Lectures On Fourier Series

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1 Introduction

With the invention of calculus by Newton (1642–1727) and Leibnitz (1646–1716), there was a surge of activity in various topics of mathematical physics, notably in the study of boundary value problems associated to vibrations of strings stretched between points and vibration of bars or columns of air associated with mathematical theories of musical vibrations. Early contributors to the theory of vibrating strings include B. Taylor (1685–1731), D. Bernoulli (1700–1782), L. Euler (1707–1783) and d'Alembert (1717–1783).

By the middle of the eighteenth century, d'Alembert, Bernoulli and Euler had advanced the theory of vibrating strings to the stage where the partial differential eqaution (now called the *wave equation*)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

was known and a solution of the boundary value problem had been found from the general solution of that equation. The concept of fundamental modes of vibration led them to the notion of superposition of solutions and Bernoulli proposed a solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \cos \frac{n\pi a t}{c} \tag{1.1}$$

(where c is the length of the string). The initial position of the string $f(x)$ will then be represented by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \qquad (0 \le x \le c). \tag{1.2}$$

Later, Euler gave the formulas for the coefficients b_n.

But the general concept of a function had not been clarified and a lengthy controversy took place over the question of representing arbitrary functions on a finite interval by a series of sine functions.

The French mathematician d'Alembert gave an elegant solution in the form

$$y(x,t) = v(at + x) - v(at - x), \tag{1.3}$$

and he believed that he had solved the problem completely.

At that time, the word *function* had a very restricted meaning and was understood as something given by an analytic expression. Euler thought that the initial position of a plucked string need not always be a function, but some form where different parts could be expressed by different functions. In other words, *function* and *graph* meant different things. For every function, we can draw its graph but every graph that can be drawn need not come from a function.

Euler strongly objected to Bernoulli's claim that every solution to the problem of a plucked string could be represented in the form (1.1), on two counts. First of all the right-hand side of (1.1) was a periodic function, while the left-hand side was arbitrary. Further, the right-hand side of (1.2) was an analytic formula and hence a function, while the left-hand side, f, could be any graph. So Euler believed that d'Alembert's solution was valid when f was any graph while Bernoulli's solution was applicable only to a very restricted class of functions.

J. B. Fourier (1768–1830) presented many instructive examples of expansions of functions in trigonometric series in connection with boundary value problems associated to the conduction of heat. His book "*Théorie Analytique de la Chaleur*" (1822) is a classic. Fourier never justified the convergence of his series expansions and this was objected to by his contemporaries Lagrange, Legendre and Laplace. Fourier asserted that any periodic function could be written as a trigonometric series.

Dirichlet (1805–1859) firmly established in 1829 (nearly seventy years after the controversy started), sufficient conditions on a function f so that its Fourier series converges to its value at a point.

Since then a lot of ideas and theories grew out of a need to understand what these series meant. Amongst them are Cantor's theory of infinite sets, the rigorous notion of a function, the theories of integration due to Riemann and Lebesgue and the theories of summability of series.

In mathematical analysis, we always try to find approximation of objects by simpler objects. For example, we approximate real numbers by rationals. By truncating the Taylor series of a function, we approximate the function by a polynomial. However, for a function to admit a Taylor series, it has to be infinitely differentiable (but this is not sufficient!) in some interval but this is quite restrictive. Indeed Weierstrass' approximation theorem states that *any* continuous function defined on a finite closed interval can be approximated *uniformly* by a polynomial.

Now, consider the set of all functions $\{1\} \cup \{\cos nt, \sin nt \mid n \in \mathbb{N}\}$ on

$[-\pi, \pi]$. Given any point $t \in [-\pi, \pi]$, the constant function does not vanish at t. Further if t_1 and t_2 are distinct points in $[-\pi, \pi]$ then we can always find a function in the above set such that it takes different values at t_1 and t_2. Thus, the set of all trigonometric polynomials, *viz.* functions of the form

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt) \tag{1.4}$$

form an algebra (*i.e.* the set is closed under pointwise addition, multiplication and scalar multiplication) and it does not 'vanish at any point' and 'separates points'. By the Stone-Weierstrass theorem (which generalizes the Weierstrass' approximation theorem), every periodic continuous function on $[-\pi, \pi]$ can be approximated uniformly by trigonometric polynomials.

A trigonometric polynomial of the form (1.4) can also be written in exponential form:

$$f(x) = \sum_{n=-N}^{N} c_n \exp(inx). \tag{1.5}$$

It is easy to see that $a_0 = c_0$, $b_0 = 0$ and that

$$a_n = c_n + c_{-n}; \quad b_n = i(c_n - c_{-n})$$

or, equivalently,

$$c_n = \frac{a_n - ib_n}{2}; \quad c_{-n} = \frac{a_n + ib_n}{2}.$$

If n is a non-zero integer, then $\exp(inx)$ is the derivative of $\exp(inx)/in$, which also has period 2π. Thus

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(inx)\, dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases} \tag{1.6}$$

Mulitplying (1.5) by $\exp(-imx)$ and integrating over $[-\pi, \pi]$, we get, in view of (1.6),

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \exp(-imx)\, dx. \tag{1.7}$$

This gives us, for any positive integer m,

$$\left.\begin{aligned} a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\, dx. \end{aligned}\right\} \tag{1.8}$$

It is common practice to replace a_0 by $a_0/2$, so that (1.8) is valid for a_0 as well.

We now generalize this to define the trigonometric series

$$\sum_{n=-\infty}^{\infty} \exp(inx),$$

or, equivalently,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Given a 2π-periodic function f on $[-\pi, \pi]$, we define a_n $(n \geq 0)$ and b_n $(n \geq 1)$ by (1.8) and the resulting series is called the **Fourier series** of the function f. The a_n and b_n are called the **Fourier coefficients** of f.

The basic question now is when does the Fourier series of a function converge? If it converges, does it converge to the value of f at the given point? In other words, to what extent does the Fourier series of a function represent the function itself?

In the sequel we will try and answer some of these questions.

To begin with, we will look at an abstract situation suggested by the relations (1.6).

2 Orthonormal Sets

Let H be a Hilbert Space (over $\mathbb{R}$ or $\mathbb{C}$). We denote the inner product in H by $(\cdot, \cdot)$ and the norm it generates by $\|\cdot\|$.

Definition 2.1. *Let $S = \{u_i \mid i \in I\}$, where I is an indexing set, be a collection of elements in H. The set S is said to be* orthonormal *if*

$$\left.\begin{aligned} \|u_i\| &= 1 \textit{ for all } i \in I \\ (u_i, u_j) &= 0 \textit{ for all } i \neq j, \quad i, j \in I. \end{aligned}\right\} \tag{2.1}$$

Example 2.1. The standard basis $\{e_i\}_{1 \leq i \leq n}$ in $\mathbb{R}^n$, where e_i has 1 in the i^{th}-coordinate and zero elsewhere, is orthonormal in $\mathbb{R}^n$ with the usual inner-product and the euclidean norm.

Example 2.2. Consider the space of square summable sequences ℓ_2, *i.e.*

$$\ell_2 = \left\{ x = (x_i) \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}.$$

Then again, the set of sequences $\{e_n\}_{n=1}^{\infty}$ where e_n has 1 as the n^{th} entry and zero at all other places, is orthonormal in ℓ_2.

Example 2.3. The sequence $\{\sqrt{2}\sin n\pi x\}$ is orthonormal in $L^2(0,1)$.

Example 2.4. The sequence $\{\frac{1}{\sqrt{2\pi}}\} \cup \{\frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} \mid n \in \mathbb{N}\}$ is orthonormal in $L^2(-\pi,\pi)$.

Proposition 2.1. *Let H be a separable Hilbert space. Then any orthonormal set is at most countable.*

Proof. Let $\{x_n\}$ be a countable dense set in H. If u and v are elements in an orthonormal set, we have $\|u - v\| = \sqrt{2}$. Thus each of the balls $B_n = B(x_n; \sqrt{2}/4)$ can contain at most one element of an orthonormal set. Since the $\{x_n\}$ form a dense set, every member of H must belong to one such ball. Hence the result. □

Henceforth, we will assume that H is a separable Hilbert space over $\mathbb{R}$.

Proposition 2.2. *Let $\{e_1, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H. Then, for any $x \in H$, we have*

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2. \tag{2.2}$$

Proof. We have $\|x - \sum_{i=1}^{n}(x,e_i)e_i\|^2 \geq 0$. Expanding this, we get, using the fact that the $\{e_i\}$ are orthonormal,

$$\|x\|^2 + \sum_{i=1}^{n} |(x, e_i)|^2 - 2\sum_{i=1}^{n} |(x, e_i)|^2 \geq 0$$

which proves (2.2). □

Theorem 2.1 (Bessel's Inequality). *If $\{e_i\}$ is an orthonormal set in a Hilbert Space H, then*

$$\sum_{i} |(x, e_i)|^2 \leq \|x\|^2. \tag{2.3}$$

Proof. Since H is separable, $\{e_i\}$ is at most countable. The result, in the finite case, has already been shown. If $\{e_i\}$ is countably infinite, then for each n, we have that (2.2) is valid. Thus, since the result is true for all partial sums, it is true for the series as well and we get (2.3). □

Corollary 2.1. *If $\{e_n\}$ is an orthonormal sequence in H, then for every $x \in H$, $(x, e_n) \to 0$ as $n \to \infty$.* □

It is immediate to see that the elements of an orthonormal set are linearly independent. Further, given a set of linearly independent elements $\{x_1, \dots, x_n\}$ in H, we can produce an orthonormal set $\{e_1, \dots, e_n\}$ such that the linear spans of $\{x_1, \dots, x_k\}$ and $\{e_1, \dots, e_k\}$ coincide for all $1 \leq k \leq n$. Indeed, set $e_1 = x_1/\|x_1\|$. Define

$$e_2 = \frac{x_2 - (x_2, e_1)e_1}{\|x_1 - (x_2, e_1)e_1\|}.$$

(Notice, by the linear independence of x_1 and x_2, the vector $x_2 - (x_2, e_1)e_1$ cannot be zero.) It is easy to see that $\|e_2\| = 1$ and that $(e_1, e_2) = 0$. In general, assume that we have constructed $e_1, \dots, e_k$ such that

(i) each e_i $(1 \leq i \leq k)$ is a linear combination of $x_1, \dots, x_i$ and

(ii) the $\{e_i\}$ are orthonormal.

Now define

$$e_{k+1} = \frac{x_{k+1} - \sum_{i=1}^{k}(x_{k+1}, e_i)e_i}{\|x_{k+1} - \sum_{i=1}^{k}(x_{k+1}, e_i)e_i\|}.$$

Thus, inductively we obtain $\{e_1, \dots, e_n\}$. This procedure is called the Gram-Schmidt orthogonalization procedure.

Thus, if H is a finite dimensional space, we can construct an orthonormal basis for H.

Henceforth, we will assume that H is infinite dimensional and separable.

Definition 2.2. *An orthonormal set is* complete *if it is maximal with respect to the partial ordering on orthonormal sets in H induced by inclusion.*

Let $\{e_n\}$ be an orthonormal sequence in an infinite dimensional (separable) Hilbert space. Let $x \in H$. Define

$$y_n = \sum_{i=1}^{n}(x, e_n)e_n.$$

Then, for $m > n$,

$$\|y_n - y_m\|^2 = \sum_{i=n+1}^{m} |(x, e_i)|^2$$

and the right-hand side tends to zero, by Bessel's inequality. Thus, $\{y_n\}$ is a Cauchy sequence and so converges in H. We define the limit to be

$$\sum_{i=1}^{\infty}(x, e_i)e_i.$$

Proposition 2.3. *The vector $x - \sum_{j=1}^{\infty}(x, e_j)e_j$ is orthogonal to each e_i. Further,*

$$\|x - \sum_{i=1}^{\infty}(x, e_i)e_i\|^2 = \|x\|^2 - \sum_{i=1}^{\infty}|(x, e_i)|^2. \tag{2.4}$$

Proof. Given any n, set $y_n = \sum_{i=1}^{n}(x, e_j)e_j$. Then $y_n \to \sum_{i=1}^{\infty}(x, e_i)e_i$ in H. Now, if $1 \leq i \leq n$, clearly $(x - y_n, e_i) = 0$. Fix i, and the above relation holds for all $n \geq i$. Thus,

$$(x - \sum_{j}(x, e_j)e_j, e_i) = 0.$$

Now,

$$\begin{aligned} \|x - y_n\|^2 &= \|x\|^2 + \sum_{i=1}^{n}|(x, e_i)|^2 - 2\sum_{i=1}^{n}|(x, e_i)|^2 \\ &= \|x\|^2 - \sum_{i=1}^{n}|(x, e_i)|^2. \end{aligned}$$

Thus, passing to the limit as $n \to \infty$, we get (2.4). □

Theorem 2.2. *Let H be a Hilbert space and $\{e_i\}$ an orthonormal set in H. The following are equivalent.*

(i) *$\{e_i\}$ is complete.*

(ii) *If $x \in H$ such that $(x, e_i) = 0$ for all i, then $x = 0$.*

(iii) *If $x \in H$, then*

$$x = \sum_{j}(x, e_j)e_j. \tag{2.5}$$

(iv) **(Parseval's Identity)** *If $x \in H$, then*

$$\|x\|^2 = \sum_i |(x, e_i)|^2. \tag{2.6}$$

Proof. (i)⇒ (ii) If $(x, e_i) = 0$ for all i and $x \neq 0$, then $\{e_i\} \cup \left\{ \frac{x}{\|x\|} \right\}$ will be an orthonormal set contradicting the maximality of $\{e_i\}$.
(ii)⇒ (iii) By Proposition 2.3, $x - \sum_j (x, e_j) e_j$ is orthogonal to each e_i and so, by (ii) we get (2.5).
(iii)⇒ (iv) This is an immediate consequence of (2.4).
(iv)⇒ (i) If $\{e_i\}$ were not maximal, there exists $e \in H$ such that $\|e\| = 1$ and $(e, e_i) = 0$ for all i. This contradicts (2.6). □

Corollary 2.2. *An orthonormal set $\{e_i\}$ in a Hilbert space H is complete, if, and only if, the linear span of the $\{e_i\}$, i.e. the space of all (finite) linear combinations of the $\{e_i\}$, is dense in H.*

Proof. If $\{e_i\}$ is complete, then by (2.5) we get that each $x \in H$ is such that

$$x = \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i) e_i.$$

Thus the linear span of $\{e_i\}$ is dense in H. Conversely, if the linear span is dense, then, if x is orthogonal to all the e_i, it follows that $x = 0$. Thus $\{e_i\}$ is complete. □

Remark 2.1. In view of (2.5), a complete orthonormal set is also called an *orthonormal basis.*

Example 2.5. Consider the sequence $\{e_n\}$ in ℓ_2 (cf. Example 2.2). This sequence is complete in ℓ_2 since

$$\|x\|^2 = \sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} |(x, e_i)|^2.$$

Example 2.6. Consider the Hilbert space $L^2(-\pi, \pi)$. It is known that continuous functions with compact support are dense in this space. Such functions are periodic (they vanish at $-\pi$ and π) and we saw in §1 that, by virtue of the Stone-Weierstrass theorem, they can be uniformly approximated by trigonometric polynomials. It then follows, *a fortiori*, that they can also be approximated in the L^2-norm. Thus, the trignometric polynomials are dense in $L^2(-\pi,\pi)$ and so, by the preceding corollary, the functions

$$\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}} \mid n \in \mathbb{N}\right\}$$

form a complete orthonormal set in $L^2(-\pi,\pi)$. In particular, if $f \in L^2(-\pi,\pi)$, we have, by Parseval's identity,

$$\begin{aligned}\int_{-\pi}^{\pi} |f|^2\,dx &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(t)\,dt\right)^2 + \\ &\quad + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(t)\cos nt\,dt\right)^2 + \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(t)\sin nt\,dt\right)^2\right]\end{aligned}$$

which yields

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right) \tag{2.7}$$

where a_n, b_n are the Fourier coefficients of f given by (1.8). □

By analogy with the above example, if $\{e_i\}$ is an orthonormal basis for a separable Hilbert space H, we say that the *Fourier series* of x is the series $\sum_{i=1}^{\infty}(x,e_i)e_i$ and the quantities (x,e_i) are called the *Fourier coefficients*.

3 Variations on the Theme

Let f be a 2π-periodic function defined on $[-\pi,\pi]$. The Fourier coefficients of f are given by the formulas (1.8). These formulas also make sense if $f \in L^1(-\pi,\pi)$. By changing the values of a function at a finite number of points, we do not change the values of the Fourier coefficients. (Indeed, recall that the spaces L^p are only *equivalence classes* of functions, under the equivalence relation given by $f \sim g$ if $f = g$ a.e.; thus, it is meaningless to talk of the value of a function in L^p at a particular point). Given a function h such that $h(\pi) \neq h(-\pi)$, we can redefine it so that $h(\pi) = h(-\pi)$ and then extend it periodically as a 2π-periodic function over $\mathbb{R}$. We can also declare the function to be undefined at $-\pi$ or π. Thus we are forced to consider the Fourier series of functions with jump discontinuities.

If a function belongs to $L^1(0,\pi)$ then we can extend it either as an odd function or as an even function to $L^1(-\pi,\pi)$. In the former case, all the coefficients a_n will be zero and we get a series only involving the functions $\sin nt$. This is called the Fourier sine series of the given function. Similarly, in the latter case, only the coefficients a_n will be non-zero and the resulting series, which involves only the functions $\cos nt$, is called the Fourier cosine series of the function.

We can also rewrite the Fourier series of a function in the *amplitude-phase* form. Indeed, let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{3.1}$$

be the Fourier series of a function. Set

$$a_n = d_n \cos\phi_n, \quad b_n = d_n \sin\phi_n.$$

In other words,

$$d_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = \cos^{-1}(a_n/d_n).$$

Then the series (3.1) can be rewritten as

$$\frac{a_0}{2} + \sum_n d_n \cos(nt - \phi_n). \tag{3.2}$$

4 The Riemann-Lebesgue Lemma

We have seen earlier that if $\{e_n\}$ were an orthonormal sequence in a Hilbert space H, then, for any $x \in H$, we have

$$\lim_{n\to\infty}(x, e_n) = 0.$$

In particular, if $f \in L^2(-\pi,\pi)$ we deduce that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = 0.$$

This result is one of the forms of what is called the *Riemann-Lebesgue lemma.* We now prove a very useful generalization of this.

Theorem 4.1 (Generalized Riemann-Lebesgue Lemma). *Let $f \in L^1(a,b)$, where $-\infty \leq a < b \leq +\infty$. Let h be a bounded measurable function defined on $\mathbb{R}$, such that*

$$\lim_{c\to\pm\infty} \frac{1}{c}\int_0^c h(t)\,dt = 0. \tag{4.1}$$

Then

$$\lim_{\omega\to\infty} \int_a^b f(t)h(\omega t)\,dt = 0. \tag{4.2}$$

Proof. We extend f by zero outside (a,b) so that we can consider $f \in L^1(\mathbb{R})$. Let us consider an interval $[c,d] \subset (0,\infty)$. Then

$$\begin{aligned}\int_0^\infty \chi_{[c,d]}h(\omega t)\,dt &= \int_c^d h(\omega t)\,dt = \frac{1}{\omega}\int_{c\omega}^{d\omega} h(t)\,dt \\ &= \frac{1}{\omega}\int_0^{d\omega} h(t)\,dt - \frac{1}{\omega}\int_0^{c\omega} h(t)\,dt\end{aligned}$$

and, by hypothesis, both integrals tend to zero as $\omega \to \infty$. The result now follows, by linearity, to all step functions made up of characteristic functions of intervals. However such functions are dense in $L^1(0,\infty)$. (Indeed continuous functions with compact support are dense in $L^1(0,\infty)$; such functions are uniformly continuous and by partitioning the interval containing the support, we can approximate continuous functions with compact support by step functions uniformly and hence, *a fortiori*, in $L^1(0,\infty)$.)

Let $|h| \leq M$. Thus, if $f \in L^1(0,\infty)$, then find a step function g such that $\int_0^\infty |f-g|\,dx < \frac{\varepsilon}{2M}$, for a given $\varepsilon > 0$. Now,

$$\begin{aligned}\left|\int_0^\infty f(t)h(\omega t)\,dt\right| &\leq \int_0^\infty |f(t)-g(t)|h(\omega t)\,dt + \left|\int_0^\infty g(t)h(\omega t)\,dt\right| \\ &\leq \frac{\varepsilon}{2} + \left|\int_0^\infty g(t)h(\omega t)\,dt\right|\end{aligned}$$

and for ω large enough the second term can also be made to be less than $\frac{\varepsilon}{2}$. A similar argument holds for $\int_{-\infty}^0 f(t)h(\omega t)\,dt$ and this completes the proof. □

Corollary 4.1. *If $f \in L^1(a,b)$ then*

$$\lim_{n\to\infty}\int_a^b f(t)\cos nt\,dt = \lim_{n\to\infty}\int_a^b f(t)\sin nt\,dt = 0. \tag{4.3}$$

Proof.

$$\begin{aligned} \left|\frac{1}{c}\int_0^c \cos t\,dt\right| = \left|\frac{1}{c}(\sin c)\right| &\leq \frac{1}{|c|} \\ \left|\frac{1}{c}\int_0^c \sin t\,dt\right| = \left|\frac{1}{c}(1-\cos c)\right| &\leq \frac{2}{|c|} \end{aligned}$$

and both tend to zero as $|c| \to \infty$. □

We now give an immediate application of this result. Given a convergent series $\sum_{n=1}^{\infty} \alpha_n$, we know that $\alpha_n \to 0$ as $n \to \infty$. We now ask if a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

is convergent, whether $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

Theorem 4.2 (Cantor-Lebesgue Theorem). *If a trigonometric series* $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$ *converges on a set* E *whose (Lebesgue) measure is positive, then* $a_n \to 0$ *and* $b_n \to 0$.

Proof. Without loss of generality, we may assume that E has finite measure. We rewrite the trigonometric series as in (3.2) where $d_n = \sqrt{a_n^2 + b_n^2}$ and $\phi_n = \cos^{-1}(a_n/d_n)$. Since the series converges, it follows that for all $t \in E$,

$$d_n \cos(nt - \phi_n) \to 0$$

as $n \to \infty$. Assume that $\{d_n\}$ does not converge to zero. Then, there exists $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that

$$d_{n_k} \geq \varepsilon > 0,$$

for all k. Then, it follows that $\cos(n_k t - \phi_{n_k}) \to 0$ as $k \to \infty$ for all $t \in E$. Since E is of finite measure, it follows, from the dominated convergence theorem, that

$$\int_E \cos^2(n_k t - \phi_{n_k})\,dt \to 0. \tag{4.4}$$

Now $\cos^2(n_k t - \phi_{n_k}) = \frac{1}{2}\left[1 + \cos 2(n_k t - \phi_{n_k})\right]$. But then $\chi_E \in L^1(\mathbb{R})$ and so

$$\begin{aligned}\int_E \cos 2(n_k t - \phi_{n_k})\, dt &= \int_{\mathbb{R}} \chi_E(t) \cos 2(n_k t - \phi_{n_k})\, dt \\ &= \cos 2\phi_{n_k} \int_{\mathbb{R}} \chi_E(t) \cos 2n_k t\, dt + \\ &\quad + \sin 2\phi_{n_k} \int_{\mathbb{R}} \chi_E(t) \sin 2n_k t\, dt\end{aligned}$$

and both the integrals on the right-hand side tend to zero by Corollary 4.1. It then follows that

$$\int_E \cos^2(n_k t - \phi_{n_k})\, dt \to \frac{\mu(E)}{2} > 0$$

which contradicts (4.4). This shows that $d_n \to 0$ and so $a_n \to 0$ and $b_n \to 0$. $\square$

5 The Dirichlet, Fourier and Fejér Kernels

Let $f \in L^1(-\pi, \pi)$ with Fourier series given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{5.1}$$

where a_n, b_n are the Fourier coefficients defined as in (1.8). We wish to examine the convergence of this series. In particular, we will be interested, in the sequel, to answers (positive or negative) to the following questions.

- Does the Fourier series converge at all points $t \in [-\pi, \pi]$?
- If it converges at $t \in [-\pi, \pi]$, does it converge to $f(t)$?
- If it converges at all $t \in [-\pi, \pi]$, is the convergence uniform?

To discuss the convergence, pointwise or uniform, of the Fourier series, we need to discuss the convergence of the sequence $\{s_n\}$ of partial sums. We have

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt). \tag{5.2}$$

Equivalently, in exponential form,

$$s_n(t) = \sum_{k=-n}^{n} c_k \exp(ikt).$$

Using the expression for the c_k (cf. (1.7)) we get

$$\begin{aligned} s_n(t) &= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \exp(ik(t-x))\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(t-x)\, dx \end{aligned}$$

where

$$D_n(t) = \sum_{k=-n}^{n} \exp(ikt). \tag{5.3}$$

The 2π-periodic function $D_n(t)$ is called the **Dirichlet Kernel**.

Proposition 5.1. *Let $\{s_n\}$ be the sequence of partial sums of the Fourier series of $f \in L^1(-\pi, \pi)$ which is 2π-periodic. Then*

$$\begin{aligned} s_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(t-x)\, dx && \text{(5.4a)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x) D_n(x)\, dx && \text{(5.4b)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(x-t)\, dx && \text{(5.4c)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) D_n(x)\, dx && \text{(5.4d)} \\ &= \frac{1}{2\pi} \int_{0}^{\pi} (f(t+x) + f(t-x)) D_n(x)\, dx && \text{(5.4e)} \end{aligned}$$

Proof. We have already established (5.4a). By a change of variable $t - x = y$ we get

$$s_n(t) = \frac{1}{2\pi} \int_{-\pi+t}^{\pi+t} f(t-y) D_n(y)\, dy.$$

By the 2π-periodicity, it follows that the integral does not change as long as the length of the interval of integration is 2π. This proves (5.4b). The relation

(5.4c) follows from (5.4a) since D_n is easily seen to be an even function. Relation (5.4d) follows from (5.4c), again by a change of variable $y = x - t$ and the fact that the integrals do not change as long as the length of the interval is 2π. Finally, we split the integral in (5.4d) as the sum of integrals over $[-\pi, 0]$ and $[0, \pi]$. Now,

$$\int_{-\pi}^{0} f(t+x)D_n(x)\,dx = \int_{0}^{\pi} f(t-y)D_n(y)\,dy$$

using the change of variable $y = -x$ and the evenness of D_n. This proves (5.4e). □

Proposition 5.2. *Let $n \geq 0$ be an integer. Then*

$$D_n(t) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} & t \neq 2k\pi, \quad k \in \mathbb{N} \cup \{0\}. \\ 2n+1 & t = 2k\pi, \quad k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{5.5}$$

Further,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1. \tag{5.6}$$

Proof. When $n = 2k\pi$, clearly $D_n(t) = 2n+1$. Assume $n \neq 2k\pi$, $k \in \mathbb{N} \cup \{0\}$. Then

$$(\exp(it) - 1)D_n(t) = \exp(i(n+1)t) - \exp(-int)$$

Multiplying both sides by $\exp(-it/2)$ we immediately deduce (5.5). The relation (5.6) immediately follows from the definition of $D_n(t)$ (cf. (5.3)) and the relations (1.6). □

We now introduce two other functions which, together with the Dirichlet kernel, will play an important role in the study of the convergence of Fourier series.

Definition 5.1. *The* **continuous Fourier kernel** *is given by*

$$\Phi(\omega, t) = \begin{cases} \dfrac{\sin \omega t}{t}, & t \neq 0 \\ \omega, & t = 0 \end{cases} \tag{5.7}$$

where ω and t are real numbers. The associated **discrete Fourier kernel** *is given by*

$$\Phi_n(t) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})t}{t} & t \neq 0 \\ n + \frac{1}{2} & t = 0, \end{cases} \tag{5.8}$$

where $t \in \mathbb{R}$ and n is a non-negative integer.

Remark 5.1. Clearly D_n is 2π-periodic. It is easy to see that $\frac{\sin(n+1/2)t}{\sin(t/2)} \to 2n+1$ as $t \to \infty$. Thus D_n is continuous. Similarly, it is easy to see that the continuous and discrete Fourier kernels are also continuous.

Definition 5.2. *The* **Fejér kernel** *is defined by*

$$K_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t). \tag{5.9}$$

Proposition 5.3. *Let $n \geq 0$ be an integer. Then*

$$K_n(t) = \frac{1}{n+1} \frac{1 - \cos(n+1)t}{1 - \cos t} \tag{5.10}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t)\, dt = 1. \tag{5.11}$$

Further, $K_n \geq 0$ and if $0 < \delta \leq |t| \leq \pi$, we have

$$K_n(t) \leq \frac{2}{(n+1)(1-\cos\delta)}. \tag{5.12}$$

Proof. As before, observe that

$$\begin{aligned}(n+1)K_n(t)\big(\exp(it)-1\big)\big(\exp(-it)-1\big) &= \big(\exp(-it)-1\big) \\ &\quad \cdot \sum_{k=0}^{m} \Big(\exp\big(i(k+1)t\big) - \exp(-ikt)\Big) \\ &= 2 - \exp\big(i(n+1)t\big) - \\ &\quad - \exp\big(-i(n+1)t\big)\end{aligned}$$

from which we deduce (5.10). The relation (5.11) follows directly from the definition (cf. (5.9)) and the relation (5.6). That K_n is non-negative follows immediately from (5.10). So does relation (5.12). ∎

We now derive some estimates for integrals of the Dirichlet and Fourier kernels. First we need a technical result.

Lemma 5.1. *Let* $\{A_k\}$ *be a sequence of real numbers such that* $A_{2k-1} > 0$ *and* $A_{2k} < 0$ *and such that* $|A_{k+1}| < |A_k|$ *for all* $k \in \mathbb{N}$. *Then, for every* $k \in \mathbb{N}$, *we have*

$$0 < A_1 + \ldots + A_k < A_1. \tag{5.13}$$

Proof. Let k be odd. Then

$$A_1 + (A_2 + A_3) + (A_4 + A_5) + \ldots + (A_{k-1} + A_k)$$

is such that each term in parentheses is negative. Thus the sum is less than A_1. Again

$$(A_1 + A_2) + (A_3 + A_4) + \ldots + (A_{k-2} + A_{k-1}) + A_k$$

is such that each term in parentheses is greater than 0 and $A_k > 0$. Thus the sum is greater than 0. This proves (5.13) in the case k is odd. The proof when k is even is similar. ☐

Proposition 5.4. *Let* $0 \le a < b \le \pi$. *Let* $n \ge 0$ *be an integer. Then*

$$\left| \int_a^b \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt \right| \le 4\pi. \tag{5.14}$$

Proof. Let

$$A_k = \int_{(k-1)\pi/(n+1/2)}^{k\pi/(n+1/2)} \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt, \quad 1 \le k \le n+1.$$

Then in each such interval, the numerator of the integrand varies like $\sin t$ between $(k-1)\pi$ and $k\pi$. On the other hand $\sin(t/2)$ is positive and increases. Thus clearly $A_1 > 0$ and A_k alternates in sign and decreases in absolute value. Now let $a \in \left[\frac{(k-1)\pi}{n+1/2}, \frac{k\pi}{n+1/2}\right)$ or $a \in \left[\frac{n\pi}{n+1/2}, \pi\right]$.

If a is in the interior of any of these intervals (or, if $a \in \left(\frac{n\pi}{n+1/2}, \pi\right]$, when $k = n+1$), we have

$$\int_{(k-1)\pi/(n+1/2)}^{a} \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt$$

is either greater than 0 (because the integrand is greater than 0, when k is odd) or is less than 0 (k even) and

$$\left|\int_{(k-1)\pi/(n+1/2)}^{a} \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt\right| < |A_k|.$$

Thus

$$\int_0^a \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt = A_1 + \ldots + A_{k-1} + \int_{(k-1)\pi/(n+1/2)}^{a} \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt$$

and it follows that

$$0 < \int_0^a \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt < A_1.$$

Now, in the interval $[0, \pi/(n+1/2)]$, the integrand is positive and decreasing with maximum value $(2n+1)$ at $t = 0$. Thus

$$A_1 < (2n+1)\frac{\pi}{(n+1/2)} = 2\pi.$$

Now

$$\begin{aligned}\left|\int_a^b \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt\right| &= \left|\int_0^b \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt - \int_0^a \frac{\sin(n+1/2)t}{\sin(t/2)}\, dt\right| \\ &\leq 4\pi\end{aligned}$$

□

Proposition 5.5. *Let* $0 \leq a < b$. *Then*

$$\left|\int_a^b \frac{\sin \omega t}{t}\, dt\right| \leq 2\pi \tag{5.15}$$

for $\omega > 0$.

Proof. Define

$$A_k = \int_{(k-1)\pi/\omega}^{k\pi/\omega} \frac{\sin \omega t}{t}\, dt = \int_{(k-1)\pi}^{k\pi} \frac{\sin t}{t}\, dt.$$

Then, again, $\{A_k\}$ has alternating signs and decreases in absolute value. As in the previous lemma, we get

$$0 < \int_0^a \frac{\sin \omega t}{t} dt < A_1.$$

Since $\left|\frac{\sin t}{t}\right| \leq 1$, we get $A_1 \leq \pi$ and so

$$\left|\int_a^b \frac{\sin \omega t}{t}\, dt\right| = \left|\int_0^b \frac{\sin \omega t}{t}\, dt - \int_0^a \frac{\sin \omega t}{t}\, dt\right| \leq 2\pi.$$

□

The discrete Fourier kernel is a very good approximation of the Dirichlet kernel.

Proposition 5.6. *Let $f \in L^1(0,\pi)$. Let $0 < r \leq \pi$. Then*

$$\lim_{n\to\infty} \int_0^r f(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\, dt = \lim_{n\to\infty} \int_0^r f(t)\frac{\sin(n+\frac{1}{2})t}{\frac{t}{2}}\, dt \tag{5.16}$$

whenever either limit exists.

Proof. By L'Hospital's rule, we get

$$\lim_{t\to 0}\left(\frac{1}{\sin t} - \frac{1}{t}\right) = 0.$$

Define

$$g(t) = \begin{cases} \dfrac{1}{\sin(t/2)} - \dfrac{1}{t/2} & t \neq 0 \\ 0 & t = 0. \end{cases}$$

Then g is continuous. Thus it is bounded in $[0,r]$ for $0 < r \leq \pi$ and the function $f(t)g(t)\chi_{[0,r]}(t)$ is integrable. So are the functions

$$\phi(t) = f(t)g(t)\cos(t/2)\chi_{[0,r]}(t)$$

and

$$\psi(t) = f(t)g(t)\sin(t/2)\chi_{[0,r]}(t).$$

Thus,

$$\int_0^r f(t)\sin(n+\frac{1}{2})t\left[\frac{1}{\sin(t/2)} - \frac{1}{t/2}\right] dt = \int_0^\pi \phi(t)\sin nt\, dt + \int_0^\pi \psi(t)\cos nt\, dt$$

and, by the Riemann-Lebesgue lemma (cf. Corollary 4.1), both the integrals on the right-hand side tend to zero as $n \to \infty$ and the result follows. □

Both the functions $D_n(t)$ and $\Phi_n(t)$ enjoy the *Riemann-Lebesgue property*. More precisely, we have the following result.

Proposition 5.7. *Let $f \in L^1(0,\pi)$ and let $0 < r \leq \pi$. Then*

$$\lim_{n\to\infty} \int_r^\pi f(t)D_n(t)\,dt = \lim_{r\to\infty} \int_r^\pi f(t)\Phi_n(t)\,dt = 0. \tag{5.17}$$

Proof. Since $r > 0$, the functions

$$\begin{aligned} \phi(t) &= f(t)\sin(t/2)g(t)\chi_{[r,\pi]}(t) \\ \psi(t) &= f(t)\cos(t/2)g(t)\chi_{[r,\pi]}(t) \end{aligned}$$

where $g(t) = \frac{1}{\sin(t/2)}$ or $\frac{1}{t/2}$, are integrable and the result follows, once again, from the Riemann-Lebesgue lemma (Corollary 4.1). □

Theorem 5.1 (The localisation principle). *Let $f \in L^1(-\pi,\pi)$. Let $0 < r < \pi$. Then, for $x \in [-\pi,\pi]$,*

$$\lim_{n\to\infty} \left(\int_{-\pi}^{-r} + \int_r^\pi \right) \big(f(x-t)D_n(t)\big)\,dt = 0.$$

Proof. Fix $x \in [-\pi,\pi]$. Then, by the preceding proposition, we have

$$\int_r^\pi f(x-t)D_n(t)\,dt \to 0 \text{ as } n \to \infty.$$

Now,

$$\int_{-\pi}^{-r} f(x-t)D_n(t)\,dt = \int_r^\pi f(x+t)D_n(t)\,dt$$

using the evenness of D_n and so once again this integral also tends to zero. This completes the proof. □

Note that the Fourier coefficients depend on the values of a function f throughout the interval $[-\pi,\pi]$. However, if f and g are in $L^1(-\pi,\pi)$ and for some $t \in [-\pi,\pi]$ and $r > 0$, we have $f \equiv g$ in $(t-r, t+r)$ it follows from the above theorem that the Fourier series of f will converge at t if, and only if, the Fourier series of g converges at t and in this case the sums of the Fourier series are the same. Thus the behaviour of the Fourier series at a point t depends only on the values of the function in a neighbourhood of t.

This is in strong contrast with the behaviour of power series. If two power series coincide in an open interval, then they are identical throughout their common domain of convergence.

6 Fourier Series of Continuous Functions

A basic question that can be asked is the following: does the Fourier series of a continuous 2π-periodic function, f, converge to $f(t)$ at every point $t \in [-\pi, \pi]$?

Unfortunately, the answer is 'No!' and we will study this now.

Proposition 6.1. *We have*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(t)|\, dt = +\infty. \tag{6.1}$$

Proof. For $t \in \mathbb{R}$, we have $|\sin t| \le |t|$ and so

$$\begin{aligned} \int_{-\pi}^{\pi} |D_n(t)|\, dt &\ge 4 \int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{t}\, dt \\ &= 4 \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin t|}{t}\, dt \\ &> 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}\, dt \\ &> 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{k\pi}\, dt \\ &= \frac{8}{\pi} \sum_{k=1}^{n} \frac{1}{k} \end{aligned}$$

from which (6.1) follows immediately. □

Proposition 6.2. *Let $V = C_{per}[-\pi, \pi]$, the space of continuous 2π-periodic functions with the usual sup-norm (denoted $\|\cdot\|_\infty$) and define $\phi_n : V \to \mathbb{R}$ by*

$$\phi_n(f) = s_n(f)(0)$$

where $s_n(f)$ is the n^{th}-partial sum of the Fourier series of f. Then ϕ_n is a continuous linear functional on V and

$$\|\phi_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt. \tag{6.2}$$

Proof. On one hand,

$$\phi_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(t)\, dt$$

(cf. (5.4d)). Thus,

$$|\phi_n(f)| \leq \|f\|_\infty \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt$$

and so

$$\|\phi_n\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt.$$

Now, let $E_n = \{t \in [-\pi, \pi] \mid D_n(t) \geq 0\}$. Define

$$f_m(t) = \frac{1 - m\, d(t, E_n)}{1 + m\, d(t, E_n)}$$

where $d(t, A) = \inf\{|t - s| \mid s \in A\}$ is the distance of t from a set A. Since $d(t, A)$ is a continuous function (in fact, $|d(t, A) - d(s, A)| \leq |t - s|$) $f_m \in C_{\text{per}}[-\pi, \pi]$, (it is periodic since D_n is even and so E_n is a symmetric set about the origin). Also $\|f_m\|_\infty \leq 1$ and $f_m(t) \to 1$ if $t \in E_n$ while $f_m(t) \to -1$ if $t \in E_n^c$. By the dominated convergence theorem, it now follows that

$$\phi_n(f_m) \to \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt$$

from which (6.2) follows. □

Let us now recall a few results from topology and functional analysis.

Theorem 6.1 (Baire). *If X is a complete metric space, the intersection of every countable collection of dense open sets of X is dense in X.* □

Equivalently, Baire's theorem also states that a complete metric space cannot be the countable union of nowhere dense sets.

One of the important consequences of Baire's theorem is the **Banach-Steinhaus theorem** also known as the **uniform boundedness principle**.

Theorem 6.2 (Banach-Steinhaus). *Let X be a Banach space and Y a normed linear space and $\{\Lambda_\alpha\}_{\alpha\in A}$ a collection of bounded linear transformations from X into Y, where α ranges over some indexing set A. Then either there exists $M > 0$ such that*

$$\|\Lambda_\alpha\| \leq M, \ \text{for all } \alpha \in A \tag{6.3}$$

or,

$$\sup_{\alpha\in A} \|\Lambda_\alpha x\| = \infty \tag{6.4}$$

for all x belonging to some dense G_δ-set in X. □

(Recall that a G_δ-set is a set which is the countable intersection of open sets).

Now consider $X = V$ as defined in Proposition 6.2 and $Y = \mathbb{R}$. Let $A = \mathbb{N}$ and set $\Lambda_n = \phi_n$, again defined in above mentioned proposition. Since, by Proposition 6.1 and 6.2, we have $\|\phi_n\| \to \infty$ as $n \to \infty$, it follows from the Banach-Steinhaus theorem, that there exists a dense G_δ-set (of continuous 2π-periodic functions) in V such that the Fourier series of all these functions diverge at $t = 0$. We could have very well dealt with any other point in the interval $[-\pi, \pi]$ in the same manner.

By another application of Baire's theorem, we can strengthen this further.

Let E_x be the dense G_δ-set of continuous 2π-periodic functions in V such that the Fourier series of these functions diverge at x. Let $\{x_i\}$ be a countable set of points in $[-\pi, \pi]$ and let

$$E = \cap_{i=1}^{n} E_{x_i} \subset V. \tag{6.5}$$

Then, by Baire's theorem E is also a dense G_δ-set. (Each E_{x_i} is the countable intersection of dense open sets and so, the same is true for E). Thus for each $f \in E$, the Fourier series of f diverges at x_i for all $1 \leq i \leq \infty$. Define

$$s^*(f; x) = \sup_n |s_n(f)(x)|.$$

Then $s^*(f, \cdot)$ is a lower semi-continuous function. Hence $\{x \mid s^*(f; x) = \infty\}$ is a G_δ-set in $(-\pi, \pi)$ for each f. If we choose the x_i above so that $\{x_i\}$ is dense (take all rationals, for instance in $(-\pi, \pi)$) then we have the following result.

Proposition 6.3. *The set $E \subset V$ is a dense G_δ-set such that for all $f \in E$, the set $Q_f \subset (-\pi, \pi)$ where its Fourier series diverges, is a dense G_δ-set in $(-\pi, \pi)$.* □

Proposition 6.4. *In a complete metric space, which has no isolated points, no countable dense set can be a G_δ.*

Proof. Let $E = \{x_1, \ldots, x_n, \ldots\}$ be a countable dense set. Assume E is a G_δ. Thus $E = \cap_{n=1}^{\infty} W_n$, W_n open and dense. Then, by hypothesis, $W_n \setminus \cup_{i=1}^{n} \{x_i\} = V_n$ is also open and dense. But $\cap_{n=1}^{\infty} V_n = \emptyset$, contradicting Baire's theorem. □

Thus, there exists uncountably many 2π-periodic continuous functions on $[-\pi, \pi]$ whose Fourier series diverge on a dense G_δ-set of $(-\pi, \pi)$.

Having answered our first general question negatively, let us now prove a positive result.

Proposition 6.5. *Let f be a 2π-periodic function on $[-\pi, \pi]$ which is uniformly Lipschitz continuous, i.e. there exists $K > 0$ such that*

$$|f(x) - f(y) \leq K|x - y|$$

for all x, y. Then, the Fourier series of f converges to f on $[-\pi, \pi]$.

Proof. Choose $0 < r < \pi$ such that

$$\frac{1}{\pi} \int_{-r}^{r} \frac{|t/2|}{|\sin(t/2)|} \, dt < \frac{\varepsilon}{2K}.$$

This is possible since $\left| \frac{t/2}{\sin(t/2)} \right|$ is a bounded continuous function, and hence integrable on $[-\pi, \pi]$. If C is an upper bound for this function, we need only choose r such that $2rC < \frac{\varepsilon \pi}{2K}$. Let $x \in [-\pi, \pi]$.

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) \, dt.$$

Thus

$$|s_n(x) - f(x)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \big(f(x-t) - f(x) \big) D_n(t) \, dt \right|.$$

If r is chosen as above, then, by the localization principle (Theorem 5.1), we have, for n large enough

$$\frac{1}{2\pi}\left|\left(\int_{-\pi}^{-r}+\int_{r}^{\pi}\right)\big(f(x-t)-f(x)\big)D_n(t)\,dt\right|<\frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-r}^{r}\big(f(x-t)-f(x)\big)D_n(t)\,dt\right| &\leq \frac{K}{2\pi}\int_{-r}^{r}\frac{|t||\sin(n+1/2)t|}{|\sin t/2|}\,dt\\ &\leq \frac{K}{\pi}\int_{-r}^{r}\frac{|t|/2}{|\sin(t/2)|}\,dt<\frac{\varepsilon}{2}.\end{aligned}$$

Thus for n large we have

$$|s_n(x)-f(x)|<\varepsilon$$

which completes the proof. □

Corollary 6.1. *If $f \in C^1[-\pi,\pi]$ is 2π-periodic, then the Fourier series of f converges to f on $[-\pi,\pi]$.* □

In a later section, we will relax these conditions on f and study the pointwise convergence of Fourier series.

Remark 6.1. The convergences in Proposition 6.5 and Corollary 6.1 above are, in fact, uniform over $[-\pi,\pi]$, cf. Exercise 32.

Remark 6.2. If f is Lipschitz continuous in a neighbourhood of $t \in [-\pi,\pi]$, then we can show, using identical arguments, that the Fourier series of f converges to $f(t)$ at t.

7 Fejér's Theorem

In the previous section, we saw that there exist an uncountable number of continuous 2π-periodic functions whose Fourier series diverge over a dense set of points. Nevertheless, as we have observed earlier, such functions can be approximated *uniformly* by means of trigonometric polynomials over $[-\pi,\pi]$, thanks to the Stone-Weierstrass theorem. Indeed, we can now prove the following result.

Theorem 7.1 (Fejér). *Let f be a continuous 2π-periodic function on $[-\pi, \pi]$. Let $\{s_n\}_{n\geq 0}$ be the sequence of partial sums of its Fourier series. Define*

$$\sigma_n(x) = \frac{s_0(x) + \ldots + s_n(x)}{n+1}.$$

Then $\sigma_n \to f$ uniformly over $[-\pi, \pi]$.

Proof. It is immediate, from the definition of the Fejér kernel K_n, to see that

$$\sigma_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)\,dt.$$

Thus

$$\sigma_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x-t) - f(x)\big)K_n(t)\,dt,$$

in view of (5.11). Now, since f is continuous on $[-\pi, \pi]$, it is bounded and uniformly continuous. Let $|f(x)| \leq M$ for all $x \in [-\pi, \pi]$ and, given $\varepsilon > 0$, let $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$.

Now, choose N large enough such that, for $n \geq N$, $K_n(t) \leq \varepsilon/4M$ for all $\delta < |t| \leq \pi$. This is possible because of the estimate (5.12). Thus, since $K_n \geq 0$,

$$\left|\int_{-\delta}^{\delta} \big(f(x-t) - f(x)\big)K_n(t)\,dt\right| \leq \frac{\varepsilon}{2}\int_{-\pi}^{\pi} K_n(t)\,dt = \pi\varepsilon,$$

again by (5.11). On the other hand,

$$\left|\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)\big(f(x-t) - f(x)\big)K_n(t)\,dt\right| \leq 2M \cdot \frac{\varepsilon}{4M} \cdot 2\pi = \pi\varepsilon$$

if $n \geq N$. It now follows that for all $x \in [-\pi, \pi]$, we have

$$|\sigma_n(x) - f(x)| < \varepsilon$$

for $n \geq N$ which completes the proof. □

Corollary 7.1. *Let f and g be two 2π-periodic and continuous functions on $[-\pi, \pi]$. If they have the same Fourier series, then $f \equiv g$.*

Proof. If the two functions have the same Fourier series, then the σ_n will be the same for both functions and we know that $\sigma_n \to f$ and $\sigma_n \to g$ uniformly on $[-\pi, \pi]$. Hence the result. □

Given a series $\sum_n a_n$, we say that it is **Cesàro summable** or $(C, 1)$ summable to a if $\sigma_n \to a$ as $n \to \infty$, where

$$\sigma_n = \frac{s_1 + \ldots + s_n}{n},$$

s_k being the partial sums of the series. Thus the Fourier series of a continuous 2π-periodic function is always Cesàro summable to the function.

Thus if f is a continuous 2π-periodic function whose Fourier series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

then f is uniformly approximated over $[-\pi, \pi]$ by the trigonometric polynomials

$$\sigma_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)\big(a_k \cos kt + b_k \sin kt\big). \tag{7.1}$$

Starting from this, we can deduce Weierstrass' approximation theorem. Indeed, let $f \in C[-1, 1]$. Define $g(t) = f(\cos t)$, for $t \in [-\pi, \pi]$. Then g is 2π-periodic and continuous. Further, g is an even function and hence its Fourier series will only consist of cosine terms. Let the Fourier series of g be given by

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt.$$

Then

$$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right) a_k \cos kt.$$

Thus, given $\varepsilon > 0$, there exists N such that, for all $n \geq N$,

$$\left| f(\cos t) - \frac{a_0}{2} - \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right) a_k \cos kt \right| < \varepsilon$$

for all $t \in [-\pi, \pi]$. This is the same as

$$\left| f(t) - \frac{a_0}{2} - \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right) a_k \cos\big(k(\cos^{-1} t)\big) \right| < \varepsilon$$

for all $t \in [-1, 1]$. Now it only remains to show that

$$P_k(t) = \cos(k \cos^{-1} t)$$

is a polynomial in t for every non-negative integer k. Indeed, $P_0(t) \equiv 1$ and $P_1(t) \equiv t$. Assume that $P_k(t)$ is a polynomial in t, of degree k, for every $1 \leq k \leq n-1$, $n \geq 2$. Then

$$\begin{aligned}
\cos ns &= \cos ns + \cos(n-2)s - \cos(n-2)s \\
&= \cos\big((n-1)s + s\big) + \cos\big((n-1)s - s\big) - \cos(n-2)s \\
&= 2\cos(n-1)s \, \cos s - \cos(n-2)s \\
&= 2\cos s \, P_{n-1}(\cos s) - P_{n-2}(\cos s).
\end{aligned}$$

It then follows that

$$P_n(t) = 2tP_{n-1}(t) - P_{n-2}(t) \tag{7.2}$$

and so $P_n(t)$ is a polynomial of degree n in t.

The polynomials $\{P_n\}$ defined recursively via (7.2) where $P_0 \equiv 1$ and $P_1(t) = t$, are called the **Chebychev polynomials** and play an important role in numerical analysis, especially in numerical quadrature.

8 Regularity

In this section, we will briefly discuss various regularity assumptions to be made on functions when discussing the pointwise convergence of Fourier series.

Let $f : [a, b] \to \mathbb{R}$ be a given function which is differentiable on (a, b). Assume that

$$f'(a+) = \lim_{t \to a} f'(t)$$

exists. Then f' is bounded in a subinterval $(a, a+\delta)$ (where $\delta > 0$) and so, by the mean-value theorem, f is uniformly continuous on $(a, a+\delta)$. Thus it can be continuously extended to $[a, a+\delta]$. Consequently $f(a+) = \lim_{t\to a} f(t)$, exists.

Again by the mean-value theorem, applied to $\tilde{f}$ where $\tilde{f}(a) = f(a+)$ and $\tilde{f}(t) = f(t)$ for $a < t \leq a + \delta$, we have

$$f(t) = f(a+) + f'(a + \theta(t-a))(t-a)$$

where $0 < \theta < 1$. Thus

$$\lim_{t \to a} \frac{f(t) - f(a+)}{t - a} = f'(a+). \tag{8.1}$$

Notice that $f'(a+)$ is different from the right-sided derivative of f at a (if it exists), which is given by

$$D^+ f(a) = \lim_{t \downarrow a} \frac{f(t) - f(a)}{t - a}.$$

We also define, $f'(b-) = \lim_{t \to b} f'(t)$.

Definition 8.1. *We say that a function $f : [a, b] \to \mathbb{R}$ is* **piecewise smooth** *if there exist a finite number of points*

$$a = a_0 < a_1 < \ldots < a_n = b$$

such that f is continuously differentiable in each subinterval (a_k, a_{k+1}), $0 \leq k \leq n - 1$ and $f'(c+)$ and $f'(c-)$ exist at all points $c \in [a, b]$ except at a, where $f'(a+)$ exists, and at b, where $f'(b-)$ exists.

Let $f : [a, b] \to \mathbb{R}$ (or $\mathbb{C}$) be a given function. Let $\mathcal{P} = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be any partition of $[a, b]$. Define

$$V(\mathcal{P}; f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

and

$$V(f; a, b) = \sup_{\mathcal{P}} V(\mathcal{P}; f)$$

the supremum being taken over all possible partitions of $[a, b]$. The quantity $V(f; a, b)$ is called the *total variation* of f over the interval $[a, b]$.

Definition 8.2. *A real (or complex) valued function defined on $[a, b]$ is said to be of* **bounded variation** *on $[a, b]$ if $V(f; a, b) < \infty$.*

Example 8.1. Any monotonic function defined on $[a, b]$ is of bounded variation. In this case

$$V(f; a, b) = |f(b) - f(a)|.$$

Example 8.2. If f is uniformly Lipschitz continuous, then it is of bounded variation. For,

$$\sum_i |f(x_i) - f(x_{i-1})| \leq L \sum_i (x_i - x_{i-1}) = L(b-a) < \infty$$

Example 8.3. Define

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & 0 < x < 1 \\ 0, & x = 0. \end{cases}$$

Then, f is not of bounded variation. To see this, choose a partition of $[0,1]$ as follows:

$$\{0,1\} \cup \left\{ \sqrt{\frac{2}{\pi(2j+1)}} \mid 0 \leq j \leq n \right\}.$$

$$\begin{aligned} |f(x_j) - f(x_{j-1})| &= \frac{2}{\pi(2j+1)} + \frac{2}{\pi(2j-1)} = \frac{2}{\pi}\frac{4j}{4j^2-1}. \\ &\geq \frac{2}{\pi}\frac{4j}{4j^2} = \frac{2}{\pi j}. \end{aligned}$$

Thus for all n, $V(f;0,1) \geq \frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{j}$ and so $V(f;0,1) = \infty$. □

Given a real number r, define $r^+ = \max\{r, 0\}$ and $r^- = -\min\{r, 0\}$. Then $r = r^+ - r^-$ and $|r| = r^+ + r^-$. Thus, if $f : [a,b] \to \mathbb{R}$, and $\mathcal{P} = \{a = x_0 < \ldots < x_n = b\}$ is any partition, set

$$p(\mathcal{P}; f) = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)^+$$

and

$$n(\mathcal{P}; f) = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)^-.$$

Then $V(\mathcal{P}; f) = p(\mathcal{P}; f) + n(\mathcal{P}; f)$ and

$$f(b) - f(a) = p(\mathcal{P}; f) - n(\mathcal{P}; f).$$

Define

$$P(f;a,b) = \sup_{\mathcal{P}} p(\mathcal{P}; f) \text{ and } N(f;a,b) = \sup_{\mathcal{P}} n(\mathcal{P}; f);$$

where, again, the supremum are taken over all possible partitions of $[a,b]$.

Proposition 8.1. *Let f be of bounded variation on $[a,b]$. Then*

$$V(f;a,b) = P(f;a,b) + N(f;a,b) \tag{8.2}$$

$$f(b) - f(a) = P(f;a,b) - N(f;a,b). \tag{8.3}$$

Proof. We know that, for any partition $\mathcal{P}$,

$$\begin{aligned} p(\mathcal{P};f) &= n(\mathcal{P};f) + f(b) - f(a) \\ &\leq N(f;a,b) + f(b) - f(a). \end{aligned}$$

Then, taking the supremum over all possible partitions, we get

$$P(f;a,b) \leq N(f;a,b) + f(b) - f(a). \tag{8.4}$$

Similarly $n(\mathcal{P};f) = p(\mathcal{P};f) + f(a) - f(b)$ yields

$$N(f;a,b) \leq P(f;a,b) + f(a) - f(b) \tag{8.5}$$

Relations (8.4) and (8.5) yield (8.3), since $V(f;a,b) < \infty$ implies that $P(f;a,b) < \infty$ and $N(f;a,b) < \infty$. Now

$$V(\mathcal{P};f) = p(\mathcal{P};f) + n(\mathcal{P};f)$$

gives us

$$V(f;a,b) \leq P(f;a,b) + N(f;a,b). \tag{8.6}$$

On the other hand,

$$\begin{aligned} V(f;a,b) &\geq p(\mathcal{P};f) + n(\mathcal{P};f) \\ &= 2p(\mathcal{P};f) - \big(f(b) - f(a)\big) \\ &= 2p(\mathcal{P};f) + N(f;a,b) - P(f;a,b) \end{aligned}$$

using (8.3). Again, taking the supremum over all partitions, we get

$$\begin{aligned} V(f;a,b) &\geq 2P(f;a,b) + N(f;a,b) - P(f;a,b) \\ &= P(f;a,b) + N(f;a,b) \end{aligned} \tag{8.7}$$

Relations (8.6) and (8.7) yield (8.2). ☐

Proposition 8.2. *If f and g are of bounded variation on $[a, b]$, then $f + g$ is of bounded variation and*

$$V(f + g; a, b) \leq V(f; a, b) + V(g; a, b). \tag{8.8}$$

Proof. If $\mathcal{P}$ is any partition,

$$V(\mathcal{P}; f + g) \leq V(\mathcal{P}; f) + V(\mathcal{P}; g)$$

by the triangle inequality and the result follows. □

We now have an important characterization of functions of bounded variation.

Theorem 8.1. *A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if, and only if, f is the difference of two monotonic increasing functions.*

Proof. If f is the difference of two monotonic increasing functions, since each of these is of bounded variation (cf. Example 8.1), it follows from the preceding proposition that f is of bounded variation.

Conversely, if f is of bounded variation on $[a, b]$, by Proposition 8.1, we have, for any $x \in [a, b]$,

$$f(x) - f(a) = P(f; a, x) - N(f; a, x).$$

Let $g(x) = P(f; a, x)$ and $h(x) = N(f; a, x) - f(a)$. Clearly both functions are monotonic increasing and $f(x) = g(x) - h(x)$. This completes the proof. □

From the properties of monotonic functions, we can now deduce the following result.

Corollary 8.1. *If $f : [a, b] \to \mathbb{R}$ is of bounded variation, then f is differentiable a.e. Further, for all $x \in (a, b)$, $f(x+)$ and $f(x-)$ exist. So do $f(a+)$ and $f(b-)$. The function f has at most a countable number of jump discontinuities.* □

The fundamental theorem of Lebesgue integration states that if f is integrable on $[a, b]$ and if

$$F(x) = c + \int_a^x f(t)\, dt,$$

then $F' = f$ a.e. on $[a, b]$. We now ask the question as to when a function can be expressed as an indefinite integral of an integrable function.

Proposition 8.3. *Let f be integrable on $[a, b]$. Then set*

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is of bounded variation on $[a, b]$.

Proof. If $\mathcal{P} = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is any partition of $[a, b]$, then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,dt = \int_a^b |f(t)|\,dt < \infty.$$

Thus

$$V(F; a, b) \leq \int_a^b |f(t)|\,dt < \infty.$$

Hence the result. □

Thus a function must be at least of bounded variation on $[a, b]$ for it to be expressed as an indefinite integral of an integrable function. But this is not enough.

Proposition 8.4. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \subset [a, b]$ is a measurable set with $\mu(E) < \delta$, then*

$$\left| \int_E f(x)\,dx \right| < \varepsilon.$$

Proof. If $|f| \leq M$ on $[a, b]$, the result is trivially true, since

$$\left| \int_E f\,dx \right| \leq M\mu(E),$$

where $\mu(E)$ is the Lebesgue measure of E.

In the general case, define

$$f_n(x) = \begin{cases} |f(x)|, & \text{if } |f(x)| \leq n \\ n, & \text{if } |f(x)| > n. \end{cases}$$

Then f_n is bounded and $f_n \to |f|$ pointwise. Further $\{f_n\}$ is an increasing sequence of non-negative functions. By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_a^b f_n\,dx = \int_a^b |f|\,dx < \infty.$$

Given $\varepsilon > 0$, choose N such that

$$\int_a^b (|f| - f_N)\ dx < \varepsilon/2.$$

Now choose $\delta > 0$ such that $\mu(E) < \delta$ implies that

$$\int_E f_N\, dx < \varepsilon/2.$$

Then $|\int_E f\, dx| \leq \int_E |f|\, dx \leq \int_a^b (|f| - f_N)\, dx + \int_E f_N\, dx < \varepsilon$. This completes the proof. □

Definition 8.3. *A function $f : [a, b] \to \mathbb{R}$ is said to be* **absolutely continuous** *on $[a, b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever we have a finite collection of disjoint intervals $\{(x_i, x_i')\}_{i=1}^n$ satisfying*

$$\sum_{i=1}^n (x_i' - x_i) < \delta$$

we have

$$\sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon.$$

Clearly, any absolutely continuous function is uniformly continuous on $[a, b]$.

Example 8.4. Any uniformly Lipschitz continuous function is absolutely continuous, since

$$\sum |f(x_i') - f(x_i)| \leq L \sum (x_i' - x_i) < L\delta.$$

Example 8.5. Any indefinite integral of an integrable function is absolutely continuous by virtue of Proposition 8.4.

We will show presently that a function can be written as an indefinite integral if, and only if, it is absolutely continuous.

Proposition 8.5. *An absolutely continuous function is of bounded variation.*

Proof. Let δ correspond to $\varepsilon = 1$ in the definition of absolute continuity. Let K be the integral part of $1 + (b - a)/\delta$, where $[a, b]$ is the given interval. Given any partition $\mathcal{P}$ of $[a, b]$, we can refine it to a partition consisting of K sets of sub-intervals each of total length less than δ. Thus $V(\mathcal{P}; f) \leq K$ and so $V(f; a, b) \leq K$. □

Consequently, any absolutely continuous function is differentiable a.e.. on $[a, b]$.

Proposition 8.6. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous. Assume that $f' = 0$ a.e. in $[a, b]$. Then f is a constant function.*

Proof. Let $c \in (a, b)$. Let

$$E = \{x \in (a, c) \mid f'(x) = 0\}$$

Then $\mu(E) = c - a$. Let $\varepsilon, \eta > 0$ be arbitrary. If $x \in E$, there exists a sufficiently small $h > 0$ such that $[x, x+h] \subset [a, c]$ and $|f(x+h) - f(x)| < \eta h$. By the Vitali covering lemma, there exists a finite disjoint collection of such intervals which cover all of E except possibly a subset of measure less than δ, where δ corresponds to ε in the definition of absolute continuity. We label these intervals $[x_k, y_k]$, with x_k increasing. Thus

$$y_0 = a \leq x_1 < y_1 \leq x_2 < y_2 \leq \ldots \leq y_n = c = x_{n+1}.$$

Then,

$$\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta.$$

Now,

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \eta \sum (y_k - x_k) < \eta(c - a).$$

Also

$$\sum_{k=1}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon$$

by absolute continuity. Thus

$$|f(c) - f(a)| \leq \varepsilon + \eta(c - a)$$

or $f(c) = f(a)$ for all $c \in (a, b)$. Hence the result. □

Theorem 8.2. *A function can be expressed as an indefinite integral of an integrable function if, and only if, it is absolutely continuous. The derivative of this function (which exists a.e.) is equal a.e. to the integrand.*

Proof. If f were an indefinite integral, it is absolutely continuous (cf. Example 8.5).

Conversely, let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Then it is of bounded variation and

$$F(x) = F_1(x) - F_2(x)$$

where $F_i(x), i = 1, 2$ are monotonic increasing. Thus $F' = F_1' - F_2'$ a.e. and $F_1' \geq 0$, $F_2' \geq 0$. Thus,

$$\begin{aligned} \int_a^b |F'| \, dx &\leq \int_a^b |F_1'| \, dx + \int_a^b |F_2'| \, dx \\ &= \int_a^b F_1' \, dx + \int_a^b F_2' \, dx \\ &\leq F_1(b) - F_1(a) + F_2(b) - F_2(a) < \infty. \end{aligned}$$

Thus F' is integrable. Now let $G(x) = \int_a^x F'(t) \, dt$. Then G, being an indefinite integral of an integrable function, is absolutely continuous and $G' = F'$ a.e. Thus $G - F$ is absolutely continuous and $(G - F)' = 0$ a.e.. Thus, by Proposition 8.6, $(G - F)(x) = (G - F)(a)$ for all $x \in [a, b]$. Hence,

$$F(x) = F(a) + \int_a^x F'(t) \, dt.$$

This completes the proof. □

Absolutely continuous functions share many properties of continuously differentiable functions. In particular, we have the following result.

Theorem 8.3 (Integration by parts). *Let f, g be absolutely continuous on $[a, b]$. Then*

$$\int_a^b f(t)g'(t) \, dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t) \, dt. \tag{8.9}$$

Proof. Consider the function $f'(x)g'(y)$ on $[a,b]\times[a,b]$. Consider the integral

$$\int_a^b \int_a^x f'(x)g'(y)\,dy\,dx.$$

It is a routine verification to check that Fubini's theorem applies and so

$$\int_a^b \int_a^x f'(x)g'(y)\,dy\,dx = \int_a^b \int_y^b f'(x)g'(y)\,dx\,dy.$$

The left-hand side gives us

$$\begin{aligned}\int_a^b \left(\int_a^x g'(y)\,dy\right) f'(x)\,dx &= \int_a^b \big(g(x)-g(a)\big)f'(x)\,dx \\ &= \int_a^b g(x)f'(x)\,dx - g(a)\big[f(b)-f(a)\big]\end{aligned}$$

by repeated use of Theorem 8.2. The right-hand side gives

$$\begin{aligned}\int_a^b \left(\int_y^b f'(x)\,dx\right) g'(y)\,dy &= \int_a^b \big(f(b)-f(y)\big)g'(y)\,dy \\ &= f(b)\big[g(b)-g(a)\big] - \int_a^b f(y)g'(y)\,dy.\end{aligned}$$

Equating the two, we deduce (8.9). This completes the proof. □

9 Pointwise Convergence

In this section, we will prove the convergence theorems of Dirichlet and Jordan.

Proposition 9.1. *Let $f \in L^1(0,\pi)$ and assume that $f'(0+)$ exists. Then*

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi f(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = \frac{1}{2}f(0+). \tag{9.1}$$

Proof. Adding and subtracting $f(0+)$ in the integrand, we get

$$\int_0^\pi f(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = \int_0^\pi (f(t)-f(0+))\frac{\sin(n+\frac{1}{2})t}{t}\,dt + f(0+)\int_0^\pi \frac{\sin(n+\frac{1}{2})t}{t}\,dt.$$

The second integral on the right-hand side becomes (after a change of variable),

$$\int_0^{(n+\frac{1}{2})\pi} \frac{\sin t}{t}\, dt$$

which converges to $\pi/2$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \frac{1}{\pi} f(0+) \int_o^{\pi} \frac{\sin(n+\frac{1}{2})t}{t}\, dt \;=\; \frac{1}{2} f(0+).$$

Hence we need to show that the first term tends to zero as $n \to \infty$. Let $\varepsilon > 0$ be an arbitrarily small number. Choose $0 < r < \pi$ such that for $0 < t \leq r$,

$$\left| \frac{f(t) - f(0+)}{t} - f'(0+) \right| \;<\; \varepsilon$$

(cf. (8.1)). Then,

$$\int_0^{\pi} (f(t) - f(0+)) \frac{\sin(n+\frac{1}{2})t}{t}\, dt \;=\; \begin{cases} \int_0^r (f(t) - f(0+)) \frac{\sin(n+\frac{1}{2})t}{t}\, dt \\ + \int_r^{\pi} (f(t) - f(0+)) \frac{\sin(n+\frac{1}{2})t}{t}\, dt. \end{cases}$$

The second term tends to zero, by the Riemann-Lebesgue property for Φ_n (cf. Proposition 5.7), since $f(t) - f(0+) \in L^1(0, \pi)$. Now, the first term above can be written as

$$\int_0^r \left(\frac{f(t) - f(0+)}{t} - f'(0+) \right) \sin(n + \frac{1}{2})t\; dt + f'(0+) \int_0^r \sin(n + \frac{1}{2})t\; dt.$$

But,

$$\left| \int_0^r \left(\frac{f(t) - f(0+)}{t} - f'(0+) \right) \sin(n + \frac{1}{2})t\; dt \right| \;<\; r\varepsilon \;<\; \pi\varepsilon$$

and

$$\int_0^r \sin(n + \frac{1}{2})t\; dt \;=\; \int_0^r \sin\frac{t}{2} \cos nt\; dt + \int_0^r \cos\frac{t}{2} \sin nt\; dt$$

and both these integrals tend to zero, as $n \to \infty$, by the Riemann-Lebesgue lemma (cf. Corollary 4.1). This completes the proof. □

Theorem 9.1. *(**Dirichlet**) Let $f \in L^1(-\pi, \pi)$ be a piecewise smooth function. Then its Fourier series converges to $\frac{1}{2}(f(t+) + f(t-))$ at all points $t \in [-\pi, \pi]$. In particular, if f is continuous at a point t, then its Fourier series converges to $f(t)$ at that point.*

Proof. Recall that (cf. (5.4e))

$$s_n(t) \;=\; \frac{1}{2\pi}\int_0^\pi (f(t+x) + f(t-x))D_n(x)\;dx.$$

Now, by (9.1), we have

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi f(t+x)\frac{\sin(n+\frac{1}{2})x}{x}\;dx \;=\; \frac{1}{2}f(t+).$$

Thus,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^\pi f(t+x)\frac{\sin(n+\frac{1}{2})x}{\frac{x}{2}}\;dx \;=\; \frac{1}{2}f(t+).$$

Hence, by Proposition 5.6, we get

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^\pi f(t+x)\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}\;dx \;=\; \frac{1}{2}f(t+).$$

Similarly,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^\pi f(t-x)\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}\;dx \;=\; \frac{1}{2}f(t-)$$

from which the result follows. □

Example 9.1. Consider the function

$$f(t) \;=\; \begin{cases} -1, & -\pi \le t < -\pi/2 \\ \;\;0, & -\pi/2 \le t \le \pi/2 \\ \;\;1, & \pi/2 < t \le \pi. \end{cases}$$

This function is piecewise smooth and is odd. Thus, its Fourier series consists only of sine functions. Now,

$$b_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\;dt \;=\; \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin nt\;dt.$$

Thus,

$$b_n = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n = 4k, \ k \in \mathbb{N} \\ -\frac{4}{n\pi}, & n = 4k+2, \ k \in \mathbb{N} \end{cases}$$

At $t = \pi/2$, the series must thus converge to $1/2$. Thus,

$$\frac{1}{2} = \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

which yields the well known *Gregory series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

In order to prove the next result, we recall a version of the mean value theorem for integrals.

Proposition 9.2. *Let $g : [a,b] \to \mathbb{R}$ be continuous and let $f : [a,b] \to \mathbb{R}$ be non-negative and monotonic increasing. Then, there exists $c \in [a,b]$ such that*

$$\int_a^b f(t)g(t)\ dt = f(b)\int_c^b g(t)\ dt. \tag{9.2}$$

Remark 9.1. The result is false if f is not non-negative. To see this, take $f(t) = g(t) = t$ on $[-1,1]$. Then the left-hand side will be

$$\int_{-1}^1 t^2\ dt = \frac{2}{3}.$$

The right-hand side is

$$f(1)\int_c^1 t\ dt = \frac{1-c^2}{2}$$

and we can never have $c \in [-1,1]$ such that the two are equal.

Remark 9.2. In the mean value theorem of differential calculus, the point c will be in the interior of the interval. In case of mean value theorems for integrals, this need not be necessarily the case. For instance, if $f \equiv 1$ on $[a,b]$ and if g is strictly positive on that interval, then we cannot have (9.2) with $c \in (a,b)$.

We are now in a position to prove a convergence theorem for a larger class of functions. We need a preliminary result analogous to Proposition 9.1.

Proposition 9.3. *Let f be of bounded variation on $[0,\pi]$. Then*

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi f(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt \;=\; \frac{1}{2}f(0+). \tag{9.3}$$

Proof. Since f is of bounded variation, it can be written as the difference of two monotonic increasing functions. Hence we may assume, without loss of generality, that f is monotonic increasing. Now,

$$\int_0^\pi f(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt \;=\; \begin{cases} \int_0^\pi (f(t)-f(0+))\frac{\sin(n+\frac{1}{2})t}{t}\,dt \\ \\ +\,f(0+)\int_0^\pi \frac{\sin(n+\frac{1}{2})t}{t}\,dt. \end{cases}$$

As already shown in the proof of Proposition 9.1, the second integral converges to $\pi/2$. Thus, as before, it is enough to show that the first integral tends to zero as $n\to\infty$.

Let $\varepsilon>0$ be an arbitrarily small number. Choose $0<r\le\pi$ such that

$$|f(t)-f(0+)| \;<\varepsilon/4,\; 0<t\le r. \tag{9.4}$$

Then, splitting the first integral over $[0,r]$ and $[r,\pi]$, we see that the integral on $[r,\pi]$ tends to zero by the Riemann-Lebesgue property for Φ_n (cf. Proposition 5.7) since $f(t)-f(0+)$ is integrable over that interval. Thus, for n sufficiently large, we have

$$\left|\frac{1}{\pi}\int_r^\pi (f(t)-f(0+))\frac{\sin(n+\frac{1}{2})t}{t}\,dt\right| < \frac{\varepsilon}{2}.$$

On the interval $[0,r]$, we redefine the function f as $f(0+)$ at $t=0$, so that the function $f(t)-f(0+)$ remains non-negative and monotonic increasing without altering the value of the integral over that interval. Hence, by the mean value theorem (cf. Proposition 9.2), there exists $c\in[0,r]$ such that

$$\int_0^r (f(t)-f(0+))\frac{\sin(n+\frac{1}{2})t}{t}\,dt \;=\; (f(r)-f(0+))\int_c^r \frac{\sin(n+\frac{1}{2})t}{t}\,dt.$$

Thus, by Proposition 5.5 and by (9.4), it follows that

$$\left|\frac{1}{\pi}\int_0^r (f(t)-f(0+))\frac{\sin(n+\frac{1}{2})t}{t}\,dt\right| < \frac{\varepsilon}{2}.$$

This completes the proof. $\square$

Theorem 9.2. *(***Jordan***) Let f be a function of bounded variation on $[-\pi, \pi]$. Then, its Fourier series at any point t in this interval converges to $\frac{1}{2}(f(t+)+f(t-))$. In case f is continuous at t, then the series converges to $f(t)$.*

Proof. The proof is identical to that of Theorem 9.1, except that we appeal to Proposition 9.2 in place of Proposition 9.1. ☐

Remark 9.3. Dirichlet proved his theorem on the convergence of Fourier series in 1829. Jordan's result was proved in 1881. In 1904, Fejér showed that if $f \in L^1(-\pi, \pi)$, and if $f(t+)$ and $f(t-)$ exist at $t \in [-\pi, \pi]$, then the Fourier series is $(C, 1)$-summable at t to $(f(t+) + f(t-))/2$. One of the greatest triumphs in the history of Fourier series is the result of Carlesson (1966) that the Fourier series of any function in $L^2(-\pi, \pi)$ converges to the value of that function a.e..This was extended by Hunt in 1968 to all functions in $L^p(-\pi, \pi)$ for $1 < p < \infty$.

10 Termwise Integration

In general, when we have a series $f(x) = \sum_n f_n(x)$, we can integrate the series term-by-term if the series is uniformly convergent. However, we have seen that Fourier series (under appropriate hypotheses) converge to $(f(t+)+f(t-))/2$ at a discontinuity. Thus, if the function is discontinuous, uniform convergence is ruled out and the above principle does not apply. However, Fourier series are special and enjoy special properties. Vis-à-vis integration, we have the following result.

Theorem 10.1. *Let $f \in L^1(-\pi, \pi)$ be extended periodically over $\mathbb{R}$ and have the Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

Then,
(i) the series obtained by termwise integration, viz.

$$\frac{a_0}{2}x + \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin x - \frac{b_n}{n}\cos x\right) + C$$

where

$$C = \sum_{n=1}^{\infty}\frac{b_n}{n},$$

converges to $\int_0^x f(t)\ dt$.
(ii) This convergence is uniform if $f \in L^2(-\pi,\pi)$.

Proof. Let

$$F(x) \;=\; \int_0^x \left(f(t) - \frac{a_0}{2}\right)\ dt.$$

Since f is integrable, it follows that F is absolutely continuous. It is also 2π-periodic. For,

$$\begin{aligned} F(x+2\pi) &= \textstyle\int_0^x \left(f(t) - \frac{a_0}{2}\right)\ dt + \int_x^{x+2\pi} \left(f(t) - \frac{a_0}{2}\right)\ dt \\ &= \textstyle F(x) + \int_{-\pi}^{\pi} \left(f(t) - \frac{a_0}{2}\right)\ dt \end{aligned}$$

since, by 2π-periodicity, the integration can be done over any interval of length 2π without affecting its value. But the last integral vanishes by the definition of a_0. Thus $F(x+2\pi) = F(x)$.

Since it is absolutely continuous, it is continuous and of bounded vaiation and so its Fourier series converges to its value at each point. Let

$$F(t) \;=\; \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos nt + d_n \sin nt).$$

We can also use integration by parts (available for absolutely continuous functions (cf. (8.9)) to obtain

$$\begin{aligned} c_n &= \textstyle\frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\ dt \\ &= \textstyle\frac{1}{n\pi}F(t)\sin nt\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\left(f(t) - \frac{a_0}{2}\right)\sin nt\ dt. \end{aligned}$$

The boundary terms vanish by periodicity. So does the integral of $\frac{a_0}{2}\sin nt$. Thus we deduce that $c_n = -b_n/n$. Similarly, $d_n = a_n/n$. Thus,

$$F(t) \;=\; \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin nt - \frac{b_n}{n}\cos nt\right).$$

Evaluating this at $t = 0$, we get

$$\frac{c_0}{2} \;=\; \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

This proves (i).

If $f \in L^2(-\pi,\pi)$, then the series $\sum_n a_n^2$ and $\sum_n b_n^2$ are convergent, by virtue of Parseval's identity (cf. (2.7)). Hence, by the Cauchy-Schwarz inequality, the series $\sum_n a_n/n$ and $\sum_n b_n/n$ are absolutely convergent. Thus, by the Weierstrass' M-test, the above series for $F(t)$ is uniformly convergent. □

Remark 10.1. Since we have shown that

$$\frac{c_0}{2} = \sum_{n=1}^{\infty} \frac{b_n}{n},$$

it follows that for *any* $f \in L^1(-\pi,\pi)$, the Fourier sine coefficients b_n are such that $\sum_n b_n/n$ is convergent. The absolute convergence of this series, however, is true under the additional hypothesis that $f \in L^2(-\pi,\pi)$.

Remark 10.2. If f is continuous and 2π-periodic, then it is in $L^2(-\pi,\pi)$ and so, for any continuous 2π-periodic function, the termwise integrated Fourier series is uniformly convergent.

We now give an application of the observation made in Remark 10.1 by presenting an example of a uniformly convergent trigonometric series which cannot be a Fourier series.

Example 10.1. Consider the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nt}{\log n}. \tag{10.1}$$

The sequence $\{(\log n)^{-1}\}$ is non-negative and monotonically decreases to zero. Further,for $t \neq 2n\pi$,

$$\sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}$$

and so

$$\left|\sum_{k=1}^{n} \sin kt\right| \le \frac{1}{|\sin\frac{t}{2}|} \le \frac{1}{\sin\frac{a}{2}}$$

for $t \in [a, 2\pi - a]$ for any $a \in (0, \pi)$. Thus, by Dirichlet's test, the given series in (10.1) converges uniformly in $[a, 2\pi - a]$ for any $a \in (0, \pi)$. However, the series

$$\sum_{n=2}^{\infty} \frac{b_n}{n} = \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

is divergent and so the given series cannot be a Fourier series.

Remark 10.3. Note that in a Fourier series, it is necessary that $\sum_n b_n/n$ is convergent. However, there is no such condition on $\sum_n a_n/n$. Indeed, Stromberg (1981) has shown that the series

$$\sum_{n=2}^{\infty} \frac{\cos nt}{\log n}$$

is a Fourier series!

11 Termwise Differentiation

Consider the Fourier series of the 2π-periodic extension of the function $f(t) = t$ on $[-\pi, \pi]$. Of course, such an extension has jump discontinuities at all odd multiples of π. Nevertheless,

$$f(t) = 2 \sum_{n-1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$$

and, by Jordan's theorem, the series converges to $f(t)$ at all points in $(-\pi, \pi)$ and to zero at $-\pi$ and π. The termwise derivative of the series is

$$2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos nt$$

which is divergent everywhere since $\cos nt$ does not converge to zero, as $n \to \infty$, for any t.

Thus, while any Fourier series may be integrated termwise meaningfully, extra hypotheses are needed for termwise differentiation.

Theorem 11.1. *Let f be a continuous and piecewise smooth 2π-periodic function on $[-\pi, \pi]$ with Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

If its derivative f' is also piecewise smooth, then

$$\frac{f'(t+) + f'(t-)}{2} = \sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt). \tag{11.1}$$

Proof. Let f' have the Fourier series expansion

$$\frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos nt + d_n \sin nt).$$

Since f is 2π-periodic, we have

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\ dt = 0.$$

Again, for $n \geq 1$,

$$\begin{aligned} c_n &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f'(t)\cos nt\ dt \\ &= \tfrac{1}{\pi} f(t)\cos nt\Big|_{-\pi}^{\pi} + \tfrac{n}{\pi}\textstyle\int_{-\pi}^{\pi} f(t)\sin nt\ dt, \end{aligned}$$

by the absolute continuity of f, which is piecewise smooth. The boundary terms cancel out by the 2π-periodicity and so we get $c_n = nb_n$. Similarly, $d_n = -na_n$. The relation (11.1) follows from the Dirichlet convergence theorem (Theorem 9.1) for piecewise smooth functions. □

Remark 11.1. By the Riemann-Lebesgue lemma, c_n and d_n tend to zero as $n \to \infty$. Thus, na_n and nb_n tend to zero. More generally, the smoother the function, the faster its Fourier coefficients tend to zero. If $f, f', f'', \cdots f^{(k-1)}$ are all continuous and $f^{(k)}$ is piecewise smooth, then $n^k a_n$ and $n^k b_n$ tend to zero as $n \to \infty$. Also, in this case, since $f^{(k)} \in L^2(-\pi, \pi)$, we have, by Parseval's identity, that

$$\sum_{n=1}^{\infty} n^{2k} a_n^2 < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} n^{2k} b_n^2 < \infty.$$

We conclude with a nice appliation of Fourier series to a problem in the calculus of variations.

One of the earliest known problems in the calculus of variations is known as the *classical isoperimetric problem* and can be traced to Virgil's *Aeneid*, which describes the problem of Dido, future queen of Carthage. In mathematical terms, the problem can be stated as follows.

- Of all simple closed curves in the plane with a fixed length, which one encloses the maximum area?
  Or, equivalently:

- Of all simple closed curves in the plane enclosing a fixed area, which one has the least length?

Given the existence of such a curve, it is possible to give a fairly elementary geometric argument to identify it as the circle. However, the existence of an optimal solution requires extra proof.

We can deal with the existence and the uniqueness in one stroke if we prove the *classical isoperimetric inequality*: if L is the length of a simple closed plane curve and if A is the enclosed area, then

$$L^2 \ \geq \ 4\pi A, \tag{11.2}$$

with equality if, and only if, the curve is a circle. For the circle, we do indeed have equality since $L = 2\pi r$ and $A = \pi r^2$, where r is its radius. On the other hand, for any curve of given length L, the maximum possible value for the enclosed area is $L^2/4\pi$, which is achieved for the circle. Thus, this establishes the circle as the optimal solution. The uniqueness follows from the 'only if' part of the proof.

Let us assume that a simple closed curve C is parametrized by the equations $x = x(s),\ y = y(s)$, where s is the arc length, which varies over the interval $[0, L]$. We assume that the functions $x(s)$ and $y(s)$ verify the hypotheses of Theorem 11.1. We reparametrize the equations using the parameter

$$t \ = \ \frac{2\pi}{L} s$$

so that $t \in [0, 2\pi]$ and x and y are 2π-periodic smooth functions in t with piecewise smooth derivaives. Let

$$\left.\begin{aligned} x(t) &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \\ y(t) &= \tfrac{c_0}{2} + \textstyle\sum_{n=1}^{\infty}(c_n \cos nt + d_n \sin nt). \end{aligned}\right\} \tag{11.3}$$

Thus,

$$\begin{aligned} x'(t) &= \textstyle\sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt) \\ y'(t) &= \textstyle\sum_{n=1}^{\infty}(nd_n \cos nt - nc_n \sin nt). \end{aligned} \tag{11.4}$$

Since

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

we get that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{L}{2\pi}\right)^2.$$

Then, by Parseval's identity, it follows that

$$2\left(\frac{L}{2\pi}\right)^2 = \frac{1}{\pi}\int_0^{2\pi}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2). \tag{11.5}$$

Now, the enclosed area A is given by

$$A = \int_C x\ dy = \int_0^{2\pi} x(t)\frac{dy}{dt}(t)\ dt = \pi\sum_{n=1}^{\infty} n(a_n d_n - b_n c_n). \tag{11.6}$$

From (11.5) and (11.6), we get

$$L^2 - 4\pi A = 2\pi^2 \sum_{n=1}^{\infty}\left[(na_n - d_n)^2 + (nb_n + c_n)^2 + (n^2-1)(c_n^2 + d_n^2)\right]$$

which is non-negative, thus proving (11.2).

If equality occurs in (11.2), then the above relation shows that for all $n \geq 1$,

$$na_n = d_n \text{ and } nb_n = -c_n.$$

Also, it shows that, for $n > 1$, $c_n = d_n = 0$, and hence it follows that $a_n = b_n = 0$ as well for those n. Thus,

$$\begin{aligned} x(t) &= \tfrac{a_0}{2} + a_1 \cos t + b_1 \sin t \\ y(t) &= \tfrac{c_0}{2} - b_1 \cos t + a_1 \sin t \end{aligned}$$

from which we get

$$\left(x(t) - \frac{a_0}{2}\right)^2 + \left(y(t) - \frac{c_0}{2}\right)^2 = a_1^2 + b_1^2$$

and so the curve has to be a circle, thus proving the uniqueness of the optimal solution.

The isoperimetric inequality exists in all dimensions. The n-dimensional ball is the unique domain with given '$(n-1)$-dimensional surface area' and maximizing the enclosed (n-dimensional) volume amongst all possible domains.

In three dimensions, the analogue of (11.2) reads as

$$S^3 \ \geq \ 36\pi V^2$$

where S is the surface area and V is the enclosed volume.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, then let us denote its n-dimensional (Lebesgue) measure by $|\Omega|_n$ and the '$(n-1)$-dimensional surface measure' of the boundary $\partial\Omega$ (which has to be suitably defined) by $|\partial\Omega|_{n-1}$. The classical isoperimetric inequality now reads as

$$|\partial\Omega|_{n-1} \ \geq \ n\omega_n^{\frac{1}{n}}|\Omega|_n^{1-\frac{1}{n}}$$

where

$$\omega_n \ = \ \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

is the volume of the unit ball in $\mathbb{R}^n$. Equality occurs in the isoperimetric inequality if, and only if, Ω is a ball.

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