MA2030: Linear Algebra and Numerical Analysis
Assignment Sheet 7

In the following V denotes an inner product spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

(1) Let $\dim(V) = n$ and $E = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V. Let $T : V \to V$ be a linear operator. Prove the following:
  (a) $[T]_{E,E} = (a_{ij})$, with $a_{ij} = \langle Tu_j, u_i \rangle$.
  (b) Let T be self-adjoint. Then $[T]_{E,E}$ is symmetric if $\mathbb{F} = \mathbb{R}$ and $[T]_{E,E}$ is hermitian if $\mathbb{F} = \mathbb{C}$.

(2) Let $\mathbb{F} = \mathbb{C}$ and $T : V \to V$ be a self-adjoint operator. Then prove the following:
  (a) $\langle Tx, x \rangle \in \mathbb{R}$ for every $x \in V$,
  (b) Eigenvalues of T are real numbers.

(3) Let $T : V \to V$ be a self-adjoint operator. Prove the following:
  (a) If V_0 is subspace such that $T(V_0) \subseteq V_0$, then $T(V_0^{\perp}) \subseteq V_0 \perp$.
  (b) Eigenvectors associated with distinct eigenvalues of T are orthogonal.
  (c) If λ and μ are scalars such that $\lambda \neq \mu$, then $N(T - \lambda I) \perp N(T - \lambda \mu)$.

(4) Let $P : V \to V$ be an *projection*, i.e., P is a linear transformation such that $P^2 = P$. Prove that P is a self-adjoint operator if and only if $R(P) \perp N(P)$.

(5) Let $\dim(V) = n$ and $T : V \to V$ be a self adjoint operator. Justify: There exist distinct scalars $\lambda_1, \ldots, \lambda_k$ and orthogonal projections $P_1, \ldots, P_k$ such that

$$T = \sum_{j=1}^{k} \lambda_k P_k.$$

[Recall: A projection operator $P : V \to V$ on an inner product space V is called an *orthogonal projection* if $R(P) \perp N(P)$.]

(6) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Justify: There exists an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ (i.e., $u^T U = I$) such that $U^T A U$ is a diagonal matrix.

(7) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Find an orthogonal matrix $U \in \mathbb{R}^{3 \times 3}$ and a diagonal matrix D such that $U^T A U = D$.

(8) Let V be an inner product space, V_0 be a subspace of V, $x \in V_0$ and $x_0 \in V_0$. Prove that

$$\|x - x_0\| \leq \|x - u\| \quad \forall\, u \in V_0 \iff \langle x - x_0, u \rangle = 0 \quad \forall\, u \in V_0.$$

(9) Let V be an inner product space, V_0 be a subspace of V with $\dim(V_0) = k$. Let $\{u_1, \ldots, u_k\}$ be an orthonormal basis of V_0. For $x \in V_0$, let $x_0 = \sum_{j=1}^{k} \langle x, u_j \rangle u_j$. Prove that

$$\|x - x_0\| \leq \|x - u\| \quad \forall\, u \in V_0.$$

(10) Let V be an inner product space and $\{u_1, \ldots, u_n\}$ be a linearly independent subset of V. Prove that the colums of the square matrix (a_{ij}) with $a_{ij} = \langle u_j, u_i \rangle$ are linearly independent, and hence it is invertible.

(11) Let V be an inner product space and $\{u_1, \ldots, u_n\}$ be a linearly independent subset of V. For $x \in V$, let $\alpha_1, \ldots, \alpha_n$ be the scalars satisfying the system of equations

$$\sum_{j=1}^{n} \langle u_j, u_i \rangle \alpha_j = \langle x, u_i \rangle, \qquad i \in \{1, \ldots, n\}.$$

Prove that $x_0 = \sum_{j=1}^{n} \alpha_j u_j$ is the best approximation of x from the subspace $V_0 := \text{span}\{u_1, \ldots, u_n\}$.

(12) Let $V = \mathbb{R}^2$ with usual inner product, and

$$V_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}.$$

Find the best approximation of $x = (0, 1)$ from the subspace V_0.

(13) Let $V = \mathbb{R}^3$ with usual inner product, and

$$V_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

Find the best approximation of $x = (1, -1, 1)$ from V_0.

(14) Let V be the vector space $C[0, 1]$ over $\mathbb{R}$ with the inner product: $\langle x, u \rangle = \int_0^1 x(t)u(t)dt$, and let $V_0 = \mathcal{P}_1$. Find the best approximation of x from the subspace V_0, where

(i) $x(t) = t^2$, (ii) $x(t) = t^3$, (iii) $x(t) = 1 + t^2$

(15) Let $V = C[0, 1]$ over $\mathbb{R}$ with inner product $\langle x, u \rangle = \int_0^1 x(t)u(t)dt$. Let $V_0 = \mathcal{P}_3$. Find best approximation for x from the subspace V_0, where $x(t)$ is given by

(i) e^t, (ii) $\sin t$, (iii) $\cos t$, (iv) t^4.