DEPARTMENT OF MATHEMATICS
IIT MADRAS

MA1010 - Calculus I: Functions of one variable

Problem Sheet I: Sequences

1. Suppose that $x_n \longrightarrow l$ as $n \longrightarrow \infty$. Prove that

   (a) $|x_n - l| \longrightarrow 0$ as $n \longrightarrow \infty$.

   (b) $|x_n| - |l| \longrightarrow 0$ as $n \longrightarrow \infty$.

2. Establish convergence or divergence of the following sequences; if converges, then find the limit:
   (i)$\frac{n}{n+1}$; $(ii)\frac{(-1)^n}{n+1}$; $(iii)\frac{2n}{3n^2+1}$; (iv) $\frac{2n^2+3}{3n^2+1}$.

3. Suppose that (a_n) is a sequence of positive real numbers. Show the following:

   (a) If $\lim\limits_{n\to\infty} a_{n+1}/a_n < 1$, then $\{a_n\}$ converges to 0.

   (b) If $\lim\limits_{n\to\infty} a_{n+1}/a_n > 1$, then $\{a_n\}$ diverges.

4. Let $x_n > 0,\ n \in \mathbb{N}$. Prove that $x_n \longrightarrow 0$ as $n \longrightarrow \infty$ if and only if $\frac{1}{x_n} \longrightarrow \infty$ as $n \longrightarrow \infty$.

5. Suppose that (x_n) is increasing and unbounded.

   (a) Prove that $x_n \longrightarrow +\infty$ as $n \longrightarrow \infty$.

   (b) If (x_n) is decreasing and unbounded, prove that $x_n \longrightarrow -\infty$ as $n \longrightarrow \infty$.

6. Let $x_n = \sum_{k=1}^{n} \frac{1}{n+k},\ n \in \mathbb{N}$. Show that (x_n) is convergent.

7. Let (x_n) be a sequence defined recursively by $x_{n+2} = x_{n+1} + x_n,\ n \geq 1$ with $x_1 = x_2 = 1$. Show that (x_n) diverges to ∞.

8. Let $x_n = \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that (x_n) and $(\sqrt{n}x_n)$ are both convergent. Find their limits.

9. Suppose (a_n) is a sequence such that the subsequences (a_{2n-1}) and (a_{2n}) converge to the same limit, say a. Show that (a_n) also converges to a.

10. Let (x_n) be a monotonically increasing sequence such that (x_{3n}) is bounded. Is (x_n) convergent? Justify your answer.

11. Let $0 < r < 1$ and (x_n) be a sequence such that $|x_{n+1} - x| \leq r|x_n - x|$ for $n \in \mathbb{N}$. Prove that (x_n) converges to x. Does the above conclusion follow if the assumption on (x_n) is replaced by $|x_{n+1} - x| < |x_n - x|$ for $n \in \mathbb{N}$?

12. Suppose that (a_n) is a real sequence such that $a_n \to 0$ as $n \to \infty$. Show the following:

    (a) $a_n^2)$ converges to 0.

    (b) If $a_n > 0$ for all n, then $(1/a_n)$ diverges to ∞.

13. If (a_n) converges to x and $a_n \geq 0$ for all $n \in \mathbb{N}$, then show that $x \geq 0$ and $\{\sqrt{a_n}\}$ converges to $\sqrt{x}$.

14. Let $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \in \mathbb{N}$. Show that (a_n) converges. Also, find its limit.

15. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{4}(2a_n + 3)$ for all $n \in \mathbb{N}$. Show that (a_n) is monotonically increasing and bounded above. Find its limit.

16. Let $a_1 = 1$ and $a_{n+1} = \dfrac{a_n}{1 + a_n}$ for all $n \in \mathbb{N}$. Show that (a_n) converges. Find its limit.

17. If $0 < a < 1$, then show that the sequence $\{na^n\}$ converges to 0.

18. Show that, if a sequence (a_n) converges, then $a_{n+1} - a_n \to 0$ as $n \to \infty$. Is the converse true? Why?

19. If $0 < a < b$ and $a_n = (a^n + b^n)^{1/n}$ for all $n \in \mathbb{N}$, then show that (a_n) converges to b. *Hint:* Note that $(a^n + b^n)^{1/n} = (b(1 + \frac{a}{b})^n)^{1/n}$.

20. If $0 < a < b$ and $a_{n+1} = (a^n b^n)^{1/2}$ and $b_{n+1} = (a_n + b_n)/2$ for all $n \in \mathbb{N}$ with $a_1 = a$, $b_1 = b$, then show that (a_n) and (b_n) converge to the same limit.