DEPARTMENT OF MATHEMATICS
IIT MADRAS

MA1010: Calculus I : Functions of one variable[1]

Problem Sheet III

1. Find $\lim\limits_{x\to 0-} \dfrac{|x|}{x}$ and $\lim\limits_{x\to 0+} \dfrac{|x|}{x}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim\limits_{x\to\infty} f(x) = \ell$. Prove that $f(n) \to \ell$ as $n \to \infty$.

3. Let $[x]$ denote the greatest integer not exceeding x. Prove that for $n \in \mathbb{N}$, $\lim\limits_{x\to n+} [x] = n$ and $\lim\limits_{x\to n-} [x] = n - 1$.

4. Use the sequential characterization, show that if $\lim_{x\to a} f(x) = A$ and $\lim\limits_{y\to A} g(y) = B$, then $\lim\limits_{x\to a}(g \circ f)(x)) = B$, provided $g \circ f)(x) := g(f(x))$ is well defined and f does not attain A in a deleted neighbourhood of A.

5. Let $f : (0, \infty) \to \mathbb{R}$. Show that $\lim\limits_{x\to 0} f(x) = b$ if and only if $\lim\limits_{x\to\infty} f(x^{-1}) = b$.

6. Find all the values of b such that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{x^3-1}{x-1}, & x \neq 1, \\ b, & x = 1 \end{cases}$ is continuous at $x_0 = 1$.

7. Find all the values of b such that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ b, & x = 0 \end{cases}$ is continuous at $x_0 = 0$.

8. Use the definition of continuity and the properties of limits to show that each of the following functions is continuous (at the given number a or the given interval):

   (a) $f(x) = (x + 2x3)4; \quad a = -1.$

   (b) $f(x) = \dfrac{(2x - 3x)^2}{1 + x^3}; \quad a = 1.$

   (c) $2\sqrt{3 - x}; \quad a \in (-\infty, 3].$

9. There exists $x \in \mathbb{R}$ such that $\sin x = x^2 - x$. Why?

10. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If $c \in (a, b)$ is such that $f(c) > 0$, and if $0 < \beta < f(c)$, then show that there exists $\delta > 0$ such that $f(x) > \beta$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the relation $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$. If f is continuous at 0, then show that f is continuous at every $x \in \mathbb{R}$, and in that case $f(x) = xf(1)$ for every $x \in \mathbb{R}$.

12. There does not exist a continuous function f from $[0, 1]$ onto $\mathbb{R}$. Why?

[1]Some of the problems are taken from the book *Calculus of One Variable* by M.T. Nair.

13. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous on the intervals $[a, c]$ and $[c, b]$ for some $c \in (a, b)$, then f is continuous on $[a, b]$.

14. Suppose $f : [a, b] \to [a, b]$ is continuous. Show that there exists $c \in [a, b]$ such that $f(c) = c$.

15. Suppose $f : [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [a, b]$ for some $K \ge 0$. Then f is a continuous function. Why?

16. Suppose $f : [a, b] \to [a, b]$ is such that there exists r satisfying $0 < r < 1$ and $|f(x) - f(y)| \le r|x - y|$ for all $x, y \in [a, b]$. Let $x_1 \in [a, b]$ and for $n \in \mathbb{N}$, let $x_{n+1} := f(x_n)$. Prove that $x_n \to c$ for some c and $f(c) = c$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 + x + x^2$. Without solving a quadratic equation, can you assert that there is some x_0 such that $f(x_0) = 2$?

18. If $p(x)$ is a polynomial of odd degree, then there exists at least one $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$. Why?

19. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(x) \to 0$ as $|x| \to \infty$. Prove that f attains either a maximum or a minimum.

20. Let $f : [a, b] \to [a, b]$ be such that $|f(x) - f(y)| \le |x - y|/2$ for all $x, y \in [a, b]$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

21. Prove the following.

    (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $\lim_{x \to -\infty} f(x) = c$ and $\lim_{x \to \infty} f(x) = d$, where $c < d$, then for every $y \in (c, d)$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

    (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $\lim_{x \to -\infty} f(x) \to c$ and $\lim_{x \to \infty} f(x) = \infty$, where $c < d$, then for every $y \in (c, \infty)$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

    (c) From (b) above, deduce that for every $y \in (0, \infty)$, there exists $x \in \mathbb{R}$ such that $e^x = y$.

22. Prove that if f is strictly monotonic on an interval I, then f is injective on I.

23. Let $n \in \mathbb{N}$. Prove that for every $y \ge 0$, there exists a unique $x \ge 0$ such that $x^n = y$.

24. Suppose f is a continuous function defined on an interval I and x_0 is an interior point of I. Prove the following.

    (a) If f is increasing on $(x_0 - h, x_0)$ and decreasing on $(x_0, x_0 + h)$ for some $h > 0$, then f attains local maximum at x_0.

    (b) If "increasing" and "decreasing" in (a) above are interchanged, then in the conclusion "maximum" can be replaced by "minimum".

    (c) If "increasing" and "decreasing" in (a) are replaced by "strictly increasing" and "strictly decreasing", respectively, then we obtain "strict local maximum".

25. Let f be a continuous function defined on an interval I. Show that, if f is injective, then it is strictly monotonic on I [Hint: Use Intermediate Value Theorem].