MA2030: Linear Algebra and Numerical Analysis
Assignment Sheet 8

(1) Find a best approximate solution for the system $Ax = b$ for the matrix A and vector b as in the following. Also, check whether the best approximate solution is unique or not.

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(2) Let $A \in \mathbb{R}^{m\times n}$. Prove that columns of A are orthonormal (with respect to the usual inner product) if and only if $A^T A = I$.

(3) Prove that if columns of $A \in \mathbb{R}^{m\times n}$ are linearly independent, then there exist $Q \in \mathbb{R}^{m\times n}$ with $Q^T Q = I$ and an upper triangular matrix $R \in \mathbb{R}^{m\times n}$ such that $A = QR$.

(4) Suppose columns of $A \in \mathbb{R}^{m\times n}$ are linearly independent. Prove that $x_0 \in \mathbb{R}^n$ is a best approximate solution of $Ax = b$ if and only if $Rx_0 = Q^T b$, where Q is as in Problem 3.

Is the above best approximate solution unique? Why?

(5) Suppose columns of $A \in \mathbb{R}^{m\times n}$ are linearly independent and $v_1, \ldots, v_n$ are orthonormal vectors obtained from the columns of A by Gram-Schmidt orthogonalization process. Let $Q = [v_1\, v_2\, \cdots\, v_n]$ and $R = Q^T A$. Prove that $x_0 \in \mathbb{R}^n$ is a best approximate solution of $Ax = b$ if and only if $Rx_0 = Q^T b$.

(6) Let $A \in \mathbb{R}^{m\times n}$ and for $x \in \mathbb{R}^n$, let

$$\|x\|_1 := \sum_{j=1}^{n} |x_j|, \quad \|x\|_2 := \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{1/2}, \quad \|x\|_\infty := \max_{1\le j\le n} |x_j|.$$

Prove that $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ are norms on $\mathbb{R}^n$, that is, if $\|\cdot\|$ denotes any of $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$, then

(a) $\|x\| \ge 0 \quad \forall\, x \in \mathbb{R}^n$ and $\|x\| = 0 \iff x = 0$,

(b) $\|x + y\| \le \|x\| + \|y\| \qquad \forall\, x, y \in \mathbb{R}^n$,

(c) $\|\alpha x\| = |\alpha|\,\|x\| \qquad \forall\, x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$.

(7) Let $A \in \mathbb{R}^{m\times n}$ and let

$$\eta_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|, \quad \eta_\infty = \max_{1\le i\le m} \sum_{i=1}^{n} |a_{ij}|, \quad \eta_F = \Big(\sum_{j=1}^{n}\sum_{i=1}^{m} |a_{ij}|^2\Big)^{1/2}.$$

Let $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ be as in Problem 6. Prove the following:
(a) $\|Ax\|_1 \le \eta_1 \|x\|_1 \quad \forall\, x \in \mathbb{R}^n$,
(b) $\|Ax\|_\infty \le \eta_\infty \|x\|_\infty \quad \forall\, x \in \mathbb{R}^n$,
(c) $\|Ax\|_2 \le \eta_F \|x\|_2 \quad \forall\, x \in \mathbb{R}^n$.

(8) Let $A \in \mathbb{R}^{m\times n}$ and $\|\cdot\|$ denote a norm on $\mathbb{R}^n$ or $\mathbb{R}^m$. Let

$$\eta := \max_{\|x\|=1} \|Ax\|.$$

Prove that
(a) $\|Ax\| \le \eta \|x\| \quad \forall\, x \in \mathbb{R}^n$,
(b) If $\beta \ge 0$ is such that $\|Ax\| \le \beta\|x\|$ for all $x \in \mathbb{R}^n$, then $\eta \le \beta$.

(9) For $A \in \mathbb{R}^{m\times n}$, let η_1 and η_∞ be as in Problem 7. Prove that there exist non-zero vectors $u, v \in \mathbb{R}^n$ such that

$$\|Au\|_1 = \eta_1 \|u\|_1, \qquad \|Av\|_\infty = \eta_\infty \|v\|_\infty.$$

Deduce that

$$\eta_1 = \max_{\|x\|_1=1} \|Ax\|_1, \qquad \eta_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty.$$

(10) Let $A \in \mathbb{R}^{m\times n}$. Prove that the eigenvalues of A^TA are non-negative.

(11) Let $A \in \mathbb{R}^{m\times n}$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A^TA. Prove that

$$\|Ax\|_2 \le \left(\max_{1\le j\le k} \sqrt{\lambda_j}\right) \|x\|_2 \quad \forall\, x \in \mathbb{R}^n.$$

Prove also that

$$\max_{\|x\|_2=1} \|Ax\|_2 = \max_{1\le j\le k} \sqrt{\lambda_j}.$$

(12) Suppose $A \in \mathbb{R}^{n\times n}$ is an invertible matrix and $\kappa(A) = \|A\|\,\|A^{-1}\|$. For nonzero $b, \tilde{b} \in \mathbb{R}^n$, let $x, \tilde{x} \in \mathbb{R}^n$ be such that $Ax = b$ and $A\tilde{x} = \tilde{b}$.
(a) Prove that

$$\frac{1}{\kappa(A)} \frac{\|b-\tilde{b}\|}{\|b\|} \le \frac{\|x-\tilde{x}\|}{\|x\|} \le \kappa(A)\frac{\|b-\tilde{b}\|}{\|b\|}.$$

(b) Prove that there exist nonzero vectors $b, \tilde{b} \in \mathbb{R}^n$ such that

$$\frac{\|x-\tilde{x}\|}{\|x\|} = \kappa(A)\frac{\|b-\tilde{b}\|}{\|b\|}.$$

(13) For the matrix $A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}$ compute $\kappa_\varepsilon := \kappa(A)$. What is $\lim_{\varepsilon\to 0} \kappa_\varepsilon$?

Following problems are recommended, but not mandatary:

For $A \in \mathbb{R}^{m\times n}$, let $\|A\| = \max_{\|x\|=1} \|Ax\|$, where for $u \in \mathbb{R}^k$, $\|u\|$ denotes any of $\|u\|_1$, $\|u\|_2$, $\|u\|_\infty$.

(1) Suppose A, B in $\mathbb{R}^{n\times n}$ are invertible matrices, and $b, \tilde{b}$ are in $\mathbb{R}^n$. Let $x, \tilde{x}$ are in $\mathbb{R}^n$ be such that $Ax = b$ and $B\tilde{x} = \tilde{b}$. Show that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \|A\|\,\|B^{-1}\|\Big(\frac{\|A-B\|}{\|A\|} + \frac{\|b-\tilde{b}\|}{\|b\|}\Big).$$

Hint: Use the fact that $B(x-\tilde{x}) = (B-A)x + (b-\tilde{b})$, and use the fact that $\|(B-A)x\| \leq \|B-A\|\,\|x\|$, and

$$\|b-\tilde{b}\| = \|b-\tilde{b}\|\frac{\|Ax\|}{\|b\|} \leq \|b-\tilde{b}\|\|A\|\,\frac{\|x\|}{\|b\|}.$$

(2) Let $B \in \mathbb{R}^{n\times n}$. If $\|B\| < 1$, then show that $I - B$ is invertible, and

$$\|(I-B)^{-1}\| \leq \frac{1}{(1-\|B\|)}.$$

Hint : Show that $I - B$ is injective, by showing that

$$\|(I-B)x\| \geq (1-\|B\|)\|x\| \qquad \forall x \in \mathbb{R}^n$$

and then deduce the result.

(3) Let $A, B \in \mathbb{R}^{n\times n}$ be such that A is invertible, and $\|A-B\| < 1/\|A^{-1}\|$. Then, show that, B is invertible, and

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A-B\|\,\|A^{-1}\|}.$$

[Hint: Observe that $B = A-(A-B) = [I-(A-B)A^{-1}]A$, and use the previous exercise.]

(4) Let $A, B \in \mathbb{R}^{n\times n}$ be such that A is invertible, and $\|A-B\| < 1/2\|A^{-1}\|$. Let $b, \tilde{b}, x, \tilde{x}$ be as in Exercise 1. Then, show that, B is invertible, and

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq 2\kappa(A)\Big(\frac{\|A-B\|}{\|A\|} + \frac{\|b-\tilde{b}\|}{\|b\|}\Big).$$

[Hint: Apply conclusion in Exercise 3 to that in Exercise 1.]