Linear Algebra: Assignment Sheet-IV

In the following V is a vector space over $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$, and $T : V \to T$ is a linear operator.

1. Let $A \in \mathbb{R}^{n\times n}$, and consider it as a linear operator from $\mathbb{R}^n$ to itself. Prove that $\lambda \in \sigma_{\text{eig}}(A) \iff \det(A - \lambda I) = 0$.

2. Show that $\sigma_{\text{eig}}(T) = \varnothing$ in the following cases:

   (a) Let $V = \mathcal{P}$, the space of all polynomials over $\mathbb{F}$ and let $Tp(t) = tp(t)$, $p(t) \in \mathcal{P}$.

   (b) Let $V = c_{00}$ and T be the right shift operator on V.

3. Find the eigenvalues and some corresponding eigenvectors for the following cases:

   (a) $V = \mathcal{P}$ and $Tf = f''$.

   (b) $V = C(\mathbb{R}))$ and $Tf = f''$.

4. Let $V = \mathcal{P}_2$. Using a matrix representation of T, find eigenvalues of $T_1 f = f'$ and $T_2 f = f''$.

5. Find eigenspectrum of T if $T^2 = T$.

6. Prove that eigenvectors corresponding to distinct eigenvalues of T are linearly independent.

7. Prove that, for every polynomial $p(t)$ and $\lambda \in \mathbb{F}$ and $x \in V$, $Tx = \lambda x \Longrightarrow p(T)x = p(\lambda)x$.

8. Suppose V is an inner product space and T is a normal operator, i.e., $T^*T = TT^*$. Prove that vector x is an eigenvector of T corresponding to an eigenvalue λ if and only if x is an eigenvector of T corresponding to the eigenvalue $\bar{\lambda}$.

9. Prove that, if V is a finite dimensional inner product space and T is a self adjoint operator, then $\sigma_{\text{eig}}(T) \neq \varnothing$.

10. Let V be a finite dimensional vector space.

    (a) Prove that T is diagonalizable if and only if there are distinct $\lambda_1, \ldots, \lambda_k$ in $\mathbb{F}$ such that $V = N(T - \lambda_1 I) + \cdots + N(T - \lambda_k I)$.

    (b) Prove that, if T has an eigenvalue λ such that $N(T - \lambda I)$ is a proper subspace of $N(T - \lambda I)^2$, then T is not diagonalizable. Is the converse true?

(c) Give an example of a non-diagonalizable operator on a finite dimensional vector space.

11. Let V be a finite dimensional vector space and T be diagonalizable. If $p(t)$ is a polynomial which vanishes at the eigenvalues of T, then prove that $p(T) = 0$.

12. Let V be a finite dimensional vector space.

    (a) Let $\lambda \neq \mu$. Prove that $N(T - \lambda I)^i \cap N(T - \mu I)^j = \{0\}$ for every $i, j \in \mathbb{N}$.

    (b) Prove that generalized eigenvectors associated with distinct eigenvalues are linearly independent.

    (c) Prove Cayley-Hamilton theorem for operators.

13. Let V be finite dimensional over $\mathbb{C}$ and λ be an eigenvalue of T with ascent ℓ. Prove that $m := \dim[N(T - \lambda I)^\ell]$ is the algebraic multiplicity of λ.

14. Let V finite dimensional, $k \in \mathbb{N}$ be such that $\{0\} \neq N(T^k) \neq N(T^{k+1})$, and let Y_k be a subspace of $N(T^{k+1})$ such that $N(T^{k+1}) = N(T^k) \oplus Y_k$. Prove that $\dim(Y_k) \leq \dim[N(T^k)]$.

15. Let V be a finite dimensional vector space and T be diagonalizable. Let $u_1, \ldots, u_n$ be eigenvectors of T which for a basis of T, and let $\lambda_1, \ldots, \lambda_n$ be such that $Tu_j = \lambda_j u_j$, $j = 1, \ldots, n$. Let f be an $\mathbb{F}$-valued function defined on an opens set $\Omega \subseteq \mathbb{F}$ such that $\Omega \supset \sigma_{\text{eig}}(T)$. For $x = \sum_{j=1}^n \alpha_j u_j \in V$, define

$$f(T)x = \sum_{j=1}^{n} \alpha_j f(\lambda_j) u_j.$$

Prove that there is a polynomial $p(t)$ such that $f(T) = p(T)$ [Hint: Lagrange interpolation].