DEPARTMENT OF MATHEMATICS
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MA1010: Calculus I : Functions of one variable[1]

Problem Sheet V

1. Let f be a bounded function on $[a, b]$. Prove that, if there is a partition P of $[a, b]$ such that $L(P, f) = U(P, f)$, then f is a constant function.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If P and Q are partitions of $[a, b]$ such that Q is a refinement of P (i.e., Q is obtained from P by including some more points), then show that

$$L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f)$$

for all $n \in \mathbb{N}$.

3. For a bounded function $f : I \to \mathbb{R}$, let $m_f := \inf_{x \in I} f(x)$ and $M_f := \sup_{x \in I} f(x)$. Prove that, if f and g are bounded functions on I, then

$$m_f + m_g \leq m_{f+g} \leq M_{f+g} \leq M_f + M_g.$$

Deduce that, if $f : [a, b] \to \mathbb{R}$ is a bounded function, then

$$L(P, f) + L(P, g) \leq L(P, f + g) \leq U(P, f + g) \leq U(P, f) + U(P, g)$$

for any partition P of $[a, b]$.

4. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Justify the following statements:

   (a) For every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$0 \leq \int_a^b f(x)dx - L(P, f) < \varepsilon.$$

   (b) For every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$0 \leq U(P, f) - \int_a^b f(x)dx < \varepsilon.$$

   (c) For every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$0 \leq \left| S(P, f, T) - \int_a^b f(x)dx \right| < \varepsilon$$

   for every tag T on P.

[1]Problems are taken from the book *Calculus of One Variable* by M.T. Nair.

5. If f is either monotonically increasing or monotonically decreasing function on $[a, b]$, then show that f is integrable.

6. Prove that every piecewise constant function on $[a, b]$ is integrable.

7. Prove that every bounded piecewise continuous function on $[a, b]$ is integrable.

8. Let f be a bounded function on $[a, b]$ such that $|f|$ is integrable. Is it necessary that f is integrable?

9. Let f be integrable on $[a, b]$ and $g(x) = \int_a^x f(t)dt$. Show the following:

   (a) If $f(x) \geq 0$ for all $x \in [a, b]$, then g is monotonically increasing.

   (b) If $f(x) \leq 0$ for all $x \in [a, b]$, then g is monotonically decreasing.

10. Suppose f is integrable on $[a, b]$. Prove the following.

    (a) $\displaystyle\lim_{t \to b} \int_a^t f(x)dx = \int_a^b f(x)dx.$

    (b) If (a_n) and (b_n) are in $[a, b]$ which converge to a and b, respectively, then $\displaystyle\lim_{n \to \infty} \int_{a_n}^{b_n} f(x)dx = \int_a^b f(x)dx.$

11. Using Newton-Leibnitz formula, evaluate the following integrals.

$$\text{(a)} \int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}, \qquad \text{(b)} \int_0^1 \sqrt{3x+1}dx,$$
$$\text{(c)} \int_1^2 (x^3 + x + 1)dx, \quad \text{(d)} \int_0^{2\pi/\omega} \cos^2(\omega t)dt, \quad \omega > 0.$$

12. Find the following limits by interpreting the sequence as a Riemann sum and then applying Newton-Leibnitz formula.

    (a) $\displaystyle\lim_{n \to \infty} \frac{1}{n}\Big[\sin(\pi/n) + \sin(2\pi/n) + \cdots + \sin(n\pi/n)\Big].$

    (b) $\displaystyle\lim_{n \to \infty} \Big[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}\Big].$

    (c) $\displaystyle\lim_{n \to \infty} \Big[\frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}\Big].$

    (d) $\displaystyle\lim_{n \to \infty} \Big[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+(n-1)n}}\Big].$

13. Justify the statement: Given a continuous function f on $[a, b]$ and $c \in \mathbb{R}$, there exists a function g which is continuous on $[a, b]$ and differentiable on (a, b) such that $g' = f$ on (a, b) and $g(a) = c$.

14. State the required properties of the function $f : [a, b] \to \mathbb{R}$ such that $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \int_a^x \frac{d}{dt} f(t)dt.$

15. Find explicit expressions for the following.

$$\text{(a)}\ \frac{d}{dx}\int_1^{e^x} \ln t\,dt \qquad \text{(b)}\ \frac{d}{dx}\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$$

$$\text{(a)}\ \frac{d}{dx}\int_1^{x^2} \frac{1}{1+t^3}\,dt \qquad \text{(b)}\ \frac{d}{dx}\int_{x^2}^{x} \sqrt{\sqrt{1+t^2}}$$

16. Evaluate the following integral by using appropriate change of variable, and properties of integrals.

$$\text{(a)}\ \int_0^1 \sqrt{1-x^2}dx \qquad \text{(b)}\ \int_0^1 xe^{x^2}dx \qquad \text{(c)}\ \int_0^1 x^3e^{x^2}dx$$

$$\text{(d)}\ \int_1^4 \frac{x^2}{\sqrt{1+x^3}}dx \qquad \text{(e)}\ \int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}dx \qquad \text{(f)}\ \int_0^1 x\sqrt{1+x^2}dx$$

17. Prove the following:

(a) $\displaystyle\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx$ for any $m, n \in \mathbb{N}$.

(b) $\displaystyle\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

18. Show that, if f is twice differentiable and f' and f'' are continuous in an open interval containing $[a,]$, then

$$\int_a^b f'(x)dx + \int_a^b xf''(x)dx = [bf'(b) - af'(a)].$$

Geometric and mechanical applications

19. Find the area of the portion of the circle $x^2+y^2=1$ which lies inside the parabola $y^2 = 1-x$. *Ans*: $\frac{\pi}{2}+\frac{4}{3}$

20. Find the area common to the cardioid $\rho = a(1+\cos\theta)$ and the circle $\rho = \frac{3a}{2}$.
Ans: $\frac{7}{4}\pi - \frac{9\sqrt{3}}{8}$.

21. For $a, b > 0$, find the area included between the parabolas given by $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$. *Ans*: $\frac{8}{3}\sqrt{ab}(a+b)$.

22. Find the area of the loop of the curve $r^2\cos\theta = a^2\sin 3\theta$

23. Find the area of the region bounded by the curves $x-y^3=0$ and $x-y=0$.
Ans: 1/2

24. Find the area of the region that lies inside the circle $r = a\cos\theta$ and outside the cardioid $r = a(1-\cos\theta)$. *Ans*: $\frac{a^3}{3}(3\sqrt{3}-\pi)$

25. Find the area of the loop of the curve $x = a(1 - t^2)$, $y = at(1 - t^2)$ for $-1 \le t \le 1$.
Ans: $8a^2/15$

26. Find the length of an arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$. *Ans*: $8a$.

27. For $a > 0$, find the length of the loop of the curve $3a\, y^2 = x(x - a)^2$.

28. Find the length of the curve $r = \frac{2}{1+\cos\theta}$, $0 \le \theta \le \pi/2$. *Ans*: $\sqrt{2} + \ln(\sqrt{2} + 1)$.

29. Find the volume of the solid obtained by revolving the curve $y = 4\sin 2x$, $0 \le x \le \pi/2$, about y-axis. *Ans*: $2\pi^2$.

30. Find the area of the surface obtained by revolving a loop of the curve $9ax^2 = y(3a - y)^2$ about y-axis. *Ans*: $3\pi a^2$.

31. Find the area of the surface obtained by revolving about x-axis, an arc of the catenary $y = c \cosh(x/c)$ between $x = -a$ and $x = a$ for $a > 0$.

32. The lemniscate $\rho^2 = a^2 \cos 2\theta$ revolves about the line $\theta = \frac{\pi}{4}$. Find the area of the surface of the solid generated.