LINEAR ALGEBRA FOR ENGINEERS: ASSIGNMENT PROBLEMS

Contents

1. **Vector Space**

1.1. **Assignment-Problems: Vector spaces.** In the following a set V, a field $\mathbb{F}$, which is either $\mathbb{R}$ or C, and operations of addition $+$ and scalar multiplication $\cdot$, are given. For $\alpha \in F$ and $x \in V$, we write their multiplication $\alpha \cdot x$ as αx. Check whether V is a vector space over $\mathbb{F}$, with these operations.

(1) $V = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, $\mathbb{F} = \mathbb{R}$, with $+$ and $\cdot$ as in $\mathbb{R}^2$.
(2) $V = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + 3x_2 = 0\}$, $\mathbb{F} = \mathbb{R}$, with $+$ and $\cdot$ as in $\mathbb{R}^2$.
(3) $V = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$, $\mathbb{F} = \mathbb{R}$, with $+$ and $\cdot$ as in $\mathbb{R}^2$.
(4) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$ with $+$ as in $\mathbb{R}^2$ and for $x = (x_1, x_2) \in V$ and $\alpha \in \mathbb{R}$, $\alpha x := (x_2, \alpha x_1)$.
(5) $V = \mathbb{C}^2$, $\mathbb{F} = \mathbb{C}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $\alpha \in \mathbb{C}$,
$$x + y := (x_1 + 2y_1, x_2 + 3y_2), \quad \alpha x := (\alpha x_1, \alpha x_2).$$
(6) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$ with $+$ as in $\mathbb{R}^2$ and for $x = (x_1, x_2) \in V$ and $\alpha \in \mathbb{R}$, $\alpha x := (x_1, 0)$.
(7) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$ with $+$ as in $\mathbb{R}^2$ and for $x = (x_1, x_2) \in V$ and $\alpha \in \mathbb{R}$, $\alpha x := (\alpha x_1, -\alpha x_2)$.
(8) $V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}$, $\mathbb{F} = \mathbb{R}$, with $+$ and $\cdot$ as in $\mathbb{R}^2$.
(9) $V = \{x \in \mathbb{R} : x \geq 0\}$, the set of non-negative real numbers, and $\mathbb{F} = \mathbb{R}$. For $x, y \in V$, $\alpha \in \mathbb{R}$,
$$x + y := xy, \quad \alpha x := |\alpha| x.$$
(10) $V = \mathcal{R}([a,b], \mathbb{R})$, the set of all real valued Riemann integrable functions on $[a,b]$, and $\mathbb{F} = \mathbb{R}$, with usual addition and scalar multiplication of functions.
(11) V is the set of all polynomials of degree 5 with real coefficients and $\mathbb{F} = \mathbb{R}$, with usual addition and scalar multiplication of polynomials.
(12) Let S be a non-empty set and $s_0 \in S$. Let V be the set of all functions $f : S \to \mathbb{R}$ with $f(s_0) = 0$ and $\mathbb{F} = \mathbb{R}$, with usual addition and scalar multiplication of functions.
(13) V is the set of all functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $f(-t) = \overline{f(t)}$ and $\mathbb{F} = \mathbb{R}$, with usual addition and scalar multiplication of functions.

1.2. **Assignment-Problems: Subspaces.**

(1) In the following, vector space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is given with usual operations on it. Check whether the given subset U is a subspace of V.
 (a) $V = \mathcal{P}_3$ and $U = \{a_1t + a_2t^2 + a_3t^3 \ : \ a_0 = 0\}$.
 (b) $V = \mathcal{P}_3$ and $U = \{a_0 + a_1t + a_2t^2 + a_3t^3 : a_2 = 0\}$.
 (c) $V = \mathcal{P}_3$ and $U = \{a_0 + a_1t + a_2t^2 + a_3t^3 : a_1 + a_2 + a_3 + a_4 = 0\}$.
 (d) V is the plane $\mathbb{R}^2$ and U is any straight line passing through the origin.
 (e) $V = \mathbb{R}^2$ and $U = \{(x_1, x_2) : x_2 = 2x_1 - 1\}$.
 (f) $V = \mathbb{R}^3$ and $U = \{(x_1, x_2, x_3) : 2x_1 - x_2 - x_3 = 0\}$.
 (g) $V = \mathcal{C}[-1, 1]$ and $U = \{f \in V : f$ is an odd function$\}$.
 (h) $V = \mathcal{C}[0, 1]$ and $U = \{f \in V : f(t) \geq 0$ for all $t \in [0, 1]\}$.
 (i) $V = \mathcal{C}^k[a, b]$ and $U = \mathcal{P}[a, b]$ for each $k \geq 1$.
 (j) $V = \mathcal{C}[0, 1]$ and $U = \{f \in V : f$ is differentiable$\}$.
 (k) $V = \mathcal{C}[-1, 1;$ and $U = \{f \in V : f$ is an odd function $\}$.
 (l) $V = \mathcal{P}_3(\mathbb{R})$ and $U = \{p \in V : \ p(0) = 0\}$.
 (m) $V = \mathcal{P}_3(\mathbb{C})$ and $U = \{p \in V : \ p(1) = 0\}$.
 (n) $V = \mathcal{P}_3(\mathbb{C})$ and $U = \{a + bt + ct^2 + dt^3 : a, b, c, d$ integers $\}$.

(2) For $\alpha \in \mathbb{F}$, let $V_\alpha = \{(a, b, c) \in \mathbb{F}^3 : a + b + c = \alpha\}$. Show that V_α is a subspace of $\mathbb{F}^3$ iff $\alpha = 0$.

(3) Give an example of a nonempty subset of $\mathbb{R}^2$ which is closed under scalar multiplication but is not a subspace of $\mathbb{R}^2$.

(4) Suppose U is a subspace of V and V is a subspace of W. Show that U is a subspace of W.

(5) Give an example of two subspaces of $\mathbb{C}^3$ whose union is not a subspace of $\mathbb{C}^3$.

(6) Show by a counter-example that if $U + W = U + X$ for subspaces U, W, X of V, then W need not be equal to X.

(7) Let $m \in \mathbb{N}$. Does the set $\{0\} \cup \{p \in \mathcal{P} :$ degree of p is equal to $m\}$ form a subspace of $\mathcal{P}$?

(8) Let $u_1, \ldots, u_n$ be vectors in a vector space V. Let $U := \{\alpha_1 u_1 + \ldots + \alpha_n u_n : \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}$. Show that U is a subspace of V.

(9) Prove that the only non-trivial proper subspaces of $\mathbb{R}^2$ are straight lines passing through the origin.

1.3. **Assignment-Problems: Linear span.**

(1) Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Show that $\text{span}\{e_1+e_2, e_2+e_3, e_3+e_1\} = \mathbb{R}^3$
(2) What is $\text{span}\{t^n : n = 0,2,4,6,\dots\}$?
(3) Do the polynomials $t^3-2t^2+1,\ 4t^2-t+3$, and $3t-2$ span $\mathcal{P}_3$?
(4) Let $u_1(t) = 1$, and for $j = 2,3,\dots,$ let $u_j(t) = 1+t+\dots+t^j$. Show that span of $\{u_1,\dots,u_n\}$ is $\mathcal{P}_n$, and span of $\{u_1,u_2,\dots\}$ is $\mathcal{P}$.
(5) We know that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. Does $\sin x \in \text{span}\{1,x,x^2,x^3,\dots\}$?
(6) Let $U = \{(a,b,c,d) \in \mathbb{R}^4 : 5a+2b-c = 3a+2c-d = 0\}$. Find a finite subset S of U such that $\text{span}(S) = U$.
(7) Let x,y,z be vectors in a vector space V. Show that if $x+y+z = 0$, then $\text{span}\{x,y\} = \text{span}\{x,z\} = \text{span}\{y,z\}$.
(8) Let S be a subset of a vector space V. Show that S is a subspace of V iff $S = \text{span}(S)$.
(9) Let U be a subspace of a vector space V. Let $w \in V \setminus U$. Show that for every $x \in \text{span}(U \cup \{w\})$, a unique pair $(\alpha,u) \in \mathbb{F} \times U$ exists such that $x = u + \alpha w$.
(10) Let V be a vector space; $x,y \in V$; U is a subspace of V; $X = \text{span}(U \cup \{x\})$; $Y = \text{span}(U \cup \{y\})$. Suppose that $y \in X \setminus U$. Show that $x \in Y$.
(11) Let $u,v,w_1,\dots,w_n$ be distinct vectors in a vector space V; $B = \{u,w_1,\dots,w_n\}$; and $E = \{v,w_1,\dots,w_n\}$. Prove that $\text{span}(B) = \text{span}(E)$ iff $u \in \text{span}(B)$ iff $v \in \text{span}(E)$.
(12) Let S be a proper subset of a vector space. Show that if x is a vector such that $x \in \text{span}(S \setminus \{x\})$, then $\text{span}(S \setminus \{x\}) = \text{span}(S)$.
(13) Let A and B be subsets of a vector space V. Show that $\text{span}(A \cap B) \subseteq \text{span}(A) \cap \text{span}(B)$. Give an example to show that $\text{span}(A) \cap \text{span}(B)$ need not be a subset of $\text{span}(A \cap B)$.
(14) Let $\mathcal{P}_e = \{p(t) \in \mathcal{P} : p(-t) = p(t)\}$; $\mathcal{P}_o = \{p(t) \in \mathcal{P} : p(-t) = -p(t)\}$. Show that both P_e and P_o are subspaces of $\mathcal{P}$ and that $\mathcal{P} = \mathcal{P}_e \oplus \mathcal{P}_o$.
(15) Let $U = \{0,\dots,0,a_{n+1},\dots,a_{2n}) : a_i \in \mathbb{R}\}$; $V = \{(a_1,\dots,a_{2n}) \in \mathbb{R}^{2n} : a_i = a_{n+i}$ for $i = 1,\dots,n\}$. Does it follow that $\mathbb{R}^{2n} = U \oplus V$?
(16) Construct three subspaces U,W,X of a vector space V so that $V = U \oplus W$ and $V = U \oplus X$ but $W \neq X$.
(17) Suppose U and W are subspaces of a vector space V. Show that $U + W = U$ iff $W \subseteq U$.

1.4. **Assignment-Problems: Linear independence.**

(1) Is $\{(1,1,0,2),(1,1,3,2),(4,2,1,2)\}$ linearly independent in $\mathbb{R}^4$?
(2) Is $\{6,2,1),(4,3,-1),(2,4,1)\}$ linearly independent in $\mathbb{C}^3$?
(3) Let $\{u,v,w,x\}$ be linearly independent in a vector space V. Does it imply that $\{u+v,v+w,w+x,x+u\}$ is linearly independent in V?
(4) Show that the vectors $(1,0,0),(0,2,0),(0,0,3)$ and $(1,2,3)$ are linearly dependent, but any three of them are linearly independent.
(5) Give three vectors in $\mathbb{R}^2$ such that none of the three is a scalar multiple of another.
(6) In each of the following, a vector space V and $S \subseteq V$ are given. Determine whether S is linearly dependent and if it is, express one of the vectors in S as a linear combination of some or all of the remaining vectors.
   (a) $V = \mathbb{R}^3,\ S = \{(1,0,-1),(2,5,1),(0,-4,3)\}$.
   (b) $V = \mathbb{R}^3,\ S = \{(1,2,3),(4,5,6),(7,8,9)\}$.
   (c) $V = \mathbb{R}^3,\ \ S = \{(1,-3,-2),(-3,1,3),(2,5,7)\}$.
   (d) $V = \mathbb{P}^3,\ \ S = \{t^2-3t+5,\ t^3+2t^2-t+1,\ t^3+3t^2-1\}$.
   (e) $V = \mathbb{P}^3,\ \ S = \{-2t^3-11t^2+3t+2,\ t^3-2t^2+3t+1,\ 2t^3+t^2+3t-2\}$.
   (f) $V = \mathbb{P}^3,\ \ S = \{6t^3-3t^2+t+2,\ t^3-t^2+2t+3,\ 2t^3+t^2-3t+1\}$.
   (g) V is the set of all real valued functions defined on $\mathbb{R}$ and $S = \{2,\sin^2 t,\cos^2 t\}$.
   (h) V as above and $S = \{1,\sin t,\sin 2t\}$.
   (i) $V = C([-\pi,\pi]), S = \{\sin t,\sin 2t,\ldots,\sin nt\}$ for some $n \in \mathbb{N}$.
(7) Let $p_1(t) = 1+t+3t^2,\ p_2(t) = 2+4t+t^2,\ p_3(t) = 2t+5t^2$. Are the polynomials p_1,p_2,p_3 linearly independent?
(8) Prove that in the vector space of all real valued functions defined on $\mathbb{R}$, the set $\{e^x,\ xe^x,\ x^3e^x\}$ of functions is linearly independent.
(9) Let $V = \mathbb{C}^2$. Determine conditions on $\alpha,\beta \in \mathbb{C}$ such that the vectors $(\alpha,1),(\beta,1)$ in $\mathbb{C}^2$ are linearly dependent.
(10) Let $V = \mathbb{C}^3$. Determine conditions on $\alpha,\beta,\gamma \in \mathbb{C}$ such that the vectors $(1,\alpha,\alpha^2),\ (1,\beta,\beta^2)$ and $(1,\gamma,\gamma^2)$ in $\mathbb{C}^3$ are linearly dependent.
(11) Let E be any subset of a vector space V. Prove that E is linearly dependent iff there exist finite number of vectors in E which are linearly dependent.
(12) Let E be any subset of a vector space V. Prove that E is linearly independent iff there exist finite number of vectors in E which are linearly independent.
(13) Prove that 0 does not belong to any linearly independent set.
(14) Prove that all supersets of a linearly dependent sets are linearly dependent.
(15) Prove that all subsets of a linearly independent set are linearly independent.
(16) Let U be a subspace of V. Let E be a linearly independent set in U. Prove that E is a linearly independent in V.
(17) Suppose S is a set of vectors and some $v \in S$ is not a linear combination of other vectors in S. Does it follow that S linearly independent?
(18) Show that two vectors (a,b) and (c,d) in $\mathbb{R}^2$ are linearly independent if and only if $ad-bc \neq 0$.

1.5. Assignment-Problems: Basis, Dimension and Quotient space.

(1) Is $\{-1-t-2t^2, 2+t-2t^2, 1-2t+4t^2\}$ a basis for $\mathcal{P}_2$?
(2) Is $\{1+2t+t^2, 3+t^2, t+t^2\}$ a basis for $\mathcal{P}_2$?
(3) Is $\{1+2t+3t^2, 4-5t+6t^2, 3t+t^2\}$ a basis for $\mathcal{P}_2$?
(4) Let $\{x,y,z\}$ be a basis for a vector space V. Check whether $\{x+y, y+z, z+x\}$ is a basis for V?
(5) Extend the set $\{1+t^2, 1-t^2\}$ to a basis of $\mathcal{P}_3$.
(6) Find a basis for the subspace $\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1+x_2+x_3 = 0\}$ of $\mathbb{R}^3$.
(7) Construct a basis for $\{(x_1,\ldots,x_6) \in \mathbb{R}^6 : x_2 = 2x_1, x_4 = 4x_3, x_6 = 6x_5\}$.
(8) Does there exist a basis for $\mathcal{P}^4$ none of the vectors of which is of degree 3?
(9) Under what conditions on α, $\{(1,\alpha,0),(\alpha,0,1),(1+\alpha,\alpha,1)\}$ is a basis of $\mathbb{R}^3$?
(10) Is $\{1+t^n, t+t^n, \ldots, t^{n-1}+t^n, t^n\}$ a basis for $\mathcal{P}_n$?
(11) Let $u_1 = 1$ and let $u_j = 1+t+t^2+\cdots+t^{j-1}$ for $j \geq 2$. Prove or disprove: $\operatorname{span}\{u_1,\ldots,u_n\} = \mathcal{P}_n$ and $\operatorname{span}\{u_1,u_2,\ldots\} = \mathcal{P}$.
(12) Let $v_1,\ldots,v_n$ be linearly independent vectors in a vector space V. Suppose $w \in V$ is such that the vectors $w+v_1,\ldots,w+v_n$ are linearly dependent. Show that $w \in \operatorname{span}\{v_1,\ldots,v_n\}$.
(13) Find bases and dimensions of the following subspaces of $\mathbb{R}^5$:
  (a) $\{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_1-x_3-x_4 = 0\}$.
  (b) $\{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4, x_1+x_5 = 0\}$.
  (c) $\operatorname{span}\{(1,-1,0,2,1),(2,1,-2,0,0),(0,-3,2,4,2),$
  $(3,3,-4,-2,-1),(2,4,1,0,1),(5,7,-3,-2,0)\}$.
(14) Find $\dim(\operatorname{span}\{1+t^2, -1+t+t^2, -6+3t, 1+t^2+t^3, t^3\})$.
(15) Find a basis, and hence dimension, for each of the following subspaces of the vector space V of all twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$:
  (a) $\{x \in V : x''+x = 0\}$.
  (b) $\{x \in V : x''-4x'+3x = 0\}$.
  (c) $\{x \in V : x'''-6x''+11x'-6x = 0\}$.
(16) Show that if U and W are subspace of $\mathbb{R}^9$ such that $\dim U = 5 = \dim W$, then $U \cap W \neq \emptyset$.
(17) Let $U = \{(a,b,c,d) \in \mathbb{R}^4 : b = -a\}$ and $W = \{(a,b,c,d) : c = -a\}$. Find the dimensions of the subspaces $U, W, U+W$, and $U \cap W$ of $\mathbb{R}^4$.
(18) Is $\operatorname{span}\{e_1+e_2, e_2+e_3, e_3+e_1\}$ a proper subspace of $\mathbb{R}^3$? Why?
(19) Prove that the only nonzero proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin.
(20) Show that $U := \{\alpha t^3 + \beta t^7 : (\alpha,\beta) \in \mathbb{R}^2\}$ is a subspace of $\mathcal{P}$. Find a subspace W of $\mathcal{P}$ such that $U \oplus W = \mathcal{P}$.
(21) In each of the following subspaces U and W of a vector space V, determine the bases and dimensions of $U, W, U \cap W$ and of $U+W$.
  (a) $V = \mathbb{R}^3$, $U = \operatorname{span}\{(1,2,3), (2,1,1)\}$, $W = \operatorname{span}\{(1,0,1), (3,0,-1)\}$.
  (b) $V = \mathbb{R}^4$, $U = \operatorname{span}\{(1,0,2,0),(1,0,3,0),\}$,
  $W = \operatorname{span}\{(1,0,0,0),(0,1,0,0),(0,0,1,1)\}$.

(c) $V = \mathbb{C}^4$, $U = \text{span}\{(1,0,3,2),(10,4,14,8),(1,1,-1,-1)\}$,
$W = \{(1,0,0,2),(3,1,0,2),(7,0,5,2)\}$.

(22) Consider each polynomial in $\mathcal{P}$ as a function from the set $\{0,1,2\}$ to $\mathbb{R}$. Is the set of vectors $\{t,t^2,t^3,t^4,t^5\}$ linearly independent?

(23) Given real numbers $a_0,a_1,\ldots,a_k$, let V be the set of all solutions $x \in C^k[a,b]$ of the differential equation

$$a_0\frac{d^kx}{dt^k} + a_1\frac{d^{k-1}x}{dt^{k-1}} + \cdots + a_kx = 0.$$

Show that V is a vector space over $\mathbb{R}$. What is $\dim(V)$?

(24) Consider the set S of all vectors in $\mathbb{R}^4$ whose components are either 0 or 1. How many subsets of S are bases for $\mathbb{R}^4$?

(25) Suppose the vectors $v_1,\ldots,v_n$ span the space V. Show that the vectors $v_1, v_2 - v_1, \ldots, v_n - v_1$ also span V. Show also that if $v_1,\ldots,v_n$ are linearly independent, then $v_1, v_2 - v_1, \ldots, v_n - v_1$ are linearly independent.

(26) Let V be a vector space. Suppose that $v_1,\ldots,v_n,\ldots$ are in V such that for each $m \in \mathbb{N}$, the vectors $v_1,\ldots,v_m$ are linearly independent. Show that $\dim(V) = \infty$.

(27) Show that $\dim(\mathbb{F}^\infty) = \infty$.

(28) Show that $\dim(\mathcal{C}[0,1]) = \infty$.

(29) Let U and W be subspaces of $\mathbb{F}^7$ with $\dim(U) = 4$ and $\dim(W) = 3$. Show that $U + W = \mathbb{F}^7$ iff $U \cap W = \{0\}$ iff $\mathbb{F}^7 = U \oplus W$.

(30) Let U be a subspace of a vector space V. Show that $\dim(U) \le \dim(V)$. Further, if U is a finite dimensional proper subspace of V, then show that $\dim(U) < \dim(V)$.

(31) Let U and W be subspaces of a vector space of dimension $2n+1$. Show that if $\dim(U) = \dim(W) \ge n+1$, then $U \cap W$ contains a nonzero vector.

(32) Determine the quotient space $\mathcal{P}/U$ for each of the following cases:
(a) $U = \mathcal{P}_n$.
(b) U is the subspace of all polynomials of even degree.
(c) U is the subspace of all polynomials which are divisible by t^n for a fixed n.

(33) Let U and W be subspaces of a finite dimensional vector space V. Prove that any basis of $(U+W)/U$ is in one-one correspondence with any basis of $W/(U \cap W)$.

(34) Let U be a subspace of a vector space V. Let $u,v,x,y \in V$. Let $\alpha, \beta \in \mathbb{F}$. Show that if $u \equiv_U x$ and $v \equiv_U y$, then $\alpha u + \beta v \equiv_U \alpha x + \beta y$.

1.6. **Assignment-Problems: Additional problems.**

(1) Suppose S is a set consisting of n elements and V is the set of all real valued functions defined on S. Show that V is a vector space of dimension n.

(2) Let U and W be subspaces of a vector space V. Prove the following:
 (a) $U \cup W = V$ iff $U = V$ or $W = V$.
 (b) $U \cup W$ is a subspace of V iff $U \subseteq W$ or $W \subseteq U$.
 (c) Let $U \cap W = \{0\}$. If $v \in U + W$, then there are unique $u \in U, w \in W$ such that $v = u + w$.

(3) Let V be a vector space. Prove that V is not a finite union of proper subspaces.

(4) Let $\ell^1(\mathbb{N}) := \left\{x \in \mathcal{F}(\mathbb{N},\mathbb{F}) : \sum_{j=1}^{\infty} |x(j)| < \infty\right\}$, where $\mathcal{F}(\mathbb{N},\mathbb{F})$ is the space of all scalar sequences. Show that $\ell^1(\mathbb{N})$ is a subspace of $\mathcal{F}(\mathbb{N},\mathbb{F})$.

(5) For a nonempty set Ω, let $\ell^\infty(\Omega) := \left\{x \in \mathcal{F}(S,\mathbb{F}) : \sup_{s \in S} |x(s)| < \infty\right\}$. Note that $\ell^\infty(\Omega)$ is the set of all bounded functions on S. Thus, $x \in \ell^\infty(\Omega)$ if and only there exists $M_x > 0$ such that $|x(s)| \le M_x$ for all $s \in S$. In particular, $\ell^\infty(\mathbb{N})$ is the set of all bounded sequences of scalars.
 Show that $\ell^\infty(\Omega)$ is a subspace of $\mathcal{F}(S,\mathbb{F})$

(6) Show that

$$\begin{aligned} c_{00} &:= \{x \in \mathcal{F}(\mathbb{N},\mathbb{F}) : \exists k \in \mathbb{N} \text{ such that } x(n) = 0 \forall n \ge k\}, \\ c_0 &:= \{x \in \mathcal{F}(\mathbb{N},\mathbb{F}) : x(n) \to 0 \text{ as } n \to \infty\}, \\ c &:= \{x \in \mathcal{F}(\mathbb{N},\mathbb{F}) : (x(n)) \text{ converges } \} \end{aligned}$$

are subspaces of $\ell^\infty(\mathbb{N})$. We observe that $c_{00} \subseteq \ell^1(\mathbb{N}) \subseteq c_0 \subseteq c \subseteq \ell^\infty(\mathbb{N})$. Are these inclusions proper?

(7) Suppose Λ is a set, and for each $\lambda \in \Lambda$ let V_λ be a subspace of a vector space V. Then prove that $\cap_{\lambda \in \Lambda} V_\lambda$ is a subspace of V.

(8) If one defines $U - W = \{u - v : u \in U, v \in V\}$ for subspaces U, W of a vector space V, then which of the following would hold and which do not?

$$U - U = \{0\},\ U - W = U + W,\ (U - W) + W = U.$$

(9) Let U, W, X be subspaces of a vector space V. Prove or give a counter-example: if $U \oplus W = U \oplus X$, then $W = X$.

(10) Show that a subset $\{u_1, \ldots, u_n\}$ of V is linearly independent iff the function $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 u_1 + \cdots + \alpha_n u_n$ from $\mathbb{F}^n$ into V is injective.

(11) Answer the following questions with justification:
 (a) Is every subset of a linearly independent set linearly independent?
 (b) Is every subset of a linearly dependent set linearly dependent?
 (c) Is every superset of a linearly independent set linearly independent?
 (d) Is every superset of a linearly dependent set linearly dependent?
 (e) Is union of two linearly independent sets linearly independent?
 (f) Is union of two linearly dependent sets linearly dependent?
 (g) Is intersection of two linearly independent sets linearly independent?
 (h) Is intersection of two linearly dependent sets linearly dependent?

(12) Let U be a finite dimensional subspace of a vector space V. Show that any subspace W of V that satisfies $U+W=V$ and $U \cap W = \{0\}$ has dimension $\dim(V)-\dim(U)$.

(13) Let $U_1,\dots,U_k$ be subspaces of a vector space V. Prove that $V = U_1 \oplus \cdots \oplus U_k$ iff for all $x_1 \in U_1,\dots,x_k \in U_k$, $x_1+\cdots+x_k = 0$ implies that $x_1 = \cdots = x_k = 0$.

(14) Let V be an infinite dimensional vector space. Show that there exists a sequence $v_1, v_2, \dots,$ of vectors in V such that for each $n \in \mathbb{N}$, the vectors $v_1,\dots,v_n$ are linearly independent.

(15) For $\lambda \in [a,b]$, let $u_\lambda(t) = \exp(\lambda t)$, $t \in [a,b]$. Show that $\{u_\lambda : \lambda \in [a,b]\}$ is an uncountable linearly independent subset of $\mathcal{C}[a,b]$.

(16) Given $a_0, a_1, \dots, a_n \in \mathbb{R}$, let

$$V = \{f \in C^k[0,1] : a_n f^{(n)}(t) + \cdots + a_1 f^{(1)}(t) + a_0 f(t) = 0 \,\forall t \in [0,1]\}.$$

Show that V is a subspace of $C^k[0,1]$, and find its dimension.

(17) Show that if $\{u_1,\dots,u_n\}$ is a linearly independent subset of a vector space V, and if W is a subspace of V such that $\{u_1,\dots,u_n\} \cap W = \emptyset$, then every x in the span of $\{u_1,\dots,u_n,W\}$ can be written uniquely as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n + y$ with $(\alpha_1,\dots,\alpha_n) \in \mathbb{F}^n$, $y \in W$.

(18) Show that if E_1 and E_2 are linearly independent subsets of V such that $\text{span}(E_1) \cap \text{span}(E_2) = \{0\}$, then $E_1 \cup E_2$ is linearly independent.

(19) Let $p_1(t),\dots,p_{n+1}(t) \in \mathcal{P}_n$ be such that $p_1(1) = \cdots = p_{n+1}(1) = 0$. Show that the polynomials $p_1(t),\dots,p_{n+1}(t)$ are linearly dependent in $\mathcal{P}_n$.

(20) Let $U_1,\dots,U_n$ be finite dimensional subspaces of a vector space V. Prove that $\dim(U_1 + \cdots + U_n) \le \dim(U_1) + \cdots + \dim(U_n)$.

(21) Let U, W and X be subspaces of a finite dimensional vector space V. Are the following true?
   (a) If $U \oplus W = V$, then $W \oplus U = V$?
   (b) If $U \oplus (W \oplus X) = V$, then $(U \oplus W) \oplus X = V$.

(22) Let $U_1,\dots,U_n$ be subspaces of a finite dimensional vector space V. Prove the following.
   (a) If $\dim(U_1) + \cdots + \dim(U_n) = V$, then $V = U_1 \oplus \cdots \oplus U_n$.
   (b) $\dim(U_1 \oplus \cdots \oplus U_n) = \dim(U_1) + \cdots + \dim(U_n)$.
   (c) Each vector in V can be written as a unique sum of vectors from the subspaces $U_1,\dots,U_n$ iff $V = U_1 + \cdots + U_n$ and for each $i \in \{2,\dots,n\}$, $U_i \cap (U_1 + \cdots + U_{i-1}) = \{0\}$.
   (d) $V = U_1 \oplus \cdots \oplus U_n$ iff $V = U_1 + \cdots + U_n$ and whenever $0 = u_1 + \cdots + u_n$ for $u_1 \in U_1,\dots,u_n \in U_n$, we have $u_1 = \cdots = u_n = 0$.

(23) Let V be a vector space. Show that $\dim(V) = n \ge 1$ iff there exist one dimensional subspaces $U_1,\dots,U_n$ such that $V = U_1 \oplus \cdots \oplus U_n$.

2. **Linear Transformations**

2.1. **Assignment-Problems: Linearity.**

(1) In each of the following determine whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation:
  (a) $T(a,b) = (1,b)$
  (b) $T(a,b) = (a,a^2)$
  (c) $T(a,b) = (|a|,b)$
  (d) $T(a,b) = (a+1,b)$

(2) Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. Which of the following $f : \mathbb{C} \to \mathbb{R}$ are linear functionals?
  (a) $f(a+ib) = a$,
  (b) $f(a+ib) = b$,
  (c) $f(a+ib) = a^2$,
  (d) $f(a+ib) = a - ib$,
  (e) $f(a+ib) = \sqrt{a^2+b^2}$.
  What happens if you consider $\mathbb{C}$ as a vector space over $\mathbb{C}$?

(3) Which of the following $f : \mathbb{C}^3 \to \mathbb{C}$ are linear functionals?
  (a) $f(a,b,c) = a+b$,
  (b) $f(a,b,c) = b - c^2$,
  (c) $f(a,b,c) = a+2b-3c$.

(4) Which of the following $f : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ are linear functionals?
  (a) $f(p) = \int_{-1}^{1} p(t)dt$,
  (b) $f(p) = \int_0^1 (p(t))^2 dt$,
  (c) $f(p) = \int_0^1 p(t^2)dt$,
  (d) $f(p) = \int_{-1}^{1} t^2 p(t)dt$,
  (e) $f(p) = dp/dt$ evaluated at $t = 0$.
  (f) $f(p) = d^2p/dt^2$ evaluated at $t = 1$.

(5) Which of the following T is a linear transformation?
  (a) $T : C^1[0,1] \to \mathbb{R}$ with $T(u) = \int_0^1 (u(t))^2\,dt$.
  (b) $T : C^1[0,1] \to \mathbb{R}^2$ with $T(u) = (\int_0^1 u(t)\,dt, u'(0))$.
  (c) $T : \mathcal{P}_n(\mathbb{R}) \xrightarrow{onto} \mathbb{R}$ with $T(p(x)) = p(\alpha)$, for a fixed $\alpha \in \mathbb{R}$.

(6) In each of the following, determine whether a linear transformation T with the given properties exists:
  (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$.
  (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$.
  (c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1,1,0) = (0,0), T(0,1,1) = (1,1)$ and $T(1,0,1) = (1,0)$.
  (d) $T : \mathcal{P}_3 \to \mathbb{R}$ with $T(a+bt^2) = 0$ for any $a,b \in \mathbb{R}$.

(7) Let $S : C^1[0,1] \to C[0,1]$ and $T : C[0,1] \to \mathbb{R}$ be defined by

$$S(u) = u' \quad \text{and} \quad T(v) = \int_0^1 v(t)\,dt.$$

  Find, if possible, ST and TS. Are they linear transformations?

(8) Check whether there exist a linear map $T : \mathbb{C}^3 \to \mathbb{C}^3$ satisfying

$$\begin{aligned} T(1,i,-i) &= (3i,2i,-i), \\ T(i,2i,-i) &= (5,i,1+i), \\ T(-1,2i-2,1-2i) &= (11i,4i-1,1-2i)? \end{aligned}$$

(9) Determine all operators on $\mathbb{R}^2$ which map the line $y = x$ into the line $y = 3x$.

(10) Denote by $\mathbb{C}_R$ the vector space $\mathbb{C}$ over the field $\mathbb{R}$. Give an example of an operator on $\mathbb{C}_R$ which is not an operator on $\mathbb{C}$.

2.2. **Rank and nullity.**

(1) Let D be the differentiation operator on $\mathcal{P}_n(\mathbb{R})$. What is rank$D$?

(2) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T(a,b,c,d) = (a-b, b+c, c-d, b+d).$$

What is nullity of T? Is it surjective?

(3) Find the rank and nullity of the linear transformation $T : \mathbb{C}^4 \to \mathbb{C}^4$ given by

$$T(a,b,c,d) = (a+b-3c+d,\ a-b-c-d,\ 2a+b+2c+d,\ a+3b+3d).$$

Give a basis for $R(T)$. Also, find $\dim(R(T) \cap \operatorname{span}\{(1,1,2,2),(2,0,0,5)\})$.

(4) Find the rank and nullity of the linear transformation $T : \mathcal{P}_6 \to \mathcal{P}_6$ given by $T(p(t)) = p'(t)$.

(5) Let $T : \mathbb{F}^6 \to \mathbb{F}^3$ be a linear transformation such that

$$N(T) = \{(x_1,\ldots,x_6) \in \mathbb{F}^6 : x_2 = 2x_1, x_4 = 4x_3, x_6 = 6x_5\}.$$

Show that T is surjective.

(6) Show that there does not exist a linear transformation from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space is equal to $\{(x_1,\ldots,x_5) \in \mathbb{F}^5 : x_2 = 2x_1, x_4 = 4x_3, x_5 = 6x_4\}$.

(7) Let U and V be finite dimensional vector spaces. Let $T : U \to V$ be a linear transformation. Give reasons for the following:
   (a) $\operatorname{rank}(T) \le \dim(U)$.
   (b) T surjective implies $\dim(V) \le \dim(U)$.
   (c) T injective implies $\dim(U) \le \dim(V)$.
   (d) $\dim(U) > \dim(V)$ implies T is not injective.
   (e) $\dim(U) < \dim(V)$ implies T is not surjective.

(8) Prove that if $S : U \to V$ and $T : V \to W$ are linear transformations such that TS is bijective, then T injective and S is surjective.

(9) Let f be a linear functional on an n-dimensional vector space V. What is the nullity of f?

(10) Let T be a linear operator on a finite dimensional vector space V. Is it true that $V = R(T) \oplus N(T)$?

(11) Find operators S and T on $\mathbb{R}^2$ such that $ST = 0$ but $TS \neq 0$.

(12) Let $S, T : C[a,b] \to C[a,b]$ be defined by

$$[S(x)](t) = \int_a^t x(s)ds, \quad [(T(x)](t) = tx(t) \quad \text{for} \quad x \in C[a,b], t \in [a,b].$$

Show that the map $x \mapsto S(x)T(x)$ is not a linear transformation.

2.3. **Matrix representations.**

(1) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a,b,c) = (b+c,\, c+a,\, a+b)$. Find $[T]_{B,E}$ in each of the following cases.
  (a) $B = \{(1,0,0),(0,1,0),(0,0,1)\}, \quad E = \{(1,0,0),(1,1,0),(1,1,1)\}$
  (b) $B = \{(1,0,0),(1,1,0),(1,1,1)\}, \quad E = \{(1,0,0),(0,1,0),(0,0,1)\}$

(2) Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(a_0 + a_1t + a_2t^2 + a_3t^3) = a_1 + 2a_2t + 3a_3t^2$. Find $[T]_{B,E}$ in each of the following cases.
  (a) $B = \{1,t,t^2,t^3\}, \quad E = \{1+t, 1-t, t^2\}$
  (b) $B = \{1, 1+t, 1+t+t^2, t^3\}, \quad E = \{1, 1+t, 1+t+t^2\}$

(3) Let T be the operator on $\mathbb{C}^2$ defined by $T(a,b) = (a,0)$. Let B be the standard basis of $\mathbb{C}^2$. Let $E = \{(1,i),(-i,2)\}$ be another basis of $\mathbb{C}^2$. Determine the matrices $[T]_B,\ [T]_E,\ [T]_{B,E}$ and $[T]_{E,B}$.

(4) Denote by $\mathbb{C}_R$ the vector space of all complex numbers over the field $\mathbb{R}$, and let $T : \mathbb{C}_\mathbb{R} \to \mathbb{C}_\mathbb{R}$ be defined by $Tz = \bar{z}$. What is $[T]_E$ with respect to the basis $E = \{1, i\}$?

(5) Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$S(a,b,c) = (a+2b+c, a-b-c, b+3c), \quad T(a,b,c) = (c,b,a).$$

Consider bases $B = \{e_1, e_2, e_3\}$ and $E = \{(1,2,3),(1,0,1),\ (1,1,0)\}$ for $\mathbb{R}^3$. Determine the matrices $[ST]_{B,E}, [ST]_{E,B}, [TS]_{B,E}$, and $[TS]_{E,B}$.

(6) Let $\theta \in (0,\pi)$. Let $A, B \in \mathbb{R}^{2\times 2}$ be given by

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Explain why A represents rotation and B represents reflection. By using matrix products, show that
  (a) rotation following rotation is rotation,
  (b) rotation following reflection is reflection,
  (c) reflection following rotation is reflection, and
  (d) reflection following reflection is rotation.

(7) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\alpha,\beta,\gamma) = (\beta+\gamma, \gamma+\alpha, \alpha+\beta)$. Determine $[T]_{B,E}$, where
  (a) $B = \{(1,0,0),(0,0,1),(0,1,0)\},\ E = \{(0,0,1),(1,0,0),(0,1,0)\}$.
  (b) $B = \{(1,1,-1),(-1,1,1),(1,-1,1)\}$,
  $E = \{(-1,1,1),(1,-1,1),(1,1,-1)\}$.

(8) Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \{1,\ t,\ t^2\}$ and $F = \{1\}$.
  (a) Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(A) = A^T$. Compute $[T]_{B,B}$.

(b) Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ by $T(f) = \begin{bmatrix} f'(0) & 2f(1) \\ 0 & f'(3) \end{bmatrix}$. Compute $[T]_{F,E}$.

(c) Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}$ by $T(A) = tr(A)$. Compute $[T]_{B,F}$.

(9) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a,b,c) = (a+b, 2a-b-c, a+b+c)$. Consider the bases $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ and $E = \{(1,2,1),(2,1,,0),(3,2,1)\}$ for $\mathbb{R}^3$. Determine the matrices $[T]_{B,B}, [T]_{B,E}, [T]_{E,B}$, and $[T]_{E,E}$. Also, find the rank(s) of all these matrices.

(10) Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be given by $P(q)(t) = q(t+1)$. Determine $[T]$ with respect to the standard basis of $\mathcal{P}_n$.

(11) Let $B = \{u_1,\ldots,u_n\}$ be an ordered basis of a vector space V. Let f be a linear functional on V. Prove that there exists a unique $(\beta_1,\ldots,\beta_n) \in \mathbb{F}^n$ such that $f(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n$. Conclude that the matrix representation $[f]_{B,\{1\}}$ is the row vector $[\beta_1 \ \cdots \ \beta_n]$.

(12) Let $\{v_1,\ldots,v_n\}$ be a basis for a vector space V. Let E be the standard basis of $\mathbb{F}^{n\times 1}$. Define a map $f : V \to \mathbb{F}^{n\times 1}$ by $f(v) = [v]_E$. Show that f is an invertible linear transformation.

(13) Let $B = \{v_1,\ldots,v_n\}$ be a basis for a vector space V. Let T be an operator on V. Show that T is invertible iff $[T]_{B,B}$ is an invertible matrix.

(14) Let B and E be bases for the finite dimensional vector spaces U and V, respectively. If $S,T : U \to V$ are linear transformations and $\alpha \in \mathbb{F}$, then show that $[S+T]_{B,E} = [S]_{B,E} + [T]_{B,E}$ and $[\alpha T]_{B,E} = \alpha[T]_{B,E}$.

(15) Let S and T be operators on a vector space of dimension n. Show that

$$\text{rank}(S) + \text{rank}(T) - n \le \text{rank}(ST) \le \min\{\text{rank}(S), \text{rank}(T)\}.$$

Give examples, where strict inequalities hold.

2.4. **Change of basis.**

(1) Let T be a linear operator on $\mathbb{C}^2$ such that all entries of $[T]$ with respect to the ordered basis $\{(1,0),(0,1)\}$ are 1. What is $[T]_{B,B}$ where $B = \{(1,-1),(1,1)\}$?

(2) Let T be an linear operator on $\mathbb{C}^3$ such that $Te_1 = u = (0,1,-1)$, $T(e_2) = v = (1,0,-1)$ and $T(e_3) = w = (-1,-1,0)$. What is $[T]_{B,B}$ where $B = \{u,v,w\}$?

(3) Consider the standard basis B and $E = \{(1,2,3),(3,2,1),(0,0,1)\}$ for $\mathbb{R}^3$. Determine the change of basis matrices $[I]_{B,E}$ and $[I]_{E,B}$. For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a,b,c) = (6a+b, a-b-c, 2a-b+3c)$, determine the matrices $[T]_{B,B}, [T]_{B,E}, [T]_{E,B}$, and $[T]_{E,E}$.

(4) Determine the change of basis matrix in each of the following cases, considering the vector space as $\mathbb{R}^n$:
  (a) Old basis is $\{e_1,\ldots,e_n\}$ and new basis is $\{e_n,\ldots,e_1\}$.
  (b) Old basis is $\{e_1,\ldots,e_n\}$ and new basis is $\{e_1+e_2,\ldots,e_{n-1}+e_n\}$.
  (c) Old basis is $\{e_1-e_2,\ldots,e_{n-1}-e_n\}$ and new basis is $\{e_1,\ldots,e_n\}$.

(5) Show that each $n\times n$ matrix is equivalent to its transpose.

(6) Given any invertible matrix $A \in \mathbb{F}^{n\times n}$, show that ordered bases B and E can be chosen for $\mathbb{F}^{n\times 1}$ such that $A = [I]_{B,E}$.

(7) Let $A, B \in \mathbb{F}^{n\times n}$. Does it follow that if $AB = 0$, then $BA = 0$?

(8) Let $B := \{u_1, \ldots, u_n\}$ and $E := \{v_1, \ldots, v_n\}$ be bases for a vector space V. Let T be an operator on V defined by $Tu_1 = v_1, \ldots, Tu_n = v_n$. Show that $[T]_{B,B} = [I]_{E,B}$.

(9) Prove that the matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ are similar iff $a \neq d$.

(10) Let V be a vector space of dimension n. Let S and T be operators on V. Prove that there exist bases B and E for V such that $[S]_B = [T]_E$ iff there exists an invertible operator P on V such that $T = PSP^{-1}$.

2.5. **Space of linear transformations.**

(1) Let $B = \{u_1, \ldots, u_n\}$ and $E = \{v_1, \ldots, v_m\}$ be bases of the vector spaces U and V, respectively. Let $B' = \{f_1, \ldots, f_n\}$ and $E' = \{g_1, \ldots, g_m\}$ be the corresponding dual bases for V' and W'. Show the following:
   (a) If $T \in \mathcal{L}(V,W)$, then $[T]_{B,E} = [(g_i(Tu_j)]$.
   (b) Let $\{A_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$ be any basis of $\mathbb{F}^{m\times n}$. If $T_{ij} \in \mathcal{L}(V,W)$ is such that $[T_{ij}]_{B,E} = A_{ij}$, then $\{T_{ij} : i = 1 \ldots, m,\ j = 1, \ldots, n\}$ is a basis of $\mathcal{L}(V,W)$.

(2) Let $B = \{u_1, \ldots, u_n\}$ and $E = \{v_1, \ldots, v_m\}$ be bases of U and V, respectively. Show the following:
   (a) If $T \in \mathcal{L}(U,V)$, then T is one-one iff columns of $[T]_{B,E}$ are linearly independent.
   (b) If $T \in \mathcal{L}(U,V)$, then T is not one-one iff $det[T]_{B,E} = 0$.

(3) For $p(t) = \sum_{j=0}^{n} \alpha_j t^j$ and any sequence of scalars $(\beta_0, \beta_1, \ldots, \beta_n, \ldots,)$ take $f(p) = \sum_{j=0} \alpha_j \beta_j$. Prove that it defines a functional f on $\mathcal{P}$. Conversely, show that each functional on $\mathcal{P}$ can be obtained this way by a suitable choice of the sequence of scalars.

(4) Let f be a nonzero functional on a vector space V. Let $\alpha \in \mathbb{F}$. Does there exist a vector $v \in V$ such that $f(v) = \alpha$?

(5) Let v be a nonzero vector in a vector space V. Let $\alpha \in \mathbb{F}$. Does there exist a functional f on V such that $f(v) = \alpha$?

(6) Let V be a finite dimensional vector space. Prove the following:
   (a) If $v \in V$ is such that $f(v) = 0$ for all $f \in V'$, then $v = 0$.
   (b) For each nonzero $v \in V$, there exists a linear functional f on V such that $f(v) \neq 0$.
   (c) For every pair of distinct vectors $u, v \in V$, there exists a linear functional f on V such that $f(u) \neq f(v)$.

(7) Consider the basis $\{(-1,-1,1), (-1,1,1), (1,1,1)\}$ of $\mathbb{C}^3$. Let $\{f_1, f_2, f_3\}$ be the corresponding dual basis. Compute $f_i(0,1,0)$ for $i = 1,2,3$.

(8) Let $f : \mathbb{C}^3 \to \mathbb{F}$ be defined by $f(a,b,c) = a+b+c$. Show that $f \in \mathbb{C}'$. Find a basis for $N(f)$.

2.6. **Additional Problems on Linear transformations.**

(1) Let $\mathbb{C}_\mathbb{R}$ be the vector space of all complex numbers over the field $\mathbb{R}$.
   Define $T : \mathbb{C}_R \to \mathbb{R}^{2\times 2}$ by $T(x+iy) = \begin{bmatrix} x+7y & 5y \\ -10y & x-7y \end{bmatrix}$. Answer the following:

(a) Is T injective?
(b) Is it true that $T(z_1z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \mathbb{C}_R$?
(c) How do you describe $R(T)$?

(2) Let $T : V \to W$ be a linear transformation. Show that rank$T < \infty$ if and only if there exists $n \in \mathbb{N}$, $\{v_1,\dots,v_n\} \subset V$ and $\{f_1,\dots,f_n\} \subset \mathcal{L}(V,\mathbb{F})$ such that

$$Tx = \sum_{j=1}^{n} f_j(x)v_j \quad \text{for each} \quad x \in V.$$

Such a linear transformation is said to be of *finite rank*.

(3) Let $A \in \mathbb{F}^{n\times n}$. Prove that if for each $B \in \mathbb{F}^{n\times n}$, $\operatorname{tr}(AB) = 0$, then $A = 0$.

(4) For $x = (x_1,\dots,x_n)^T$ and $y = (y_1,\dots,y_n)^T$ in $\mathbb{R}^{n\times 1}$, define $xy = (x_1y_1,\dots,x_ny_n)^T$. Let A and B be nonzero matrices in $\mathbb{R}^{n\times n}$. Prove that there exists $z \in \mathbb{R}^{n\times 1}$ such that $(BA)(x) \neq (Bx)(Ax)$.

(5) Let V and W be vector spaces. Let U be a subspace of V. Let $T : U \to W$ be a linear transformation. Show that there exists a linear transformation $S : V \to Q$ such that $S_{|U} = T$.

(6) Let $A_t = \begin{bmatrix} \sin 2\pi t & \sin(\pi/6)t \\ \cos 2\pi t & \cos(\pi/6)t \end{bmatrix}$ for $0 \le t \le 12$. Determine $\operatorname{rank}(A_t)$ for each t. Determine the values of t for which $\operatorname{rank}(A_t) = 1$.

(7) Let $\hat{I} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\hat{J} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ and $\hat{K} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Show that $\hat{I}^2 = \hat{J}^2 = \hat{K}^2 = -I$, $\hat{I}\hat{J} = -\hat{J}\hat{I} = K$, $\hat{J}\hat{K} = -\hat{K}\hat{J} = I$, and $\hat{K}\hat{I} = -\hat{I}\hat{K} = J$.

(8) Let $U = \{A \in \mathbb{C}^{n\times n} : A^T = A\}$ and let $W = \{A \in \mathbb{C}^{n\times n} : A^T = -A\}$. Show that U and W are subspaces of $\mathbb{C}^{n\times n}$ and that $C^{n\times n} = U \oplus W$. Matrices in U are called *symmetric matrices* and matrices in W are called *skew-symmetric matrices*.

(9) Let T be a linear operator on a finite dimensional vector space V. Show that if the matrix of T with respect to all bases of V is same, then $T = \alpha I$ for some $\alpha \in \mathbb{F}$.

(10) Prove that if $B \in \mathbb{F}^{m\times m}$ is such that $AB = BA$ for each invertible $A \in \mathbb{F}^{m\times m}$, then B is a scalar matrix.

(11) Let E_{ij} denote the matrix in $\mathbb{F}^{n\times n}$ whose (i, j)th entry s 1 and all other entries are 0. Show that $E_{ij}E_{k\ell} = 0$ if $j \neq k$, and $E_{ij}E_{j\ell} = E_{i\ell}$.

(12) Show that, if $A = [a_{ij}]$ is an $m \times n$ matrix with $a_{ij} \in \mathbb{F}$ and $n > m$, then there exist non-zero scalars $\alpha_1,\dots,\alpha_n$ such that $A[\alpha_1,\dots,\alpha_n]^T = 0$.

(13) Let $A,E \in \mathbb{F}^{m\times m}$, $B,F \in \mathbb{F}^{m\times n}$, $C,G \in \mathbb{F}^{n\times m}$ and $D,H \in \mathbb{F}^{n\times n}$. Show that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}.$$

(14) Let T be an operator on a vector space V of dimension n. Let U be a subspace of V. Prove that $\dim(T(U) \ge \dim(U) - \operatorname{nul}(T)$.

(15) Let $T : V \to \mathbb{F}$ be a linear functional. Let $v \in V \setminus N(T)$. Show that $V = N(T) \oplus \{\alpha v : \alpha \in \mathbb{F}\}$.

(16) Let $T : V \to W$ be a linear transformation where $\dim(V) < \infty$. Show that there exists a subspace U of V such that $R(T) = T(U)$ and $U \cap N(T) = \{0\}$.

(17) Let V and W be finite dimensional vector spaces. Prove the following:
 (a) An injective linear map from V to W exists iff $\dim(V) \leq \dim(W)$.
 (b) A surjective linear map from V to W exists iff $\dim(V) \geq \dim(W)$.

(18) Let $T : V \to W$ be a linear transformation. Prove the following:
 (a) Let $\dim(W) < \infty$. Then T is injective iff there exists $S : W \to V$ such that $ST = I_V$.
 (b) Let $\dim(V) < \infty$. Then T is surjective iff $\exists\ S : W \to V$ such that $TS = I_W$.

(19) Let T be an operator on a vector space such that $T^2 - T + I = 0$. Show that A is invertible.

(20) Let $A \in \mathbb{F}^{m\times n}$. Define $T : \mathbb{F}^{n\times 1} \to \mathbb{F}^{m\times 1}$ by $T(X) = AX$. Show the following:
 (a) If $m < n$, then T can be surjective but not injective.
 (b) If $m > n$, then T can be injective but not surjective.

(21) Let $k, m, n \in \mathbb{N}$. Let $A \in \mathbb{F}^{k\times m}$. Define $T : \mathbb{F}^{m\times n} \to \mathbb{F}^{k\times n}$ by $T(X) = AX$. Prove that T is invertible iff $k = m$ and A is invertible.

(22) Let $p(t)$ be any polynomial. Let S, T be operators where S is invertible. Show that $S^{-1}p(T)S = p(S^{-1}TS)$.

(23) Let S and T be linear operators on a finite dimensional vector space V. Show that ST is invertible iff both S and T are invertible.

(24) Let S and T be linear operators on a finite dimensional vector space V. Show that $ST = I$ iff $TS = I$.

(25) Let $A, B \in \mathbb{F}^{n\times n}$. Prove that if $AB = I$, then $BA = I$.

(26) Let $T : V \to W$ be a linear transformation, where both $\operatorname{rank}T$ and $\operatorname{nul}T$ are finite. Show that V is finite dimensional. (Note: Since $\dim(V) < \infty$ is not assumed, you cannot use the formula $\dim(V) = \operatorname{rank}T + \operatorname{nul}T$.)

(27) Let U be a subspace of a finite dimensional vector space V over $\mathbb{F}$. Let W be any vector space over $\mathbb{F}$. Let $T : U \to W$ be a linear transformation. Show that there exists a linear transformation $S : V \to W$ such that $Tu = Su$ for each $u \in U$.

(28) Let U and V be vector spaces. Let $\{u_1, \ldots, u_n\}$ be a basis for U. Let $v_1, \ldots, v_n \in V$. Prove or disprove:
 (a) There exists a unique linear transformation $T : U \to V$ with $T(u_i) = v_i$ for $i = 1, 2, \ldots, n$.
 (b) This T is one-one iff $\{v_1, \ldots, v_n\}$ is linearly independent.
 (c) This T is onto iff $\operatorname{span}\{v_1, \ldots, v_n\} = V$.

(29) Let U and V be vector spaces. Let $E = \{u_1, \ldots, u_n\} \subseteq U$ be linearly independent and let $v_1, \ldots, v_n \in V$. Prove the following:
 (a) A linear transformation $T : V \to W$ with $Tu_1 = v_1, \ldots, Tu_n = v_n$ exists.
 (b) T in (a) is unique if and only if E is a basis of V.

(30) Let P be an operator on a finite dimensional vector space V such that $P^2 = P$. Prove that $\operatorname{tr}(P) = \operatorname{rank}(P)$.

(31) Let V be a finite dimensional vector space. Let V'' be the dual of the dual of V. Define the map $T : V \to V''$ by $(Tv)(g) = g(v)$ for each $g \in V'$. Prove that T is an isomorphism.

(32) In general, a *hyperspace* in a vector space v is a maximal proper subspace of V. Prove the following:

(a) If f is a nonzero functional on V, then $N(f)$ is a hyperspace in V.
(b) Each hyperspace in V is the null space of some linear functional on V.

(33) Let $A, B \in \mathbb{F}^{n\times n}$. Are the following true?
(a) If A is equivalent to B, then A^* is equivalent to B^*.
(b) If A is equivalent to αI for some scalar α, then $A = \alpha I$.
(c) If A is equivalent to B, then A^2 is equivalent to B^2.
(d) If A and B are invertible and A is equivalent to B, then A^{-1} is equivalent to B^{-1}.

(34) Let $B = \{v_1, \ldots, v_n\}$ be a basis for a vector space V. Let T be the operator on V such that $T(v_1) = v_2, \ldots, T(v_{n-1} = v_n$ and $T(v_n) = 0$. What is $[T]_B$?

(35) Let D be the differentiation operator on $\mathcal{P}_n$. Let T be an operator on $\mathcal{P}_n$ satisfy $T(p(t)) = p(t+1)$. Then show that $T = I + D + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!}$.

(36) Let V be a finite dimensional vector space. Fix a vector $u \in V$ and a linear functional f on V. Define the operator T on V by $Tx = f(x)u$. Find a polynomial $p(t)$ such that $p(T) = 0$.

(37) Let V be a finite dimensional vector space. $T : \mathcal{L}(V,V)$ be defined by $T(X) = PX$ for a given $P \in \mathcal{L}(V,V)$. Under what conditions on P, is T invertible?

(38) Let $t_1, \ldots, t_n \in \mathbb{R}$ be distinct. For any $p(t) \in \mathcal{P}_{n-1}(\mathbb{R})$, let $L_i(p) = p(t_i)$ for each $i \in \{1, \ldots, n\}$. Denote by

$$p_j(t) = \frac{(t-t_1)\cdots(t-t_{j-1})(t-t_{j+1})\cdots(t-t_n)}{(t_j-t_1)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_n)} \quad \text{for } j \in \{1, \ldots, n\},$$

Prove the following:
(a) $\{p_1, \ldots, p_n\}$ is a basis of $\mathcal{P}_{n-1}(\mathbb{R})$.
(b) $\{L_1, \ldots, L_n\}$ is a basis of the dual space of $\mathcal{P}_{n-1}(\mathbb{R})$.
(c) Given $a_1, \ldots, a_n \in \mathbb{R}$, there exists a unique polynomial $p \in \mathcal{P}_{n-1}(\mathbb{R})$ such that $p(t_1) = a_1, \ldots, p(t_n) = a_n$.

The polynomials $p_j(t)$ are called the *Lagrange polynomials*. By doing this exercise you have solved the interpolation problem which asks for constructing a polynomial that takes prescribed values at prescribed points.

(39) Let $T : U \to V$ be an isomorphism. Define $\phi : \mathcal{L}(U,U) \to \mathcal{L}(V,V)$ by $\phi(S) = T^{-1}ST$. Prove that ϕ is an isomorphism.

(40) Let f and g be linear functionals on a vector space V. If h is a linear functional on V with $h(v) = f(v)g(v)$ for each $v \in V$, then prove that either $f = 0$ or $g = 0$.

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