Linear Algebra: Assignment Sheet-III

In the following, V is an inner product over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

For $i, j \in \mathbb{N}$, we denote $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$

1. Verify:

   (a) On the vector space c_{00}, $\langle x, y \rangle := \sum_{j=1}^{\infty} x(j)\overline{y(j)}$ defines an inner product.

   (b) On the vector space $C[a, b]$, $\langle, y \rangle := \int_a^b x(t)\overline{y(t)}dt$ defines an inner product.

   (c) Let $\tau_1, \ldots, \tau_{n+1}$ be distinct real numbers. On the vector space $\mathcal{P}_n$, $\langle p, q \rangle := \sum_{i=1}^{n+1} p(\tau_i)\overline{q(\tau_i)}$ defines an inner product.

2. Prove the following:

   (a) For $x \in V$, $\langle x, u \rangle = 0 \,\forall\, u \in V \Longrightarrow x = 0$.

   (b) For $u \in V$, if $f : V \to \mathbb{F}$ is defined by $f(x) = \langle x, u \rangle$ for all $x \in V$, then $f \in V'$.

   (c) Let $u_1, u_2, \ldots, u_n$ be linearly independent vectors in V and let $x \in V$. Then

   $$\langle x, u_i \rangle = 0 \quad \forall\, i \in \{1, \ldots, n\} \iff \langle x, y \rangle = 0 \quad \forall\, y \in \text{span}\{u_1, \ldots, u_n\}.$$

   In particular, if $\{u_1, u_2, \ldots, u_n\}$ is a basis of V, and if $\langle x, u_i \rangle = 0$ for all $i \in \{1, \ldots, n\}$, then $x = 0$.

3. Recall that $d : V \times V \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on V, called the metric induced by the inner product. Then, with respect to the above metric, prove the following:

   (a) The map $x \mapsto \|x\|$ is continuous on V.

   (b) For each $u \in V$, the linear functional $f : V \to \mathbb{F}$ defined by $f(x) = \langle x, u \rangle$, $x \in V$, is continuous.

   (c) For every $S \subseteq V$, the set $S^\perp$ is closed in V.

4. Consider the standard inner product on $\mathbb{F}^n$. For each $j \in \{1, \ldots, n\}$, let $e_j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{nj})$. Show that $(e_i + e_j) \perp (e_i - e_j)$ for every $i, j \in \{1, \ldots, n\}$.

5. Consider the vector space $C[0, 2\pi]$ with inner product defined by $\langle f, g \rangle := \int_0^{2\pi} f(t)\overline{g(t)}\, dt$ for $f, g \in C[0, 2\pi]$. For $n \in \mathbb{N}$, let

   $$u_n(t) := \sin(nt), \quad v_n(t) = \cos(nt), \qquad 0 \leq t \leq 2\pi.$$

Let $w_{2n-2} = v_n$ and $w_{2n-1} = u_n$ for $n \in \mathbb{N}$.Show that the sets

$$\{u_n : n \in \mathbb{N}\}, \quad \{v_n : n \in \mathbb{N}\}, \quad \{w_n : n \in \mathbb{N}\}$$

are orthogonal sets.

6. Suppose $\{u_1, \dots, u_n\}$ is an orthonormal set in an inner product space V and $x \in V$. Then

$$x - \sum_{i=1}^{n} \langle x, u_i \rangle u_i \perp \operatorname{span}\{u_1, \dots, u_n\}$$

and

$$\sum_{i=1}^{n} |\langle x, u_i \rangle|^2 \leq \|x\|^2.$$

Further, the following are equivalent:

(a) $x \in \operatorname{span}\{u_1, \dots, u_n\}$

(b) $x = \sum_{i=1}^{n} \langle x, u_i \rangle u_i$

(c) $\|x\|^2 = \sum_{i=1}^{n} |\langle x, u_i \rangle|^2$.

7. Let $V = \mathbb{F}^3$ with standard inner product. Form the given vectors $x, y, z \in \mathbb{F}^3$ in the following Construct orthonormal vectors u, v, w in $\mathbb{F}^3$ such that $\operatorname{span}\{u, v\} = \operatorname{span}\{x, y\}$ and $\operatorname{span}\{u, v, w\} = \operatorname{span}\{x, y, z\}$.

(a) $x = (1, 0, 0)$, $y = (1, 1, 0)$, $z = (1, 1, 1)$;

(b) $x = (1, 1, 0)$, $y = (0, 1, 1)$, $z = (1, 0, 1)$.

8. For $(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$ and $(\beta_1, \dots, \beta_n) \in \mathbb{F}^n$, show that

$$\sum_{j=1}^{n} |\alpha_j \beta_j| \leq \Big(\sum_{j=1}^{n} |\alpha_j|^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{n} |\beta_j|^2\Big)^{\frac{1}{2}}.$$

9. For $x, y \in \mathcal{F}(\mathbb{N})$ prove that

$$\sum_{j=1}^{\infty} |\alpha_j \beta_j| \leq \Big(\sum_{j=1}^{\infty} |\alpha_j|^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{\infty} |\beta_j|^2\Big)^{\frac{1}{2}}.$$

Hint: Use Exercise 8.

Let

$$\ell^2 = \{x \in \mathcal{F}(\mathbb{N}) : \sum_{j=1}^{\infty} |x(j)|^2 < \infty\}.$$

Prove that

(a) ℓ^2 is a subspace $\mathcal{F}(\mathbb{N})$.

(b) For $x, y \in \ell^2$, $\sum_{j=1}^{\infty} |x(j)\overline{y(j)}|$ converges.

(c) $\langle x, y\rangle := \sum_{j=1}^{\infty} |x(j)\overline{y(j)}|$ defines an inner product on ℓ^2.

10. For $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ and $(\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$, show that

$$\Big(\sum_{j=1}^{n} |\alpha_j + \beta_j|^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{j=1}^{n} |\alpha_j|^2\Big)^{\frac{1}{2}} + \Big(\sum_{j=1}^{n} |\beta_j|^2\Big)^{\frac{1}{2}}.$$

11. For $x, y \in \mathcal{F}(\mathbb{N})$ prove that

$$\Big(\sum_{j=1}^{\infty} |\alpha_j + \beta_j|^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{j=1}^{\infty} |\alpha_j|^2\Big)^{\frac{1}{2}} + \Big(\sum_{j=1}^{\infty} |\beta_j|^2\Big)^{\frac{1}{2}}.$$

Hint: Use Exercise 10.

12. Let $\dim(V) = n$ and let $E = \{u_1, \ldots, u_n\}$ be an ordered orthonormal set which is a basis of V. Let $A : V \to V$ be a linear transformation.

(a) Show that $[A]_{E,E} = (\langle Au_j, u_i\rangle)$. [Hint: Use Fourier expansion.]

(b) Define $B : V \to V$ such that $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x, y \in V$.

13. Let $\dim(V) = n$ and let $E = \{u_1, \ldots, u_n\}$ be an ordered orthonormal set which is a basis of V. Let $A, B : V \to V$ be a linear transformations satisfying $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x, y \in V$. Show that $[B]_{E,E} = \overline{[A]}^{T}_{E,E}$, conjugate transpose of $[A]_{E,E}$.

14. Let V be finite dimensional and V_0 is a subspace of V. Prove that every $x \in V$ can be written uniquely as $x = y + z$ with $y \in V_0$ and $z \in V_0^{\perp}$. [Hint: Obtain a basis of V_0, extend it to a basis of V, and consider the orthonormalization of that basis.]

15. Let V be an inner product space and V_0 be a finite dimensional subspace of V. Then for every $x \in V$, there exists a unique pair $y \in V_0$ such that

$$\|x - y\| = \inf_{u \in V_0} \|x - u\|.$$

16. Let V be an inner product space and V_0 be a subspace of V and let $x \in V$ and $y \in V_0$. Prove the following:

(a) If $\langle x - y, u\rangle = 0 \quad \forall\, u \in V_0 \implies \|x - y\| = \inf_{u \in V_0} \|x - u\|$.

(b) If $\text{span}(S) = V_0$ and $\langle x - y, u \rangle = 0 \quad \forall\, u \in S \quad \Longrightarrow \quad \|x - y\| = \inf_{u \in V_0} \|x - u\|$.

17. Let V be an inner product space, V_0 be a finite dimensional subspace of V and $x \in V$. Let $\{u_1, \ldots, u_k\}$ be a basis of V_0. Prove that for $y = \sum_{j=1}^{k} \alpha_j u_j$,

$$\langle x - y, u \rangle = 0 \quad \forall\, u \in V_0 \iff \sum_{j=1}^{k} \langle u_j, u_i \rangle \alpha_j = \langle x, u_i \rangle, \quad i = 1, \ldots, k.$$

Further, prove that there exists a unique $(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}^k$ such that

$$\sum_{j=1}^{k} \langle u_j, u_i \rangle \alpha_j = \langle x, u_i \rangle, \quad i = 1, \ldots, k,$$

and in that case $\|x - y\| = \inf_{u \in V_0} \|x - u\|$.

18. Let $V = C[0,1]$ with inner product: $\langle f, g \rangle := \int_0^1 f(t)\overline{g(t)}dt$. Let $x(t) = t^5$. Find best approximation for x from the space V_0, where

$$(i)\, V_0 = \mathcal{P}_1, \qquad (ii)\, V_0 = \mathcal{P}_2, \qquad (iii)\, V_0 = \mathcal{P}_3, \qquad (iv)\, V_0 = \mathcal{P}_4, \qquad (v)\, V_0 = \mathcal{P}_5.$$

19. Let $V = C[0, 2\pi]$ with inner product: $\langle f, g \rangle := \int_0^{2\pi} f(t)\overline{g(t)}dt$. Let $x(t) = t^2$. Find best approximation for x from the space V_0, where

$$V_0 = \text{span}\{1, \sin t, \cos t, \sin 2t, \cos 2t\}.$$

20. Let V be finite dimensional and V_0 is a subspace of V. For $x \in V$, let y, z be as in the last problem. Define $P, Q : V \to V$ by $P(x) = y$ and $Q(x) = z$. Prove that P and Q are liner transformations satisfying the following:

$$R(P) = V_0, \quad R(Q) = V_0^{\perp}, \quad P^2 = P, \quad Q^2 = Q, \quad P + Q = I,$$

$$\langle Pu, v \rangle = \langle u, Pv \rangle \quad \forall\, u, v \in V, \quad \|x - Px\| \le \|x - u\| \quad \forall\, u \in V_0.$$