Linear Algebra: Assignment Sheet-II

In the following, V_1 and V_2 are vector spaces over a filed $\mathbb{F}$.

For $i, j \in \mathbb{N}$, we denote $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$

1. Let $T : V_1 \to V_2$ be a linear transformation. Prove that

   (a) $T(0) = 0$.

   (b) T is one-one iff $N(T) = \{0\}$.

2. Verify the assertion in each of the following:

   (a) Let $A \in \mathbb{R}^{m \times n}$ and let $T : \underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^m$ be defined by

   $$T\underline{x} = A\underline{x}\,, \quad \underline{x} \in \underline{\mathbb{R}}^n.$$

   Then T is a linear transformation.

   (b) For $x \in C[a, b]$, define

   $$T(x) = \int_a^b x(t)dt.$$

   Then $T : C[a, b] \to \mathbb{R}$ is a linear transformation.

   (c) For $x \in C^1[a, b]$, define

   $$(Tx)(t) = x'(t), \quad t \in [a, b].$$

   Then $T : C^1[a, b] \to C[a, b]$ is a linear transformation.

   (d) For $\tau \in [a, b]$ and $x \in C^1[a, b]$, define

   $$T(x) = x'(\tau).$$

   Then $T : C^1[a, b] \to \mathbb{R}$ is a linear transformation.

   (e) Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and V be any of the spaces $c_{00}, \ell^1, \ell^\infty$. Recall that

   $$c_{00} = \{x \in \mathcal{F}(\mathbb{N}, \mathbb{F}) : \exists\, k \in \mathbb{N} \text{ with } x(j) = 0\,\forall\, j \geq k\},$$

   $$\ell^1 = \{x \in \mathcal{F}(\mathbb{N}, \mathbb{F}) : \sum_{j=1}^{\infty} |x(j)| \text{ converges}\},$$

   $$\ell^\infty = \{x \in \mathcal{F}(\mathbb{N}, \mathbb{F}) : (x(n)) \text{ bounded}\}.$$

i. $T : V \to V$ defined by

$$T(\alpha_1, \alpha_2, \ldots,) = (0, \alpha_1, \alpha_2, \ldots,)$$

is a linear transformation, called the **right shift operator**.

ii. $T : V \to V$ defined by

$$T(\alpha_1, \alpha_2, \ldots,) = (\alpha_2, \alpha_3, \ldots,)$$

is a linear transformation, called the **left shift operator**.

3. Let $T : V_1 \to V_2$ be a linear transformation. Prove:

   (a) If $u_1, \ldots, u_n$ are in V_1 such that $Tu_1, \ldots, Tu_n$ are linearly independent in V_2, then $u_1, \ldots, u_n$ are linearly independent in V_1.

   (b) If T is one-one and $u_1, \ldots, u_n$ are linearly independent in V_1, then $Tu_1, \ldots, Tu_n$ are linearly independent in V_2.

   Let $T : V_1 \to V_2$ be a linear transformation. Prove:

   (a) If E_1 is a basis of V_1, then $R(T) = \text{span}(T(E_1))$.

   (b) $\dim R(T) \leq \dim(V_1)$.

   (c) If T is one-one, then $\dim R(T) = \dim(V_1)$.

   (d) If V_1 and V_2 are finite dimensional such that $\dim(V_1) = \dim(V_2)$, then T is one-one if and only if T is onto.

4. Let $A \in \mathbb{R}^{m\times n}$ and let $T : \underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^m$ be the linear transformation defined by

$$T\underline{x} = A\underline{x}\,, \quad \underline{x} \in \underline{\mathbb{R}}^n.$$

   Prove:

   (a) T is one-one if and only if the columns of A are linearly independent.

   (b) $R(T)$ is the space spanned by the columns of A, and $\text{rank}(T)$ is the dimension of the space spanned by the columns of A.

5. Let V_1 and V_2 be finite dimensional vector spaces over the same field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be a basis of V_1. Let $\{v_1, \ldots, v_n\} \subseteq V_2$. Define $T : V_1 \to V_2$ be

$$T\Big(\sum_{i=1}^{n} \alpha_i u_i\Big) = \sum_{i=1}^{n} \alpha_i v, \quad (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n.$$

(a) Show that T is a linear transformation such that $T(u_j) = v_j$ for $j \in \{1, \ldots, n\}$.

(b) T is one-one if and only if $\{v_1, \ldots, v_n\}$ is linearly independent.

(c) T is onto if and only if $\text{span}(\{v_1, \ldots, v_n\}) = V_2$.

6. Let V_1 and V_2 be finite dimensional vector spaces over the same field $\mathbb{F}$ and let $E := \{u_1, \ldots, u_n\}$ be a linearly independent subset of V_1. Let $\{v_1, \ldots, v_n\} \subseteq V_2$. Show that there exists a linear transformation $T : V_1 \to V_2$ such that $T(u_j) = v_j$ for $j \in \{1, \ldots, n\}$.

Let V be a finite dimensional space and $E = \{u_1, \ldots, u_n\}$ be an order basis of V. For each $j \in \{1, \ldots, n\}$, let $f_j : V \to \mathbb{F}$ be defined by

$$f_j(x) = \alpha_j \quad \text{for} \quad x := \sum_{i=1}^{n} \alpha_i u_i.$$

Prove:

(a) $f_1, \ldots, f_n$ are in V' and they satisfy $f_i(u_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$,

(b) $\{f_1, \ldots, f_n\}$ is a basis of V'.

7. Prove: Let V be a finite dimensional space. Then V and V' are linearly isomorphic.

8. Let $E = \{u_1, \ldots, u_n\}$ be an order basis of V. If $f_1, \ldots, f_n$ are in V' such that $f_i(u_j) = \delta_{ij}$. Prove $\{f_1, \ldots, f_n\}$ is the dual basis of V.

9. Let $T_1 \in \mathcal{L}(V_1, V_2)$ and $T_2 \in \mathcal{L}(V_2, V_3)$. Show that

(a) $T_2 T_1$ one-one implies T_1 one-one.

(b) $T_2 T_1$ onto implies T_2 onto.

10. Prove: Let V be a vector space and W be a subspace of V. Then the map $\eta : V \to V/W$ defined by

$$\eta(x) = x + W, \quad x \in V,$$

is a linear transformation.

11. Let V_1 and V_2 be finite dimensional vector spaces over the same field $\mathbb{F}$ and let $E_1 := \{u_1, \ldots, u_n\}$ and $E_2 := \{v_1, \ldots, v_m\}$ be ordered bases of V_1 and V_2, respectively. Let $T : V_1 \to V_2$ be a linear transformation. Prove that for each j, $[Tu_j]_{E_2}$ is the j^{th} column of $[T]_{E_1 E_2}$.

12. Let $A \in \mathbb{R}^{m \times n}$ and let $T : \underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^m$ be defined by $T\underline{x} = A\underline{x}, \quad \underline{x} \in \underline{\mathbb{R}}^n$. If E_1 and E_2 are the standard bases for $\underline{\mathbb{R}}^n$ and $\underline{\mathbb{R}}^m$, respectively, then prove that $[T]_{E_1 E_2} = A$.

13. Prove: Let V_1 and V_2 be finite dimensional vector spaces over the same field $\mathbb{F}$ with $\dim(V_1) = n$ and $\dim(V_2) = m$ and let E_1 and E_2 be ordered bases of V_1 and V_2, respectively. Let $T : V_1 \to V_2$ be a linear transformation. Then the following hold:

    (a) $[Tx]_{E_2} = [T]_{E_1E_2}[x]_{E_1}$ for all $x \in V_1$.

    (b) T is one-one (respectively, onto) if and only if $[T]_{E_1E_2} : \underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^m$ is one-one (respectively, onto).

    (c) For $A \in \mathbb{R}^{m\times n}$,

    $$A = [T]_{E_1E_2} \iff [Tx]_{E_2} = A[x]_{E_1} \quad \forall\, x \in V_1.$$

    (d) $T = J_2^{-1}[T]_{E_1E_2}J_1$, where $J_1 : V_1 \to \underline{\mathbb{R}}^n$ and $J_2 : V_2 \to \underline{\mathbb{R}}^m$ are the canonical isomorphisms,

14. Let V_1, V_2, V_3 be finite dimensional vector spaces over the same field $\mathbb{F}$, and let E_1, E_2, E_3 be ordered bases of V_1, V_2, V_3, respectively. If $T_1 \in \mathcal{L}(V_1, V_2)$ and $T_2 \in \mathcal{L}(V_2, V_3)$. Then the

    $$[T_2T_1]_{E_1E_3} = [T_2]_{E_2E_3}[T_1]_{E_1E_2}.$$

15. For $n \in \mathbb{N}$, let $D : \mathcal{P}_n \to \mathcal{P}_{n-1}$ and $T : \mathcal{P}_n \to \mathcal{P}_{n+1}$ be defined by

    $$D(a_0 + a_1t + \cdots + a_nt^n) = a_1t + 2a_2t + \cdots + na_nt^{n-1},$$

    $$T(a_0 + a_1t + \cdot + a_nt^n) = a_0t + \frac{a_1}{2}t^2 + \cdots + \frac{a_n}{n+1}t^{n+1}.$$

    Let $E_k = \{1, t, \ldots, t^k\}$ for $k \in \mathbb{N}$. Find

    $$[D]_{E_nE_{n-1}}, \quad [T]_{E_nE_{n+1}}, \quad [TD]_{E_nE_n}, \quad [DT]_{E_nE_n}.$$

16. Let V_1 and V_2 be finite dimensional vector spaces over the same field $\mathbb{F}$ and let $T : V_1 \to V_2$ be a linear transformation. Let $E_1 = \{u_1, \ldots, u_n\}$ and $\tilde{E}_1 = \{\tilde{u}_1, \ldots, \tilde{u}_n\}$ be two bases of V_1 and $E_2 = \{v_1, \ldots, v_m\}$ and $\tilde{E}_2 = \{\tilde{v}_1, \ldots, \tilde{v}_m\}$ be two bases of V_2. Let $\Phi_1 : V_1 \to V_1$ and $\Phi_2 : V_2 \to V_2$ be the linear transformations such that

    $$\Phi_1(u_i) = \tilde{u}_i, \quad \Phi_2(v_j) = \tilde{v}_j$$

    for $i = 1, \ldots, n$; $j = 1, \ldots, m$. Prove that

    $$[T]_{\tilde{E}_1\tilde{E}_2} = [\Phi_2]_{E_2E_2}^{-1}[T]_{E_1E_2}[\Phi_1]_{E_1E_1}.$$