Assignment-2
Linear Algebra and Numerical Analysis

Jan-May, 2012

1. Consider the system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \ldots & + & \ldots & + & \ldots & + & \ldots & = & \ldots \\ a_{m1}x_1 & + & a_{m1}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

Let

$$u_1 := \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}, u_2 := \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix}, \ldots, u_n := \begin{bmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{bmatrix}.$$

(a) Show that the above system has a solution vector $x = [x_1, \ldots, x_n]^T$ if and only if $b = [b_1, \ldots, b_n]^T \in \text{span}(\{u_1, \ldots, u_n\}$.

(b) Show that the above system has atmost one solution vector $x = [x_1, \ldots, x_n]^T$ if and only if $\{u_1, \ldots, u_n\}$ is linearly independent.

2. Is union and intersection of two linearly independent sets linearly independent? Justify yoiur answer.

3. Is union (resp., intersection) of two linearly dependent sets a linearly dependent? Why?

4. Show that vectors $u = (a, c)$, $v = (b, d)$ are linearly independent in $\mathbb{R}^2$ iff $ad - bc \neq 0$. Can you think of a generalization to n vectors in $\mathbb{R}^n$.

5. Show that $V_0 := \{x = (x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$. Find a basis for V_0.

6. Show that $E := \{1 + t^n, t + t^n, t^2 + t^n, \ldots, t^{n-1} + t^n, t^n\}$ is a basis of $\mathcal{P}_n$.

7. Let $u_1(t) = 1$, and for $j = 2, 3, \ldots$, let $u_j(t) = 1 + t + \ldots + t^j$. Show that span of $\{u_1, \ldots, u_n\}$ is $\mathcal{P}_n$, and span of $\{u_1, u_2, \ldots\}$ is $\mathcal{P}$.

8. Let $p_1(t) = 1 + t + 3t^2$, $p_2(t) = 2 + 4t + t^2$, $p_3(t) = 2t + 5t^2$. Are the polynomials p_1, p_2, p_3 linearly independent?

9. Suppose V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$. Show that every $x \in V_1 + V_2$ can be written *uniquely* as $x = x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$.

10. Suppose V_1 and V_2 are subspaces of a vector space V. Show that $V_1 + V_2 = V_1$ if and only if $V_2 \subseteq V_1$.

11. Let V be a vector space.

   (a) Show that a subset $\{u_1, \ldots, u_n\}$ of V is linearly independent if and only if the function $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 u_1 + \cdots + \alpha_n u_n$ from $\mathbb{F}^n$ into V is injective.

   (b) Show that if $E \subseteq V$ is linearly dependent in V, then every superset of E is also linearly dependent.

   (c) Show that if $E \subseteq V$ is linearly independent in V, then every subset of E is also linearly independent.

   (d) Show that if $\{u_1, \ldots, u_n\}$ is a linearly independent subset of V, and if W is a subspace of V such that $(\text{span}\{u_1, \ldots, u_n\}) \cap W = \{0\}$, then every V in the span of $\{u_1, \ldots, u_n, W\}$ can be written uniquely as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n + y$ with $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$, $y \in W$.

   (e) Show that if E_1 and E_2 are linearly independent subsets of V such that $(\text{span}E_1) \cap (\text{span}E_2) = \{0\}$, then $E_1 \cup E_2$ is linearly independent.

12. For each $k \in \mathbb{N}$, let $\underline{\mathbb{F}}^k$ denotes the set of all column k-vectors, i.e., the set of all $k \times 1$ matrices. Let A be an $m \times n$ matrix of scalars with columns $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$. Show the following:

   (a) The equation $A\underline{x} = \underline{0}$ has a non-zero solution if and only if $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ are linearly dependent.

   (b) For $\underline{y} \in \underline{\mathbb{F}}^m$, the equation $A\underline{x} = \underline{y}$ has a solution if and only if $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n, \underline{y}$ are linearly dependent, i.e., if and only if $\underline{y}$ is in the span of columns of A.

13. For $i = 1, \ldots, m$; $j = 1, \ldots, n$, let E_{ij} be the $m \times n$ matrix with its (i, j)-th entry as 1 and all other entries 0. Show that

$$\{E_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$$

   is a basis of $\mathbb{F}^{m \times n}$.

14. If $\{u_1, \ldots, u_n\}$ is a basis of a vector space V, then show that every $x \in V$, can be expressed *uniquely* as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$; i.e., for every $x \in V$, there exists a unique n-tuple $(\alpha_1, \ldots, \alpha_n)$ of scalars such that $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$.

15. Suppose S is a set consisting of n elements and V is the set of all real valued functions defined on S. Show that V is a vector space of dimension n.

16. Given real numbers $a_0, a_1, \ldots, a_k$, let V be the set of all solutions $x \in C^k[a, b]$ of the differential equation

$$a_0 \frac{d^k x}{dt^k} + a_1 \frac{d^{k-1} x}{dt^{k-1}} + \cdots + a_k x = 0.$$

   Show that V is a linear space over $\mathbb{R}$. What is the dimension of V?

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