The Space $\mathcal{K}(X, Y)$ and the Ideal $\mathcal{K}(X)$

M.T.Nair
Department of Mathematics, IIT Madras

Let X and Y be normed linear spaces. One of the important property of a finite rank operator from X to Y is that the image of the closed unit ball is *relatively compact*.

Definition 1. A linear operator $A : X \to Y$ is said to be a **compact operator** if $\{Ax : \|x\| \leq 1\}$ is relatively compact, that is, $\text{cl}\{Ax : \|x\| \leq 1\}$ is compact; equivalently, for every bounded sequence (x_n) in X, the sequence (Ax_n) has a convergent subsequence. $\diamondsuit$

Clearly,

$$\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y).$$

Observe:

- Every bounded finite rank operator is compact.
- The identity operator on a normed linear space is compact if and only if the space is finite dimensional.

THEOREM 2. *Let X, Y, Z be normed linear spaces.*

(i) *$\mathcal{K}(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$.*

(ii) *If Y is a Banach space, then $\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$.*

(iii) *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, Z)$. If either $A \in \mathcal{K}(X, Y)$ or $B \in \mathcal{K}(Y, Z)$, then $BA \in \mathcal{K}(X, Z)$.*

Proof. (i) Let A and B be in $\mathcal{K}(X, Y)$ and $\alpha \in \mathbb{F}$. Let (x_n) be a bounded sequence in X. It is enough to show that the sequence $((A + \alpha B)x_n)$ has a convergent subsequence. Since A and B are compact, there exists a subsequence (x'_n) for (x_n) and a subsequence (x''_n) for (x'_n) such that (Ax'_n) and (Bx''_n) converge, say to y and z respectively. Hence,

$$Ax''_n + \alpha Bx''_n \to z + \alpha z \quad \text{as} \quad n \to \infty.$$

(ii) Suppose Y be a Banach space. Let (A_n) be a sequence in $\mathcal{K}(X, Y)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$ for some $A \in \mathcal{B}(X, Y)$. We have to show that $A \in \mathcal{K}(X, Y)$. Again, let (x_n) be a bounded sequence in X, say $\|x_n\| \leq c$ for all $n \in \mathbb{N}$. Since Y is complete, it is enough to show that (Ax_n) has a Cauchy subsequence.

Since each A_k is compact and (x_n) is bounded, (x_n) has a subsequence $(x_n^{(k)})$ such that $(A_k x_n^{(k)})$ converges. Without loss of generality, we may assume that $(x_n^{(k+1)})$ is a subsequence of $(x_n^{(k)})$. Note that by construction, $(x_{k+n}^{(k+n)})$ is a subsequence of $(x_{k+n}^{(k)})$. Therefore, by taking $u_n = x_n^{(n)}$, $n \in \mathbb{N}$, the sequence $(A_k u_n)$ converges. Note that, for every $k, m, n \in \mathbb{N}$,

$$\|Au_n - Au_m\| \le \|(A - A_k)u_n\| + \|A_k u_n - A_k u_m\| + \|(A_k - A)u_m\|. \qquad (*)$$

Now, let $\varepsilon > 0$ be given. Since $\|A_n - A\| \to 0$, there exists $k \in \mathbb{N}$ such that

$$\|A - A_k\| < \varepsilon.$$

Since $(A_k u_n)$ converges, there exists $N \in \mathbb{N}$ such that

$$\|A_k u_n - A_k u_m\| < \varepsilon \quad \forall\, n, m \ge N.$$

Hence, using the fact that $\|x_n\| \le c$ for all $n \in \mathbb{N}$, $(*)$ implies

$$\|Au_n - Au_m\| \le c\varepsilon + \varepsilon + c\varepsilon = (2c+1)\varepsilon \quad \forall\, n, m \ge N.$$

Thus, we have shown that (Au_n) is a Cauchy subsequence of (Ax_n).

(iii) Suppose (x_n) is a bounded sequence in X and atleast one of A and B is a compact operator. Assume that A is a compact operator. Then (x_n) has a subsequence (x_n') such that (Ax_n') converges. Since B is continuous, (Ax_n') also converges. Next, assume that B is a compact operator. Since $A \in \mathcal{B}(X, Y)$, (Ax_n) is a bounded sequence, and since $B \in \mathcal{K}(Y, Z)$, (x_n) has a subsequence (x_n') such that (BAx_n') converges. Thus, in both the cases, the sequence (BAx_n) has a convergent subsequence. □

Recall that for any normed linear space X, $\mathcal{B}(X)$ is a *normed algebra*, that is, $\mathcal{B}(X)$ is an algebra such that

$$\|AB\| \le \|A\|\,\|B\| \quad \forall\, A, B \in \mathcal{B}(X).$$

Also, if X is a Banach space, then $\mathcal{B}(X)$ is a Banach space so that it is a *Banach algebra*. Hence, we have the following.

COROLLARY 3. *Let X be a normed linear space.*

(i) *Then $\mathcal{K}(X)$ is an ideal of $\mathcal{B}(X)$.*

(ii) *If X is a Banach space, then $\mathcal{K}(X)$ is a closed ideal of $\mathcal{B}(X)$.*

Definition 4. The quotient space $\mathcal{B}(X)/\mathcal{K}(X)$ is called the **Calkin algebra** on X. ♢