(\partial^\alpha f)(x)|.$$

So, for $m \in \mathbb{N}$, let $K_m := \overline{B_m(0)}$ and $f_m \in \mathcal{E}(\Omega)$ such that

$$|u(f_m)| > m \sum_{|\alpha| \leq m} \sup_{x \in K_m} |(\partial^\alpha f_m)(x)|.$$

Let $g_m = f_m/[m\sum_{|\alpha|\le m}\sup_{x\in K_m}|(\partial^\alpha f_m)(x)|]$. Then for every $\beta\in\mathbb{N}_0^d$ with $|\beta|\le m$ and $K\subseteq\Omega$ with $K\subseteq K_m$, we have

$$\sup_{x\in K}|\partial^\beta g_m\| \le \sum_{|\gamma|\le m}\sup_{x\in K_m}|(\partial^\gamma g_m)(x)| = \frac{1}{m}.$$

Thus, $f_m\to 0$ in $\mathcal{E}(\Omega)$ but $|u(f_m)|>1$ for all $m\in\mathbb{N}$. This is a contradiction. □

Proof of Theorem 9.10. Let $u\in\mathcal{E}'(\Omega)$. We have already seen that $u|_{\mathcal{D}(\Omega)}$ is a distribution. Let K be as in Theorem 9.11. We claim that $\mathrm{supp}(u)\subseteq K$. To prove this claim, suppose $x\notin K$. Then there exists an open neighbourhood $G_x\subseteq\Omega$ of x such that $G_x\cap K=\emptyset$. Hence, $\varphi\in\mathcal{D}(G_x)$ implies $\mathrm{supp}(\varphi)\cap K=\emptyset$. Hence, from the relation

$$|u(f)|\le C\sum_{|\alpha|\le m}\sup_{x\in K}|(\partial^\alpha f)(x)| \quad \forall\, f\in\mathcal{E}(\Omega)$$

in Theorem 9.11, we have $u(\varphi)=0$. Therefore, $x\notin\mathrm{supp}(u)$. Thus we have proved that $x\notin K$ implies $x\notin\mathrm{supp}(u)$. Equivalently, $\mathrm{supp}(u)\subseteq K$. □

In view of Theorems 9.7 and 9.10, there is a one-one correspondence between $\mathcal{E}'(\Omega)$ and distributions with compact support. Therefore, distributions with compact support is also denoted by $\mathcal{E}'(\Omega)$.

10. Differentiation of distributions

Let $f\in C^1(0,1)\cap C[0,1]$. Then for every $\varphi\in C_c^\infty(0,1)$, we have

$$\int_0^1 f'(x)\varphi(x)dx = [\varphi(x)f(x)]_0^1 - \int_0^1\varphi'(x)f(x)dx = -\int_0^1\varphi'(x)f(x)dx.$$

Thus,

$$u_{f'}(\varphi) = -u_f(\varphi').$$

More generally, it can be seen that:

If $f\in C^1(\Omega)\cap C(\overline{\Omega})$, then for every $\varphi\in C_c^\infty(\Omega)$ and for every $\alpha\in\mathbb{N}_0^d$,

$$\int_\Omega(\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|}\int_0^1 f(x)(\partial^\alpha\varphi)(x)dx$$

so that

$$u_{\partial^\alpha f}(\varphi) = (-1)^{|\alpha|}u_f(\partial^\alpha\varphi).$$

Identifying L^1_{loc}-functions with the corresponding distributions, we may write the above as

$$\boxed{(\partial^\alpha f)(\varphi) = (-1)^{|\alpha|}f(\partial^\alpha\varphi) \quad \forall\,\varphi\in\mathcal{D}(\Omega).}$$

Theorem 10.1. *For $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, the map $\partial^\alpha u : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by*

$$(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(O),$$

is a distribution.

Definition 10.2. For $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, the distribution $\partial^\alpha u$ defined by

$$(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(O),$$

is called the **α-th derivative** of u.

Notation 10.3. *If $f \in L^1_{\rm loc}(\Omega)$, then $\partial^\alpha u_f$ is usually denoted by $\partial^\alpha f$.*

Example 10.4. Consider the *Heaveside function*:

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Then

$$\int_{\mathbb{R}} H(x)\varphi'(x)dx = \int_0^\infty \varphi'(x)dx = -\varphi(0) = -\delta_0(\varphi).$$

Thus, $H' = \delta_0$.

Suppose $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$.

(1) We say that $\partial^\alpha u$ belongs to $L^1_{\rm loc}(\Omega)$, and write as $\partial^\alpha u \in L^1_{\rm loc}(\Omega)$ if there exists a function $f \in L^1_{\rm loc}(\Omega)$ such that

$$(\partial^\alpha u)(\varphi) = u_f(\varphi) \quad \forall\, \varphi \in \mathcal{D}(O).$$

(2) We say that $\partial^\alpha u \in L^p(\Omega)$ iff there exists a function $f \in L^p(\Omega)$ such that

$$(\partial^\alpha u)(\varphi) = u_f(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Suppose $f \in L^1_{\rm loc}(\Omega)$.

(1) We say that $\partial^\alpha f \in L^1_{\rm loc}(\Omega)$ iff there exists a function $g \in L^1_{\rm loc}(\Omega)$ such that

$$(\partial^\alpha u_f)(\varphi) = u_g(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$

i.e., iff

$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx = \int_\Omega g(x)\varphi(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$

and this fact is also written as

$$\int_\Omega (\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

(2) We say that $\partial^\alpha f \in L^p(\Omega)$ iff there exists a function $g \in L^p(\Omega)$ such that

$$(\partial^\alpha u_f)(\varphi) = u_g(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$

i.e., iff

$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx = \int_\Omega g(x)\varphi(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega),$$

and this fact is also written as

$$\int_\Omega (\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Definition 10.5. (Sobolev spaces) For $r \in \mathbb{N}_0$ and $1 \le p \le \infty$, the **Sobolev space** $W^{r,p}(\Omega)$ is defined as the vector space

$$W^{r,p}(\Omega) := \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega)\ \forall\, |\alpha| \le r\}.$$

Thus, if $f \in L^p(\Omega)$, then $f \in W^{r,p}(\Omega)$ iff there exists $g \in L^p(\Omega)$ such that

$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)dx = \int_\Omega g(x)\varphi(x)dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Theorem 10.6. *For every multi-index α, $u \mapsto \partial^\alpha u$ is continuous on $\mathcal{D}'(\Omega)$, i.e.,*

$$u_n \to u \text{ in } \mathcal{D}'(\Omega) \implies \partial^\alpha u_n \to \partial^\alpha u \text{ in } \mathcal{D}'(\Omega).$$

Proof. Follows from the definitions. □

11. Convolution involving distributions

Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then we have

$$(f * \varphi)(x) = \int_{\mathbb{R}^d} f(y)\varphi(x-y), \quad x \in \mathbb{R}^d.$$

Let us introduce the notation:

$$\tilde{\varphi}(x) = \varphi(-x), \quad \varphi \in C(\mathbb{R}^d),\, x \in \mathbb{R}^d.$$

Then

$$\varphi(x-y) = \tilde{\varphi}(y-x) = (\tau_x \tilde{\varphi})(y).$$

Thus, we have

$$(f * \varphi)(x) = u_f(\tau_x \tilde{\varphi}), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),\, x \in \mathbb{R}^d.$$

Motivated by this, we have the following definition.

Definition 11.1. The convolution of $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\Omega)$ is defined by

$$(u * \varphi)(x) = u(\tau_x \tilde{\varphi}), \quad x \in \mathbb{R}^d,$$

where $\tilde{\varphi}(s) = \varphi(-s)$.

Theorem 11.2. *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi, \psi \in \mathcal{D}(\Omega)$. Then*

1. $u * \varphi \in C^\infty(\mathbb{R}^d)$,
2. $\text{supp}(u * \varphi) \subseteq \text{supp}(u) + \text{supp}(\varphi)$,
3. $\partial^\alpha (u * \varphi) = u * \partial^\alpha \varphi = (\partial^\alpha u) * \varphi$.
4. $u * (\varphi * \psi) = (u * \varphi) * \psi$.

Recall that, if $\varphi \in \mathcal{D}$ is such tha $\varphi \geq 0$ and $\int \varphi = 1$ and for $\varepsilon > 0$ if $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(x\varepsilon)$, then $\varphi_\varepsilon \in \mathcal{D}$ and $\{\varphi_\varepsilon : \varepsilon > 0\}$ is called an **approximate identity**. It is known that

1. $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies $f * \varphi_\varepsilon \in C^\infty(\mathbb{R}^d)$.
2. $f \in C_c(\mathbb{R}^d)$ implies $f * \varphi_\varepsilon \to f$ uniformly as $\varepsilon \to 0$.
3. f continuous at x implies $(f * \varphi_\varepsilon)(x) \to f(x)$ as $\varepsilon \to 0$.
4. $f \in L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ implies $f * \varphi_\varepsilon \to f$ in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$.

In the following we use the notation φ_ε for an approximate identity.

Theorem 11.3. (Regularization of distributions) *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\{\varphi_\varepsilon : \varepsilon > 0\}$ be an approximate identity. Then*

$$u * \varphi_\varepsilon \to u \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d).$$

Proof. For $\psi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\begin{aligned}(u * \varphi_\varepsilon)(\psi) &= \int (u * \varphi_\varepsilon)(y)\psi(y)dy = \int (u * \varphi_\varepsilon)(y)\tilde{\psi}(0 - y)dy = [(u * \varphi_\varepsilon) * \tilde{\psi}](0) \\ &= [u * (\varphi_\varepsilon * \tilde{\psi})](0) \to (u * \tilde{\psi})(0) \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

But,

$$(u * \tilde{\psi})(0) = u(\tau_0 \psi) = u(\psi).$$

Thus, $u * \varphi_\varepsilon \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ as $\varepsilon \to 0$. □

Corollary 11.4. *Let $u \in \mathcal{D}'(\mathbb{R})$ such that $u' = 0$. Then u is a constant.*

Proof. Let $u_\varepsilon := u * \varphi_\varepsilon$. Then $u'_\varepsilon = u' * \varphi_\varepsilon = 0$. Hence, $u_\varepsilon = C_\varepsilon$, constants. But, $u_\varepsilon \to u$. Therefore, there exists a constant C such that $u_\varepsilon \to C$ and hence $u = C$. □

Now, suppose $f, g \in L^1(\mathbb{R}^d)$. Then for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\begin{aligned}(f*g)(\varphi) &= \int (f*g)(x)\varphi(x)dx = \int \left(\int f(y)g(x-y)dy\right)\varphi(x)dx \\ &= \int f(y)\left(\int g(x-y)\varphi(x)dx\right)dy = \int f(y)\left(\int g(s)\varphi(s+y)ds\right)dy \\ &= \int f(y)\left(\int g(s)(\tau_{-y}\varphi)(s)ds\right)dy \\ &= f(\varphi_g)\end{aligned}$$

where

$$\varphi_g(y) := g(\tau_{-y}\varphi).$$

Definition 11.5. For $u, v \in \mathcal{D}'(\mathbb{R}^d)$,

$$(u*v)(\varphi) := u(\varphi_v)$$

where

$$\varphi_v(y) := v(\tau_{-y}\varphi).$$

Exercise 11.6. *Show that*

$$(u*v)(\varphi) = u*\widetilde{(v*\tilde{\varphi})}.$$

12. Schwarz space and tempered distributions

Definition 12.1. The **Schwarz space** $\mathcal{S}(\mathbb{R}^d)$ is the space of all functions in $C_b^\infty(\mathbb{R}^d)$ such that for every $\alpha, \beta \in \mathbb{N}_0^d$, $x^\alpha \partial^\beta f \in C_b(\mathbb{R}^d)$. The elements of $\mathcal{S}(R^d)$ are called the **rapidly decreasing functions**.

Thus, if $f \in C_b^\infty(\mathbb{R}^d)$, then

$$f \in \mathcal{S}(\mathbb{R}^d) \iff \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty$$

for every $\alpha, \beta \in \mathbb{N}_0^d$.

We observe that for each $\alpha, \beta \in \mathbb{N}_0^d$,

$$f \mapsto \|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$$

defines a semi norm on $\mathcal{S}(\mathbb{R}^d)$.

Note that if $f \in C_b^\infty(\mathbb{R}^d)$, $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if for every $\alpha, \beta \in \mathbb{N}_0^d$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial^\beta f(x)| \le \frac{C_{\alpha,\beta}}{|x^\alpha|} \quad \forall\, z \in \mathbb{R}^d.$$

In fact,

$$|\partial^\beta f(x)| \leq \frac{\|f\|_{\alpha,\beta}}{|x^\alpha|} \quad \forall\, z \in \mathbb{R}^d,$$

where

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|.$$

It can be seen that, for each $\alpha, \beta \in \mathbb{N}_0^d$,

$$f \mapsto \|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$$

defines a norm on $\mathcal{S}(\mathbb{R}^d)$. In view of the above observation, elements of $\mathcal{S}(\mathbb{R}^d)$ are also called **rapidly decreasing functions**.

Theorem 12.2. *For $1 \leq p \leq \infty$, $\mathcal{S}(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$. In fact, for $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|f\|_p \leq C_p \sum_{|\alpha|\leq 2d} \|f\|_{\alpha,0},$$

where $C_p := \left(\int \frac{dx}{(1+|x|^2)^p}\right)^{1/p}$ for $1 \leq p < \infty$ and $C_\infty = 1$. Further, $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$. The result is trivially true if $p = \infty$. So, let $1 \leq p < \infty$. Then

$$\int |f|^p = \int \frac{(1+|x|^2)^p |f|^p}{(1+|x|^2)^p} \leq C \sup_{x\in\mathbb{R}^d} (1+|x|^2)^p |f|^p,$$

where $C := \int \frac{dx}{(1+|x|^2)^p}$. But,

$$(1+|x|^2)|f| = \left(1 + \sum_{j=1}^d x_j^2\right)|f| = |f| + \sum_{j=1}^d |x_j^2 f| \leq \sum_{|\alpha|\leq 2d} \|f\|_{\alpha,0}.$$

Thus, we obtain $f \in L^p(\mathbb{R}^d)$, and

$$\|f\|_p \leq C^{1/p} \sum_{|\alpha|\leq 2d} \|f\|_{\alpha,0}.$$

The last part follows, because, $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. □

Definition 12.3. A sequence (f_n) in $\mathcal{S}(\mathbb{R}^d)$ is said to **converge** to $f \in \mathcal{S}(R^d)$ if

$$\|f_n - f\|_{\alpha,\beta} \to 0 \quad \text{as} \quad n \to \infty$$

for each $\alpha, \beta \in \mathbb{N}_0^d$, and in that case we write $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$.

Theorem 12.4. *The space $\mathcal{S}(\mathbb{R}^d)$ is complete, in the sense that, if (f_n) in $\mathcal{S}(\mathbb{R}^d)$ is a Cauchy sequence with respect to $\|\cdot\|_{\alpha,\beta}$ for every $\alpha, \beta \in \mathbb{N}_0^d$, then it converges to a function in $\mathcal{S}(\mathbb{R}^d)$.*

Theorem 12.5. *The space $\mathcal{D}(\mathbb{R}^d)$ is a subspace of $\mathcal{S}(\mathbb{R}^d)$ and for $\varphi_n, \varphi \in \mathcal{D}(\mathbb{R}^d)$, $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ implies $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Clearly, $\mathcal{D}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$. Let $\varphi_n \in \mathcal{D}$ such that $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$. Let K be a compact set in $\mathbb{R}^d$ such that $\operatorname{supp}(\varphi_n) \cup \operatorname{supp}(\varphi) \subseteq K$ for all $n \in \mathbb{N}$. Then for every $\alpha, \beta \in \mathbb{N}_0^d$,

$$\|\varphi_n - \varphi\|_{\alpha,\beta} = \sup_{x \in K} |x^\alpha \partial^\beta (\varphi_n - \varphi)(x)| \le C_\alpha \sup_{x \in K} |\partial^\beta (\varphi_n - \varphi)(x)|$$

for some $C_\alpha > 0$. Since $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, $\sup_{x \in K} |\partial^\beta (\varphi_n - \varphi)(x)| \to 0$ so that $\varphi_n \to \varphi$ in the space $\mathcal{S}(\mathbb{R}^d)$. □

In fact,

Theorem 12.6. *The space $\mathcal{D}(\mathbb{R}^d)$ is a dense subspace of $\mathcal{S}(\mathbb{R}^d)$.*

Definition 12.7. A linear functional u on $\mathcal{S}(\mathbb{R}^d)$ is called a **tempered distribution** if for every sequence (f_n) in $\mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ implies $u(f_n) \to u(f)$. The space of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Definition 12.8. A sequence (u_n) in $\mathcal{S}'(\mathbb{R}^d)$ is said to converge to $u \in \mathcal{S}'(\mathbb{R}^d)$ if

$$u_n(f) \to u(f)$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$.

Notation 12.9.

$$\mathcal{S} := \mathcal{S}(\mathbb{R}^d), \quad \mathcal{S}' := \mathcal{S}'(\mathbb{R}^d).$$
$$\mathcal{D} := \mathcal{D}(\mathbb{R}^d), \quad \mathcal{D}' := \mathcal{D}'(\mathbb{R}^d).$$

Theorem 12.10. *The restrictions of tempered distributions to $\mathcal{D}$ are in $\mathcal{D}'$. Further, the map $u \mapsto u|_{\mathcal{D}}$ is a continuous embedding of $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{D}'(\mathbb{R}^d)$.*

Proof. Let $u \in \mathcal{S}'$. Let $\varphi_n \in \mathcal{D}$ be such that $\varphi_n \to \varphi$ in $\mathcal{D}$. Then by Theorem 12.5, $\varphi_n \to \varphi$ in $\mathcal{S}$. Hence, $u(\varphi_n) \to u(\varphi)$. Thus, $u|_{\mathcal{D}} \in \mathcal{D}'$. Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, $u|_{\mathcal{D}} = 0$ implies $u = 0$. Clearly, for a sequence (u_n) in $\mathcal{S}'(\mathbb{R}^d)$, $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^d)$ implies that $u_n|_{\mathcal{D}} \to u|_{\mathcal{D}}$ in $\mathcal{D}'(\mathbb{R}^d)$. □

Theorem 12.11. *Let u be a linear functional on $\mathcal{S}(\mathbb{R}^d)$. Then $u \in \mathcal{S}'(\mathbb{R}^d)$ if and only if there is a constant $C > 0$ and $m \in \mathbb{N}_0$ such that*

$$|u(f)| \le C \sum_{|\alpha|,|\beta| \le m} \|f\|_{\alpha,\beta}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 12.12. *For $1 \le p \le \infty$, $L^p(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$, and $f_n \to f$ in $L^p(\mathbb{R}^d)$ implies $u_{f_n} \to u_f$ in $\mathcal{S}'(\mathbb{R}^d)$. In other words, the inclusion $L^p(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$ is a (sequentially continuous) imbedding.*

Proof. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$|u_f(\varphi)| \le \int |f|\,|\varphi| \le \|f\|_p \|f\|_q.$$

By Theorem 12.2, $\|f\|_q \le C \sum_{|\alpha|\le 2d} \|f\|_{\alpha,0}$ for some $C > 0$. Hence,

$$|u_f(\varphi)| \le C \sum_{|\alpha|\le 2d} \|f\|_{\alpha,0}$$

for some $C > 0$. Hence, by Theorem 12.11, $u \in \mathcal{S}'(\mathbb{R}^d)$.

Next, suppose $f_n, f \in L^p(\mathbb{R}^d)$ be such that $f_n \to f$ in $L^p(\mathbb{R}^d)$. Then, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$|u_n(\varphi) - u(\varphi)| \le \int |f_n(x) - f(x)|\,|\varphi(x)|dx \le \|f_n - f\|_p \|\varphi\|_q \to 0.$$

Thus, $u_{f_n} \to u_f$ in $\mathcal{S}'(\mathbb{R}^d)$. □

We have

$$\mathcal{E}'(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d) \subseteq \mathcal{D}'(\mathbb{R}^d)$$

in the sense of (sequentially) continuous embedding.

Exercise 12.13. *The space of polynomials on $\mathbb{R}^d$ is a subspace of $\mathcal{S}'(\mathbb{R}^d)$.*

13. Fourier transform of distributions

Recall that for $f \in L^1(\mathbb{R}^d)$,

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx, \quad \xi \in \mathbb{R}^d.$$

Hence, for $f \in L^1(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \hat{f}(\xi)\varphi(\xi)d\xi &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx \right) \varphi(\xi) d\xi \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(\xi) e^{-ix.\xi} d\xi \right) f(x) dx \\
&= \int_{\mathbb{R}^d} \hat{\varphi}(x) f(x) dx.
\end{aligned}$$

So, formally, we write

$$u_{\hat{f}}(\varphi) = u_f(\hat{\varphi}).$$

Formally, because,

$$\varphi \in \mathcal{D}(\mathbb{R}^d) \quad \text{does not imply} \quad \hat{\varphi} \in \mathcal{D}(\mathbb{R}^d).$$

However,

$$\varphi \in \mathcal{S}(\mathbb{R}^d) \quad \Longrightarrow \quad \hat{\varphi} \in \mathcal{S}(\mathbb{R}^d).$$

In fact, we have:

Theorem 13.1. *For every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ and the map $\varphi \mapsto \hat{\varphi}$ is a (bijective) homeomorphism (with respect to sequential continuity), and*

$$\|\hat{\varphi}\|_2 = (2\pi)^{d/2}\|f\|_2 \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Thus, for $f \in L^1(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $u_f(\hat{\varphi})$ makes sense and

$$\varphi \mapsto u_f(\hat{\varphi})$$

is a tempered distribution.

Theorem 13.2. *For $u \in \mathcal{S}'(\mathbb{R}^d)$, $\hat{u} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ defined by*

$$\hat{u}(f) := u(\hat{f}),\ f \in \mathcal{S}(\mathbb{R}^d),$$

belongs to $\mathcal{S}'(\mathbb{R}^d)$.

Proof. Exercise. □

The above theorem motivates the following definition.

Definition 13.3. The **Fourier transform** of $u \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\hat{u}(f) := u(\hat{f}) \quad \forall\, f \in \mathcal{S}(\mathbb{R}^d).$$

Exercise 13.4. *Prove the following. The following results hold:*

1. *For $u \in \mathcal{S}'(\mathbb{R}^d)$, $\hat{u}(f) := u(\hat{f})$, $f \in \mathcal{S}(\mathbb{R}^d)$, belongs to $\mathcal{S}'(\mathbb{R}^d)$.*
2. *$u \mapsto \hat{u}$ is continuous on $\mathcal{S}'(\mathbb{R}^d)$.*
3. *For $f \in L^1(\mathbb{R}^d)$, $\widehat{u_f}(\varphi) = u_{\hat{f}}(f)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.*
4. *$\hat{\delta} = 1$.*

14. Problems

Throughout, Ω denotes a nonempty open subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$.

(1) Let φ be a mollifier. For $a \in \Omega$ and $\varepsilon > 0$ be such that $\overline{B_\varepsilon(a)} \subset \Omega$, let $\psi_{\varepsilon,a}(x) := \frac{1}{\varepsilon^d}\varphi(\frac{x-a}{\varepsilon}$. Show that $\psi_{\varepsilon,a} \in \mathcal{D}(\Omega)$ such that $\mathrm{supp}(\psi_{\varepsilon,a}) \subseteq B_\varepsilon(a)$ and $\int_\Omega \psi_{\varepsilon,a} dx = 1$.

(2) Let $\psi_{\varepsilon,a}$ be as Problem 1, and let $\psi_\varepsilon := \psi_{\varepsilon,0}$. Prove that for $f \in C_c(\mathbb{R}^d)$, $f * \psi_\varepsilon \to f$ uniformly.

(3) Show that $\mathcal{D}(\Omega)$ is sequentially complete. That is, if (φ_n) in $\mathcal{D}(\Omega)$ is such that for every $\varepsilon > 0$ and for every $\alpha \in \mathbb{N}_0^d$, there exists $N \in \mathbb{N}$ such that $\|\partial^\alpha(\varphi_n - \varphi_m)\|_\infty < \varepsilon$ for all $n \geq N$, then there exists $\varphi \in \mathcal{D}(\Omega)$ such that $\|\partial^\alpha(\varphi_n - \varphi)\|_\infty \to 0$ as $n \to \infty$ for each $\alpha \in \mathbb{N}_0^d$.

(4) Corresponding to $f \in L^1_{\mathrm{loc}}(\Omega)$, let
$$u_f(\varphi) := \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega),\, x \in \Omega.$$
Show that u_f is a distribution, and it is of order 0.

(5) Show that the delta-distribution is not a regular distribution.

(6) Show every delta-distribution is a limit of a sequence of regular distributions.

(7) Let (f_n) in $L^1_{\mathrm{loc}}(\Omega)$ and $f : \Omega \to \mathbb{C}$ be such that $f_n \to f$ a.e. on Ω and for every compact $K \subseteq \Omega$, there exists $g \in L^1(\Omega)$ such that $|f_n| \leq |g|$ a.e. on K. Prove that $f \in L^1_{\mathrm{loc}}(\Omega)$ and $f_n \to f$ in the sense of distribution.

(8) Let $f_n, f \in C(\Omega)$ such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that $f_n \to f$ in the sense of distribution.

(9) Let $f_n(x) := e^{inx}$, $x \in \mathbb{R}$. Show that (u_{f_n}) converges to the zero distribution.

(10) Making use of necessary results, prove that for $f, g \in L^1_{\mathrm{loc}}(\Omega)$, $u_f = u_g$ implies $f = g$ a,e.

(11) Let u be a linear functional on $\mathcal{D}(\Omega)$. Prove that u is a distribution if and only if for each compact $K \subseteq \Omega$, there exists a constant $C > 0$ and an $N \in \mathbb{N}_0$ such that
$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty \tag{1}$$
for all $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq K$.

(12) Define $u(\varphi) := \sum_{j=0}^\infty \varphi^{(j)}(j), \quad \varphi \in \mathcal{D}(\mathbb{R})$. Show that $u \in \mathcal{D}'(\mathbb{R})$, and it is of infinite order.

(13) Prove that
   (a) $\mathrm{supp}(\delta_a) = \{a\}$.
   (b) For $f \in L^1_{\mathrm{loc}}(\Omega)$, $\mathrm{supp}(u_f) = \mathrm{supp}(f)$.
   (c) For $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, $\mathrm{supp}(u) \cap \mathrm{supp}(f) = \emptyset \implies u(\varphi) = 0$.

(14) If $f \in C^\infty(\Omega)$, then prove that $f\varphi \in \mathcal{D}(\Omega)$ for every $\varphi \in \mathcal{D}(\Omega)$.

(15) For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, prove that the map $\varphi \mapsto u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega)$, is a distribution.

(16) If $f \in C^\infty(\Omega)$ and $a \in \Omega$, show that $f\delta_a = f(a)\delta$.

(17) For $f, g \in L^1_{\rm loc}(\Omega)$, show that $fu_g = u_{fg}$.

(18) Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Prove that $\operatorname{supp}(fu) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(u)$.

(19) If u is a distribution with compact support, then prove that for any $f \in \mathcal{E}(\Omega)$, fu is also of compact support.

(20) If $u \in \mathcal{D}'(\Omega)$ is with compact support, then prove that $u \in \mathcal{E}'(\Omega)$ in the sense that for every $u \in \mathcal{D}'(\Omega)$, there exists a unique $\tilde{u} \in \mathcal{D}'(\Omega)$ such that $u|_{\mathcal{D}(\Omega)} = \tilde{u}$.

(21) If $u \in \mathcal{E}'(\Omega)$, then prove that $u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$ is with compact support.

(22) Prove that $\tau_h\delta_a = \delta_{a+h}$. (Recall: For $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, the distribution $\tau_h u$ is defined by $(\tau_h u)(\varphi) := u(\tau_{-h}\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

(23) For each $h \in \mathbb{R}^d$, show that the map $u \mapsto \tau_h u$ is continuous on $\mathcal{D}'(\mathbb{R}^d)$ in the sense that $u_n \to u$ in $\mathcal{D}'(\mathbb{R}^d)$ implies $\tau_h u_n \to \tau_h u$ in $\mathcal{D}'(\mathbb{R}^d)$.

(24) For $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, show that the map $\partial^\alpha u : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by $(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{D}(O)$, is a distribution.

(25) Let H be the *Heaviside function*, i.e., $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$ Show that $H' = \delta_0$.

(26) For $\alpha \in \mathbb{N}_0^d$, $x_0 \in \Omega$, prove that u defined by $u(\varphi) = (\partial^\alpha \varphi)(x_0)$ defines a distribution of order α.

(27) Let (x_n) be a sequence in Ω without a limit point in Ω and $(\alpha^{(n)})$ be a sequence in $\mathbb{N}_0^d$. Let $u(\varphi) := \sum_{n=1}^\infty \partial^{\alpha_n}\varphi(x_n)$. Prove that u is a distribution, and it has finite order if and only if $\sup |\alpha^{(n)}| < \infty$ and in that case the order is $\sup |\alpha^{(n)}|$.

(28) If $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp}(u) \cap \operatorname{supp}(\varphi) = \emptyset$, then prove that $u(\varphi) = 0$.

(29) Suppose u is a linear functional on $\mathcal{E}(\Omega)$ such that there exists compact $K \subseteq \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ satisfying

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{\infty,K} \quad \forall\, \varphi \in \mathcal{E}'(\Omega).$$

Prove that $u \in \mathcal{E}'(\Omega)$.

(30) Suppose $u \in \mathcal{E}'(\Omega)$ and there exists compact $K \subseteq \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ satisfying

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{\infty,K} \quad \forall\, \varphi \in \mathcal{E}'(\Omega).$$

Prove that $u|_{\mathcal{D}(\Omega)}$ is a distribution with compact support.

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