DEPARTMENT OF MATHEMATICS
IIT MADRAS

MA1010: Calculus I : Functions of one variable[1]

Problem Sheet IV

1. Let x_0 be an interior point of an interval I. Prove that $f : I \to \mathbb{R}$ is differentiable at x_0 if and only if $f'_+(x_0)$ and $f'_-(x_0)$ exist and $f'_-(x_0) = f'_+(x_0)$, and in that case $f'(x_0) = f'_-(x_0) = f'_+(x_0)$.

2. Consider a polynomial $p(x) = a_0 + a_1x^2 + \ldots + a_nx^n$ with real coefficients $a_0, a_1, \ldots, a_n$ such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0$. Show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.

   [Note that the conclusion need not hold if the condition imposed on the coefficients is dropped. To see this, consider $p(x) = 1 + x^2$.]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$ Show that f is differentiable at 0 and $f'(0) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Show that f is differentiable at every $x \in \mathbb{R}$. Is f' continuous?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2, & x \geq 0, \\ 0, & x < 0. \end{cases}$ Show that f is differentiable at every $x \in \mathbb{R}$. Is f' continuous?

6. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 + x + x^3$ has neither maximum nor minimum at any point. Why?

7. Find the points at which the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 = x - x^3$ attain local maxima and local minima.

8. Using Taylor's theorem, show that
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n.$$

9. Suppose f is differentiable on $(0, \infty)$ and $\lim_{x\to\infty} f'(x) = 0$.

   Prove that $\lim_{x\to\infty} [f(x+1) - f(x)] = 0$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 + x + x^3$. Prove that f is a bijection.

    [*Hint:* Use Intermediate value theorem and mean value theorem.]

11. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Prove the following.

[1] Problems are taken from the book *Calculus of One Variable* by M.T. Nair.

(a) If $f'(x) \geq 0$ (respectively, $f'(x) > 0$) for all $x \in (a, b)$, then f is monotonically increasing (respectively, strictly increasing) on $[a, b]$.

(b) If $\lim_{x \to a} f'(x)$ exists, then $f'(a)$ exists and $f'(a) = \lim_{x \to a} f'(x)$.

[*Hint:* Use mean value theorem.]

12. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Prove that if f' is strictly positive or strictly negative on (a, b), then it is a one-one function.

13. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Prove that if $f'(x) \neq 0$ for all $x \in (a, b)$, then f is either strictly increasing or strictly decreasing on $[a, b]$.

14. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which is differentiable on $(0, 1)$. Prove that, if $f(0) = 0$ and $f(1) = 1$, then there exists $c \in (0, 1)$ such that $f'(c) = 1$.

15. Using MVT, prove the following:

    (a) $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

    (b) $|\sin x - \sin y| \leq |x - y|$ for all $x \in \mathbb{R}$.

    (c) $\dfrac{x-1}{x} \leq \ln x \leq x - 1$ for all $x \in \mathbb{R}$.

16. Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ for all $x \in \mathbb{R}$. What is wrong with the following statement?

$$\text{Since } g'(0) \neq 0, \quad \lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = 0.$$

17. Prove that $\lim_{x \to \infty} x \ln(1 + \frac{1}{x})$ exists and it is equal to 1. [*Hint*: L'Hospital's rule]

18. Find the following limits.

$$\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2}, \quad \lim_{x \to 0} \frac{1}{x(\ln x)^2}, \quad \lim_{x \to \infty} \frac{\ln x}{x^2}, \quad \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

19. Let f be a function defined on an interval I. Prove the following.

    (a) f is convex (i.e., convex downwards) if and only if the line segment joining any tow points on the graph of f lies in the region $\{(x, y) : f(x) \leq y,\ x \in I\}$.

    (b) f is concave (i.e., concave upwards) if and only if the line segment joining any tow points on the graph of f lies in the region $\{(x, y) : y \leq f(x),\ x \in I\}$.

20. Find approximate values of $\sin x$, $\cos x, \tan^{-1} x$ at the point $x = 1/2$ using Taylor's formula and find the estimates for the errors for different values of n.