DEPARTMENT OF MATHEMATICS
IIT MADRAS

MA1010: Calculus I : Functions of one variable[1]

Problem Sheet VI

1. Suppose (a_n) in $\mathbb{R}$ is such that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Show that $\sum_{n=1}^{\infty} \frac{a_n x^{2n}}{1+x^{2n}}$ is a dominated series on $\mathbb{R}$.

2. Show that for every r with $0 < r < 1$, the series $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty} nx^n$ are dominated series on $[-r, r]$.

3. Show that $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is a dominated series on $\mathbb{R}$.

4. Show that $\sum_{n=1}^{\infty} \frac{x}{1+nx^2}$ is a dominated series on $[c, \infty)$ for any $c > 0$.

5. Show that $\sum_{n=1}^{\infty} (xe^{-x})^n$ is a dominated series on $[0, \infty)$ for any $c > 0$.

6. Show that $\sum_{n=1}^{\infty} (1-x)x^{n-1}$ is not a dominated series on $[0, 1]$.

7. Show that the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is dominated on any closed interval $[a, b]$. Is it dominated on $\mathbb{R}$?

8. Justify the statement: For each $p > 1$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$ is convergent on $[-1, 1]$ and the limit function is continuous.

9. Show that the series $\sum_{n=1}^{\infty} \{(n+1)^2 x^{n+1} - n^2 x^n\}(1-x)$ converges to a continuous function on $[0, 1]$, but it is not dominated.

10. Show that the series $\displaystyle\sum_{n=1}^{\infty} [\frac{1}{1+(k+1)x} - \frac{1}{1+kx}]$ is convergent on $[0, 1]$, but it is not a dominated series. Show also that the series can be integrated term by term over the interval $[0, 1]$.

11. Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges at a point x_0. Prove that it converges at every $x \in (-|x_0|, |x_0|)$ and diverges at every $x \notin [-|x_0|, |x_0|]$.

12. Show that the series $\sum_{n=1}^{\infty} nx^{n-1}$ converges pointwise. Is it dominated on $(-1, 1)$? Why?

13. Find the radius of convergence of the following power series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n4^n}$

(iii) $\displaystyle\sum_{n=1}^{\infty} x^{n^2}$ (iv) $\displaystyle\sum_{n=0}^{\infty} \frac{(n-1)!}{n^n} x^n$

[1]Problems are taken from the book *Calculus of One Variable* by M.T. Nair.

14. Find the interval of convergence of the following power series.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{(-1)^n}{nx^n} \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{(4n-1)x^n}$$

$$\text{(iii)} \sum_{n=1}^{\infty} \frac{n(x+5)^n}{(2n+1)^3} \qquad \text{(iv)} \sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2}$$

[Answers: (i): $(-\infty,-1) \cup [1,\infty)$; (ii): $(-\infty,-3) \cup [3,\infty)$; (iii): $[-6,-4]$; (iv): $[-\frac{\pi}{6} + k\pi, \frac{\pi}{6} + k\pi]$, $k \in \mathbb{Z}$.]

15. Find the radius of convergence and interval of convergence of the following series:

$$\text{(i)} \sum_{n=0}^{\infty} \alpha^n x^n \quad \text{for } \alpha > 0 \qquad \text{(ii)} \sum_{n=0}^{\infty} \alpha^{n^2} x^n \quad \text{for } \alpha > 0$$

$$\text{(iii)} \sum_{n=0}^{\infty} x^{n!} \qquad \text{(iv)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^{n(n+1)}$$

$$\text{(v)} \sum_{n=0}^{\infty} \frac{\sin(n\pi/6)}{2^n}(x-1)^n \qquad \text{(vi)} \sum_{n=0}^{\infty} \frac{(-i)^n}{4^n n^\alpha} x^{2n}$$

16. Show that

(i) $\log(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ for $-1 < x \le 1$,

(ii) $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $-1 < x \le 1$, and derive $\displaystyle\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$.

(iii) $\displaystyle\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$ for $-1 < x < 1$.