DEPARTMENT OF MATHEMATICS
IIT MADRAS

MA1010: Calculus I : Functions of one variable[1]

Problem Sheet VII

1. Justify the following statements:
   (a) If $\int_a^\infty f(x)\,dx$ exists, then $\int_c^\infty f(x)\,dx$ exists for any $c > a$.
   (b) If $\int_{-\infty}^b f(x)\,dx$ exists, then $\int_{-\infty}^c f(x)\,dx$ exists for any $c < b$.
   (c) If f is defined on $(-\infty, \infty)$ and integrable on every closed and bounded interval, and if $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ exist for some $c \in \mathbb{R}$, then $\int_{-\infty}^d f(x)\,dx$ and $\int_d^\infty f(x)\,dx$ exist for any $d \in \mathbb{R}$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an even function, that is, $f(-x) = f(x)$ for every $x \in \mathbb{R}$. Prove that $\int_{-\infty}^\infty f(x)dx$ exists if and only if $\int_0^\infty f(x)dx$ exists, and in that case
$$\int_{-\infty}^\infty f(x)dx = 2\int_0^\infty f(x)dx.$$

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an odd function, that is, $f(-x) = -f(x)$ for every $x \in \mathbb{R}$. Prove that $\int_{-\infty}^0 f(x)dx$ exists if and only if $\int_0^\infty f(x)dx$ exists, and in that case
$$\int_{-\infty}^\infty f(x)dx = 0.$$

4. Justify the following statements:
   (a) Suppose f is defined on $[a, \infty)$ with values in $[0, \infty)$. Then $\int_a^\infty f(x)dx$ converges if and only if $\lim_{n\to\infty} \int_a^n f(x)dx$ exists.
   (b) Suppose f is defined on $(-\infty, b]$ with values in $[0, \infty)$. Then $\int_{-\infty}^b f(x)dx$ converges if and only if $\lim_{n\to\infty} \int_{-n}^b f(x)dx$ exists.

5. Justify the following statements:
   (a) Suppose f is defined on $[a, \infty)$. Then $\int_a^\infty f(x)dx$ converges if and only if for any sequence (b_n) in $\mathbb{R}$, $b_n \to \infty$ implies $\lim_{n\to\infty} \int_a^{b_n} f(x)dx$ exists.
   (b) Suppose f is defined in $(-\infty, b]$. Then $\int_{-\infty}^b f(x)dx$ converges if and only if for any sequence (a_n) in $\mathbb{R}$, $a_n \to -\infty$ implies $\lim_{n\to\infty} \int_{a_n}^b f(x)dx$ exists.

6. Suppose f is absolutely integrable over $(-\infty, \infty)$. Show that
$$\int_{-\infty}^\infty f(x)dx = \lim_{t\to\infty} \int_{-t}^t f(x)dx.$$

[1]Problems are taken from the book *Calculus of One Variable* by M.T. Nair.

7. Suppose $f \geq 0$ on $[a,b)$ and the integral $\int_a^t f(x)dx$ exists for every $t \in [a,b)$. If $\lim_{x\to b}(b-x)^\alpha f(x)$ converges for some $\alpha < 1$, then show that $\int_a^b f(x)dx$ also converges.

   [*Hint:* Observe that for any $\varepsilon > 0$, there exists $x_0 \in [a,b)$ such that the number $\beta := \lim_{x\to b}(b-x)^\alpha f(x)$ satisfies $0 \leq f(x) \leq \frac{\beta+\varepsilon}{(b-x)^\alpha}$ for all $x \in [x_0,b)$.]

8. For $a < b$ and $p < 1$, show that the improper integral $\int_a^b \frac{dx}{(x-a)^p}$ converges.

9. Show that the improper integral $\int_1^\infty \frac{\cos x}{x^p}\, dx$ converges for all $p > 0$.

10. Show that, for $p > 1$, $\int_0^1 \frac{dx}{x^{1/p}} = \frac{p}{p-1}$.

11. Show that, for $p > 1$, $\int_0^1 \frac{dx}{x^{1/p}} - \int_1^\infty \frac{dx}{x^{1/p}} = 1$.

12. Show that $\int_1^\infty \frac{dx}{x^p} \to 0$ as $p \to \infty$ and $\int_0^1 \frac{dx}{x^p} \to 1$ as $p \to 0$.

13. Show that $\lim_{p\to 1-} \int_0^1 \frac{dx}{x^p} = \infty$.

14. Does $\int_1^\infty \sin\left(\frac{1}{x^2}\right) dx$ converge? [*Hint*: Note that $\left|\sin\left(\frac{1}{x^2}\right)\right| \leq \frac{1}{x^2}$.]

15. Does $\int_2^\infty \frac{\cos x}{x(\log x)^2} dx$ converge?

    [*Hint*: Observe $\left|\frac{\cos x}{x(\log x)^2}\right| \leq \frac{1}{x(\log x)^2}$ and use the change of variable $t = \log x$.]

16. Does $\int_0^\infty \frac{\sin^2 x}{x^2} dx$ converge?

    [*Hint*: Observe $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for $x \geq 1$ and $\frac{\sin^2 x}{x^2}$, $0 < x \leq 1$ has a continuous extension on $[0,1]$.]

17. Does $\int_0^1 \frac{\sin x}{x^2} dx$ converge? [*Hint*: Observe $\frac{\sin x}{x^2} = \left(\frac{\sin x}{x}\right)\frac{1}{x} \geq \frac{\sin 1}{1}$.]

18. Suppose $f \geq 0$ on $[a,b)$ and the integral $\int_a^t f(x)dx$ exists for every $t \in [a,b)$. If $\lim_{x\to b}(b-x)^\alpha f(x)$ converges for some $\alpha < 1$, then show that $\int_a^b f(x)dx$ also converges.

    [*Hint:* Observe that for any $\varepsilon > 0$, there exists $x_0 \in [a,b)$ such that the number $\beta := \lim_{x\to b}(b-x)^\alpha f(x)$ satisfies $0 \leq f(x) \leq \frac{\beta+\varepsilon}{(b-x)^\alpha}$ for all $x \in [x_0,b)$.]

19. Does $\int_0^\infty e^{-x^2}\, dx$ converge? [*Hint*: $e^{-x^2} \leq \frac{1}{x^2}$ for $1 \leq x \leq \infty$.]

20. Does $\int_2^\infty \frac{\sin(\log x)}{x} dx$ converge?

    [*Hint*: Change of variable $t = \log x$, and the fact that $\int_{\log 2}^\infty \sin t\, dt$ diverges.]